The effective potential of the confinement order parameter in the Hamiltonian approach

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ABSTRACT

The effective potential of the order parameter for confinement is calculated within the Hamiltonian approach by compactifying one spatial dimension and using a background gauge fixing. Neglecting the ghost and using the perturbative gluon energy one recovers the Weiss potential. From the full non-perturbative potential calculated within a variational approach a critical temperature of the deconfinement phase transition of 269 MeV is found for the gauge group SU(2).

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1. Introduction

Understanding the phase structure of QCD is one of the major challenges of particle physics [1]. Running and upcoming high-energy heavy-ion experiments call for a deeper understanding of hadronic matter under extreme conditions. The central issue is the understanding of the deconfinement phase transition from the confining hadronic phase with chiral symmetry spontaneously broken to the deconfined quark-gluon plasma with chiral symmetry restored. This transition is expected to be driven by the gluon dynamics and lattice calculations show that confinement is exclusively determined by the strongly interacting low-energy gluonic modes [2]. Therefore, understanding the deconfinement phase transition requires non-perturbative methods. In quenched QCD reliable results on the deconfinement phase transition have been obtained by means of the lattice approach [3], which, however, fails at large baryon densities due to the notorious fermion sign problem. Therefore, alternative non-perturbative methods based on the continuum formulation of QCD are required. In recent years substantial progress has been achieved within continuum approaches to QCD [4–7]. Among these is the variational approach to the Hamiltonian formulation of QCD in Coulomb gauge [7,8] (see also Refs. [9,10] for related earlier work). In this approach the energy density is minimized using Gaussian type ansätze for the Yang–Mills vacuum wave functional. Within this approach a decent

description of the infrared sector of Yang–Mills theory was obtained [11–14]. Recently, this approach was extended to finite temperatures by considering the grand canonical ensemble making a suitable quasi-particle ansatz for the density operator and minimizing the free energy [15,16]. In this Letter we present an alternative Hamiltonian approach to finite temperature Yang–Mills theory, which does not require an ansatz for the density operator. The finite temperature is introduced here by compactifying one spatial dimension.

2. Order parameter for confinement

As is well known, Euclidean quantum field theory can be extended to finite temperature L^{-1} by compactifying the Euclidean time dimension to an effective length L. At temperature L^{-1} the order parameter for confinement is the expectation value $\langle P[A_0] \rangle$ of the Polyakov loop [17] ($\mathcal{P}$ path-ordering)

$$P[A_0] = \frac{1}{N} \operatorname{tr} \mathcal{P} e^{-\int_0^L dx^0 A_0(x^0, \mathbf{x})}. \tag{1}$$

This quantity is related to the free energy of a static (infinitely heavy) quark at spatial position $\mathbf{x}$. In the absence of fermions Yang–Mills theory is invariant under gauge transformations $U(x^0, \mathbf{x}) \in SU(N)$ being periodic up to a center element $z_k \in Z(N)$

$$U(L, \mathbf{x}) = z_k U(0, \mathbf{x}). \tag{2}$$

Under such a gauge transformation the Polyakov loop transforms as $P[A_0^U] = z_k P[A_0]$ and as a consequence $\langle P[A_0] \rangle = 0$ in the

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center symmetric confining phase while $\langle P[A_0] \rangle \neq 0$ in the de-confining phase with center symmetry spontaneously broken. In Polyakov gauge, $\partial_0 A_0 = 0$, and with A_0 residing in the Cartan algebra, in the fundamental modular region $P[A_0]$ is a convex function of A_0 and by Jensen's inequality $\langle P[A_0] \rangle \leqslant P[\langle A_0 \rangle]$, instead of $\langle P[A_0] \rangle$, one can alternatively use $P[\langle A_0 \rangle]$ or $\langle A_0 \rangle$ as order parameter for confinement [18,19]. Note however, that by gauge invariance a non-vanishing $\langle A_0 \rangle$ requires the presence of an external background field a_0. Choosing a_0 in Polyakov gauge and to satisfy $\langle A_0 \rangle = a_0$ the background field becomes an order parameter for confinement whose value is determined by the minimum of the effective potential $V[\langle A_0 \rangle = a_0]$. In Polyakov gauge there are still residual gauge transformations satisfying Eq. (2), which transform A_0 to $A_0^U = A_0 + \mu_k/L$, where μ_k is a coweight satisfying $\exp(-\mu_k) = z_k$, and as a consequence of gauge invariance the effective potential of $\langle A_0 \rangle = a_0$ must obey the periodicity condition

$$V[a_0 + \mu_k/L] = V[a_0]. \tag{3}$$

This potential was first calculated in Ref. [20] in one-loop perturbation theory. It was found that $V[a_0]$ is minimal at $a_0 = 0$, so that $\langle P \rangle \simeq P[\langle A_0 \rangle = 0] = 1$ implying that the perturbative theory is in the non-confining phase with center symmetry broken. This, of course, is the expected behavior at high temperatures, where perturbation theory is reliable. In this Letter we calculate non-perturbatively the effective potential $V[\langle A_0 \rangle]$ in the Hamiltonian approach and determine from this potential the critical temperature of the deconfinement phase transition.

3. Finite temperature from compactification of a spatial dimension

Clearly the order parameter $\langle P \rangle \approx P[\langle A_0 \rangle]$ or $\langle A_0 \rangle$ is not directly accessible in Weyl gauge $A_0 = 0$, which is assumed in the canonical quantization. However, by O(4) symmetry, all four Euclidean dimensions are equivalent and instead of compactifying the time, one can equally well introduce the temperature by compactifying one of the spatial dimension, say the x_3-axis, and consider $\langle A_3 \rangle$ as order parameter for confinement. This can be seen as follows:

Consider Yang–Mills theory at finite temperature L^{-1}, which is defined by the partition function

$$Z(L) = \mathrm{Tr}\, e^{-LH(\mathbf{A})}. \tag{4}$$

Here $H(\mathbf{A})$ is the usual Yang–Mills Hamiltonian defined by canonical quantization in Weyl gauge $A^0 = 0$. The partition function (4) can be equivalently represented by the Euclidean functional integral, see for example Ref. [21]

$$Z(L) = \int\limits_{x^0-pbc} \prod_\mu \mathcal{D}A_\mu(x) e^{-S[A]}, \tag{5}$$

where

$$S[A] = \int\limits_{-L/2}^{L/2} dx^0 \int d^3x\, \mathcal{L}(A^\mu; x^\mu) \tag{6}$$

is the Euclidean action and the functional integration is performed over temporally periodic fields

$$A^\mu\left(\frac{L}{2}, \mathbf{x}\right) = A^\mu\left(-\frac{L}{2}, \mathbf{x}\right), \tag{7}$$

which is indicated in Eq. (5) by the subscript $x^0 - pbc$. This boundary condition is absolutely necessary at finite L but becomes irrelevant in the zero temperature ($L \to \infty$) limit.

We perform now the following change of variables

$$
\begin{aligned}
x^0 &\to z^3 & A^0 &\to C^3 \\
x^1 &\to z^0 & A^1 &\to C^0 \\
x^2 &\to z^1 & A^2 &\to C^1 \\
x^3 &\to z^2 & A^3 &\to C^2.
\end{aligned}
\tag{8}
$$

Due to the O(4) invariance of the Euclidean Lagrangian we have

$$\mathcal{L}(A^\mu, x^\mu) = \mathcal{L}(C^\mu, z^\mu) \tag{9}$$

and the partition function (5) can be rewritten as

$$Z(L) = \int\limits_{z^3-pbc} \prod_\mu \mathcal{D}C^\mu(z) e^{-\tilde{S}[C^\mu]}, \tag{10}$$

where the action is now given by

$$\tilde{S}[C^\mu] = \int dz^0\, dz^1\, dz^2 \int\limits_{-L/2}^{L/2} dz^3\, \mathcal{L}(C^\mu, z^\mu) \tag{11}$$

and the functional integration runs over fields satisfying periodic boundary condition in the z^3-direction

$$C^\mu(z^0, z^1, z^2, L/2) = C^\mu(z^0, z^1, z^2, -L/2). \tag{12}$$

We can now interpret z^0 as time and $\mathbf{z} = (z^1, z^2, z^3)$ as space coordinates and perform a usual canonical quantization in "Weyl gauge" $C^0 = 0$, interpreting $\mathbf{C} = (C^1, C^2, C^3)$ as spatial coordinates of the gauge field, which are defined, however, not on $\mathbb{R}^3$ but instead on $\mathbb{R}^2 \times S^1$. We obtain then the usual Yang–Mills Hamiltonian in which, however, the integration over z^3 is restricted to the interval $[-\frac{L}{2}, \frac{L}{2}]$. Let us denote this Hamiltonian by $\tilde{H}(\mathbf{C}, L)$. Obviously $\tilde{H}(\mathbf{C}, L \to \infty) = H(\mathbf{C})$. Reversing the steps which lead from (4) to (5) and taking into account the irrelevance of the temporal boundary conditions in the functional integral for an infinite time-interval we obtain from Eq. (10) the alternative representation of the partition function

$$Z(L) = \mathrm{Tr}\, e^{-\int dz^0 \tilde{H}(\mathbf{C}, L)} = \lim_{T \to \infty} \mathrm{Tr}\, e^{-T\tilde{H}(\mathbf{C}, L)}. \tag{13}$$

Due to the infinite z^0-(time-)interval $T \to \infty$ only the lowest eigenvalue of $\tilde{H}(\mathbf{C}, L)$ contributes to the partition function $Z(L)$. The calculation of $Z(L)$ is thus reduced to solving the Schrödinger equation $\tilde{H}(\mathbf{C}, L)\psi(\mathbf{C}) = E\psi(\mathbf{C})$ for the vacuum state on the space manifold $\mathbb{R}^2 \times S^1(L)$, where $S^1(L)$ is a circle with circumference L.

Let us illustrate the equivalence between Eqs. (5) and (13) by means of the free scalar field theory in $1+1$ dimension defined by the (Euclidean) Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2. \tag{14}$$

Calculating the partition function for this model from the functional integral (5) with the temporally periodic boundary condition $\phi(L/2) = \phi(-L/2)$ one finds

$$
\begin{aligned}
\ln Z(L) &= -\frac{1}{2}\mathrm{Tr}\ln(-\partial^2 + m^2) \\
&= -\frac{1}{2}\int\limits_0^L dx^0 \int dx^1 \frac{1}{L}\sum_n \int \frac{dp}{2\pi}\ln(p_n^2 + p^2 + m^2),
\end{aligned}
\tag{15}
$$

where the $p_n = 2\pi n/L$ are the usual Matsubara frequencies. Representing the logarithm by a proper-time integral, carrying out the

integral over the spatial momentum p and using the proper-time representation of the square root one obtains

$$\ln Z(L) = -\int dx^1 E_0(L), \tag{16}$$

where

$$E_0(L) = \frac{1}{2}\sum_n \sqrt{p_n^2 + m^2} \tag{17}$$

is identified as the ground state energy (the lowest eigenvalue of the corresponding Hamiltonian) of the scalar field theory (14) defined, however, on a compact spatial manifold $S^1(L)$. With the substitution (8) $x^1 \to z^0$ Eq. (16) is precisely the representation (13).

The upshot of the above consideration is that finite temperature gauge theory can be described in the Hamiltonian approach by compactifying a spatial dimension and solving the corresponding Schrödinger equation for the vacuum sector. This equivalence holds in fact for any O(4) invariant quantum field theory.

The above consideration for the partition function can be extended to the finite temperature effective potential $V[\langle A_0 \rangle]$. One finds that $V[\langle A_0 \rangle = a]$ can be calculated in the Hamiltonian approach from $V[\langle A_3 \rangle = a]$ with the z^3-axis compactified. Furthermore, as shown in Ref. [22], in the Hamilton approach the effective potential $V[\langle A_3 \rangle = a]$ is given by the energy density in the state minimizing $\langle H \rangle$ for given $\langle A_3 \rangle$.

Below we calculate the effective potential $V[\langle A_3 \rangle = a_3]$ in the Hamiltonian approach exploiting the representation (13) of the partition function.

4. Hamiltonian approach in background gauge

In the presence of an external constant background field a the Hamiltonian approach turns out to be most conveniently formulated in the background gauge

$$\hat{\mathbf{d}} \cdot \mathbf{A} = 0, \qquad \hat{\mathbf{d}} = \partial + \hat{\mathbf{a}}, \qquad \hat{\mathbf{a}}^{ab} = f^{acb}\mathbf{a}^c, \tag{18}$$

where the hat "ˆ" denotes the adjoint representation. This gauge allows for an explicit resolution of Gauss' law, which results in the gauge fixed Yang–Mills Hamiltonian

$$H = \frac{1}{2}\int d^3x \left(J_A^{-1}\boldsymbol{\Pi}(\mathbf{x}) J_A \cdot \boldsymbol{\Pi}(\mathbf{x}) + \mathbf{B}^2(\mathbf{x})\right) + H_C, \tag{19}$$

where $\Pi_k^a(\mathbf{x}) = -i\delta/\delta A_k^a(\mathbf{x})$ is the "transversal" momentum operator ($\hat{\mathbf{d}} \cdot \boldsymbol{\Pi} = 0$) and

$$J_A = \text{Det}(-\hat{\mathbf{D}} \cdot \hat{\mathbf{d}}), \quad \hat{\mathbf{D}} = \partial + \hat{\mathbf{A}} \tag{20}$$

is the Faddeev–Popov determinant of the gauge (18). Furthermore,

$$H_C = \frac{g^2}{2}\int d^3x\, d^3y\, J_A^{-1}\rho^a(\mathbf{x}) J_A F^{ab}(\mathbf{x}, \mathbf{y})\rho^b(\mathbf{y}) \tag{21}$$

arises from the kinetic energy of the "longitudinal" part of the momentum operator. Here

$$\rho^a = -\hat{\mathbf{D}} \cdot \boldsymbol{\Pi} = -(\hat{\mathbf{A}} - \hat{\mathbf{a}}) \cdot \boldsymbol{\Pi} \tag{22}$$

is the color charge density of the gluons, which interacts through the kernel

$$F = (-\hat{\mathbf{D}} \cdot \hat{\mathbf{d}})^{-1}(-\hat{\mathbf{d}} \cdot \hat{\mathbf{d}})(-\hat{\mathbf{D}} \cdot \hat{\mathbf{d}})^{-1}. \tag{23}$$

For a vanishing background field $a = 0$ the gauge (18) reduces to the ordinary Coulomb gauge and H (19) becomes the familiar Yang–Mills Hamiltonian in Coulomb gauge [23].

We are interested here in the energy density in the state $\psi_a[A]$ minimizing $\langle H \rangle_a = \langle \psi_a | H | \psi_a \rangle$ under the constraint $\langle A \rangle_a = a$. For this purpose we perform a variational calculation with the trial wave functional

$$\psi_a[A] = J_A^{-1/2}\tilde{\psi}[A - a], \qquad \tilde{\psi}[A] = \mathcal{N}e^{[-\frac{1}{2}\int A\omega A]}, \tag{24}$$

which already fulfills the constraint $\langle A \rangle = a$. For $a = 0$ this ansatz reduces to the trial wave functional used in Coulomb gauge [7]. However, due to the presence of the colored background field the variational kernel $\omega(\mathbf{p})$ is now a non-trivial color matrix. Proceeding as in the variational approach in Coulomb gauge [7], from $\langle H \rangle_a \to \min$ one derives a set of coupled equations for the gluon and ghost propagators

$$\mathcal{D} = \langle AA \rangle_0 = \frac{1}{2}\omega^{-1}, \qquad G = -\langle((\hat{\mathbf{D}} + \hat{\mathbf{a}})\hat{\mathbf{d}})^{-1}\rangle_0. \tag{25}$$

Using the same approximation as in Ref. [16] in Coulomb gauge, i.e. restricting to two loops in the energy, while neglecting H_C (21) and also the tadpole arising from the non-Abelian part of the magnetic energy, one finds from the minimization of $\langle H \rangle_a$ the gap equation

$$\omega^2 = -\hat{\mathbf{d}} \cdot \hat{\mathbf{d}} + \chi^2 \tag{26}$$

where[1]

$$\chi(1, 2) = -\frac{1}{2}\left\langle \frac{\delta^2 \ln J[A + a]}{\delta A(1)\delta A(2)}\right\rangle_0 = \frac{1}{2}\text{Tr}[G\Gamma(1)G\Gamma_0(2)] \tag{27}$$

is the ghost loop (referred to as "curvature") with Γ_0 and Γ being the bare and full ghost-gluon vertex. The gap equation (26) has to be solved together with the Dyson–Schwinger equation (DSE) for the ghost propagator

$$G^{-1} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{d}} - \Gamma_0(1)G\Gamma(2)\mathcal{D}(2, 1). \tag{28}$$

Due to the presence of the background field these equations have a non-trivial color structure. Fortunately, due to the choice of the background gauge (18), the background field enters these equations only in form of the covariant derivative $\hat{d} = \partial + \hat{a}$. Choosing the background field in the Cartan algebra the above equations can be diagonalized in color space. For simplicity, let us consider the gauge group SU(2) so that $\hat{a} = a\hat{T}_3$ (the extension to SU(N) is straightforward). The eigenvectors $|\sigma = 0, \pm 1\rangle$ of $\hat{T}_3^{ab} = \varepsilon^{a3b}$ are the spin-1 eigenstates, see e.g. [24]. Since the explicit color dependence is only due to the background field a the various propagators have to become diagonal in the basis which diagonalizes $\hat{a}$. Indeed, one can show that the above Eqs. (26) and (28) can be consistently solved for propagators of the form (in momentum space)

$$\mathcal{D}^{\sigma\tau}(\mathbf{p}) = \delta^{\sigma\tau}\mathcal{D}^\sigma(\mathbf{p}), \qquad G^{\sigma\tau}(\mathbf{p}) = \delta^{\sigma\tau}G^\sigma(\mathbf{p}). \tag{29}$$
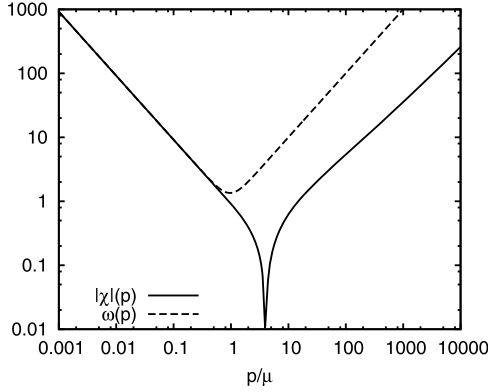
In addition, one can show that the propagators $\mathcal{D}^\sigma(\mathbf{p})$, $G^\sigma(\mathbf{p})$ are related to the propagators in Coulomb gauge in the absence of the background field, $\mathcal{D}(\mathbf{p})$, $G(\mathbf{p})$, by

$$\mathcal{D}^\sigma(\mathbf{p}) = \mathcal{D}(\mathbf{p}^\sigma), \qquad G^\sigma(\mathbf{p}) = G(\mathbf{p}^\sigma), \tag{30}$$

where

$$\mathbf{p}^\sigma = \mathbf{p} - \sigma\mathbf{a} \tag{31}$$

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[1] We use here the compact notation $A(1) \equiv A_{i_1}^{a_1}(\mathbf{x}_1)$. For Lorentz scalars like the ghost, the index "1" stands for the color index a_1 and the spatial position $\mathbf{x}_1$. Repeated indices are summed/integrated over.

Fig. 1. The gluon energy $\omega(p)$ and the curvature $\chi(p)$ resulting from the full numerical solution of the variational approach in Coulomb gauge as described in Ref. [16].

is the momentum shifted by the background field. The relation (30) implies $\chi^\sigma(\mathbf{p}) = \chi(\mathbf{p}^\sigma)$ and applies also to the transversal projector $t_{kl}(\mathbf{p}) = \delta_{kl} - p_k p_l$ of the Lorentz tensors in Coulomb gauge

$$\mathcal{D}_{kl}(\mathbf{p}) = t_{kl}(\mathbf{p})\frac{1}{2\omega(\mathbf{p})}, \qquad \chi_{kl}(\mathbf{p}) = t_{kl}(\mathbf{p})\chi(\mathbf{p}). \tag{32}$$

By these relations the gap equation (26) and the ghost DSE (28) reduces to the ones in Coulomb gauge in the absence of the background field [16]

$$\omega^2(\mathbf{p}) = \mathbf{p}^2 + \chi^2(\mathbf{p}), \tag{33}$$

$$d^{-1}(\mathbf{p}) = \frac{1}{g} - I_d(\mathbf{p}),$$

$$I_d(\mathbf{p}) = N_c \int \frac{d^3q}{(2\pi)^3}\left[1 - \left(\frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\right)^2\right]\frac{d(\mathbf{p}-\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2}\frac{1}{2\omega(\mathbf{q})}. \tag{34}$$

Here $d(\mathbf{p})$ is the ghost form factor, defined by

$$G(\mathbf{p}) = \frac{d(\mathbf{p})}{g\mathbf{p}^2}, \tag{35}$$

and we have replaced the full ghost-gluon vertex Γ by the bare one Γ_0, which is known to be a good approximation, see Ref. [25]. Lattice calculations [26] of the gluon propagator in Coulomb gauge show that the gluon energy can be nicely fitted by Gribov's formula [27]

$$\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^4/\mathbf{p}^2}. \tag{36}$$

A full self-consistent solution of the gap equation (33) and the ghost DSE (34) reveals that $\omega(\mathbf{p})$ contains in addition sub-leading UV-logs, which on the lattice are found to be small. Using Gribov's formula (36) for $\omega(\mathbf{p})$ and solving the gap equation (33) for $\chi(\mathbf{p})$ yields

$$\chi(\mathbf{p}) = M^2/|\mathbf{p}|, \tag{37}$$

which is indeed the correct IR-behavior obtained in a full solution [16] of the coupled ghost DSE and gap equation show in Fig. 1 but which misses the sub-leading UV-logs.

5. The effective potential

As explained in Section 2 the constant background field residing in the Cartan algebra can serve as order parameter for confinement when it is directed along a compactified dimension. Choosing $\mathbf{a} = a\mathbf{e}_3$ and compactifying the 3-axis to a circle with circumference L, the shifted momentum (31) becomes

$$\mathbf{p}^\sigma = \mathbf{p}_\perp + (p_n - \sigma a)\mathbf{e}_3, \quad p_n = 2\pi n/L, \tag{38}$$

where $\mathbf{p}_\perp$ is the projection of $\mathbf{p}$ into the 1–2-plane and p_n is the Matsubara frequency. In the Hamiltonian approach the effective potential of the constant background field is given by the energy density in the state minimizing $\langle H\rangle_a$ under the constraint $\langle A\rangle_a = a$ [22]. Using the gap equation one finds for the energy density per transversal degree of freedom $\langle H\rangle_a/(2V)$ (V is the spatial volume) in the present approximation

$$e(a, L) = \sum_\sigma \frac{1}{L}\sum_{n=-\infty}^{\infty}\int\frac{d^2p_\perp}{(2\pi)^2}(\omega(\mathbf{p}^\sigma) - \chi(\mathbf{p}^\sigma)). \tag{39}$$

By shifting the summation index n one verifies the periodicity

$$e(a + 2\pi/L, L) = e(a, L), \tag{40}$$

which is a necessary property for the effective potential of the confinement order parameter by center symmetry, cf. Eq. (3). Neglecting $\chi(\mathbf{p})$ Eq. (39) gives the energy of a non-interacting Bose gas with single-particle energy $\omega(\mathbf{p})$. This quasi-particle picture is a consequence of the Gaussian ansatz (24) for the wave functional. The quasi-particle energy $\omega(\mathbf{p})$ is, however, highly non-perturbative, see for example Eq. (36). The curvature $\chi(\mathbf{p})$ in Eq. (39) arises from the Faddeev–Popov determinant in the kinetic part of the Yang–Mills Hamiltonian (19).

In certain limiting cases and for $0 \leqslant aL/2\pi \leqslant 1$ the energy density (39) can be calculated analytically. Neglecting $\chi(\mathbf{p})$ and assuming the perturbative expression for the gluon energy $\omega(\mathbf{p}) = |\mathbf{p}|$ one finds from (39) the Weiss potential originally obtained in [20]

$$e_{\text{UV}}(a, L) = \frac{4}{3}\frac{\pi^2}{L^4}\left(\frac{aL}{2\pi}\right)^2\left(\frac{aL}{2\pi} - 1\right)^2. \tag{41}$$

Neglecting $\chi(\mathbf{p})$ and using the infrared expression for the gluon energy $\omega(\mathbf{p}) = M^2/|\mathbf{p}|$ (see Eq. (36)), one obtains

$$e_{\text{IR}}(a, L) = 2\frac{M^2}{L^2}\left[\left(\frac{aL}{2\pi}\right)^2 - \frac{aL}{2\pi}\right]. \tag{42}$$

This expression drastically differs from the Weiss potential (41): While $e_{\text{UV}}(a, L)$ is minimal for $a = 0$, the minimum of $e_{\text{IR}}(a, L)$ occurs at $a = \pi/L$ corresponding to a center symmetric ground state. Accordingly $e_{\text{UV}}(a, L)$ yields for the Polyakov loop $\langle P\rangle = 1$ while $e_{\text{IR}}(a, L)$ yields $\langle P\rangle = 0$.

Obviously, the deconfinement phase transition is related to a change of the effective potential from its infrared behavior $e_{\text{IR}}(a, L)$ (42) to its UV-behavior $e_{\text{UV}}(a, L)$ (41). To illustrate this let us approximate the gluon energy $\omega(\mathbf{p})$ (36) by

$$\omega(\mathbf{p}) \approx |\mathbf{p}| + M^2/|\mathbf{p}|. \tag{43}$$

This expression agrees with the Gribov formula (36) in both, the IR and UV but deviates from it in the mid-momentum regime, which influences the deconfinement phase transition. With $\omega(\mathbf{p})$ given by Eq. (43) and with $\chi(\mathbf{p}) = 0$ the energy density (39) becomes

$$e(a, L) = e_{\text{IR}}(a, L) + e_{\text{UV}}(a, L) = \frac{4}{3}\frac{\pi^2}{L^4}f\left(\frac{aL}{2\pi}\right),$$

$$f(x) = x^2(x - 1)^2 + cx(x - 1), \qquad c = \frac{3M^2L^2}{2\pi^2}. \tag{44}$$

For small temperatures L^{-1}, $e_{\text{IR}}(a, L)$ dominates and the system is in the confined phase. As L^{-1} increases the center symmetric minimum at $x = 1/2$ eventually turns into a maximum and the system undergoes the deconfinement phase transition. In the deconfined
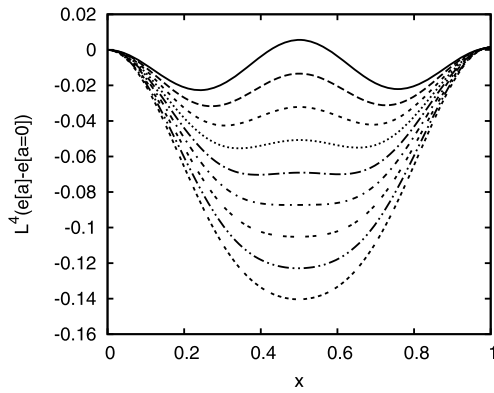
Fig. 2. The energy density (39) as a function of $x = aL/(2\pi)$. L^{-1} was varied from 260 to 280 MeV (from bottom to top).

Fig. 3. The Polyakov loop $\langle P[a] \rangle$ evaluated at the minimum $a = a_{min}$ of the full effective potential shown in Fig. 2 as a function of T/T_c.

phase $f(x)$ has two degenerate minima and, starting in the deconfined phase, the phase transition occurs when the three roots of $f'(x)$ degenerate. This occurs for $c = 1/2$, i.e. for a critical temperature
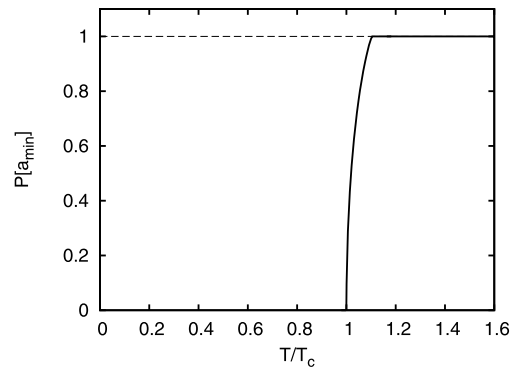
$$T_c = L^{-1} = \sqrt{3}M/\pi. \tag{45}$$

With the lattice result $M = 880$ MeV this corresponds to a critical temperature of $T_c \simeq 485$ MeV, which is much too high. This, of course, is not surprising given the approximation used to arrive at (45), i.e. neglecting the ghost loop $\chi(\mathbf{p})$ and approximating the Gribov formula by Eq. (43). Using the correct Gribov formula (36) instead of the approximation (43) only slightly reduces the critical temperature to $T_c \simeq 432$ MeV. It is the neglect of the curvature $\chi(\mathbf{p})$ which pushes the deconfinement phase transition to higher temperatures as can be easily seen: From the gap equation (33) follows that in the deep IR $\omega(\mathbf{p})$ (36) approaches $\chi(\mathbf{p})$ (37). Therefore neglecting $\chi(\mathbf{p})$ in Eq. (39) increases the contribution of the confining part $e_{IR}(a)$ (42) (relative to that of the deconfining part e_{UV} (41)) and thus pushes the deconfinement phase transition to higher temperatures as we will also explicitly see below.

6. Numerical results

We now turn to a full numerical evaluation of the effective potential (39) using for $\omega(\mathbf{p})$ and $\chi(\mathbf{p})$ the numerical solution of the variational approach in Coulomb gauge obtained in Ref. [16] by solving the gap equation (33) and the ghost DSE (34). The results for $\omega(\mathbf{p})$ and $\chi(\mathbf{p})$ are shown in Fig. 1. With these results one finds from Eq. (39) the effective potential shown in Fig. 2. From this potential one extracts a critical temperature for the deconfinement phase transition of $T_c \simeq 269$ MeV, which is close to the lattice predictions of $T_c = 290$ MeV. Let us also mention that if one uses for $\omega(\mathbf{p})$ the Gribov formula (36) and in accord with the gap equation (33) for $\chi(\mathbf{p})$ its infrared expression (37) one finds a critical temperature of $T_c \simeq 267$ MeV, which is only slightly smaller than the value $T_c \simeq 269$ MeV obtained above with the full numerical solution for $\omega(\mathbf{p})$ and $\chi(\mathbf{p})$. This shows that it is indeed the infrared part of the curvature $\chi(\mathbf{p})$ (neglected in Eq. (44), but fully included in Eq. (39) and Fig. 2), which is crucial for the critical temperature. In view of the ghost dominance in the IR this is not surprising. Fig. 3 shows the Polyakov loop $P[a]$ calculated from the minimum a_{min} of the potential (39) shown in Fig. 2. At the phase transition $P[a_{min}]$ rapidly changes from $P = 0$ to $P = 1$.

The value $T_c = 269$ MeV obtained above from the full effective potential, Fig. 2, is also close to the range of critical temperatures $T_c = 275 \ldots 290$ MeV obtained in Ref. [16] from the grand canonical ensemble of Yang–Mills theory in Coulomb gauge. It is however not surprising that the critical temperatures found in Ref. [16] differ somewhat from the value obtained in the present Letter. The reason is that in the approach of Ref. [16] an additional approximation is made by using a singe particle ansatz for the density matrix. Such an approximation is not necessary in the present approach. In this respect the present approach is superior over the variational treatment of the grand canonical ensemble given in Ref. [16].

In Ref. [19] the Polyakov loop potential was calculated from a functional renormalization group flow equation approach using the Landau gauge ghost and gluon propagators as input. For the gauge group SU(2) a critical temperature of 266 MeV was obtained, which compares well with our result of 269 MeV.

In the present approach the deconfinement phase transition is entirely determined by the zero temperature propagators, which are defined as vacuum expectation values. Consequently, the finite temperature behavior of the theory and, in particular, the dynamics of the deconfinement phase transition must be fully encoded in the vacuum wave functional, as should be clear from the considerations of Section 3. The results obtained above are encouraging for an extension of the present approach to full QCD at finite temperature and baryon density.

Acknowledgements

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