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On some fractional stochastic delay differential equations

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ABSTRACT

We consider the Cauchy problem for an abstract stochastic delay differential equation driven by fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$. We prove the existence and uniqueness for this problem, when the coefficients have enough regularity, the diffusion coefficient is bounded away from zero and the coefficients are smooth functions with bounded derivatives of any order. We prove the theorem by using the convergence of the Picard–Lindelö f iterations in $L^2(\Omega)$ to a solution of this problem which admits a smooth density with respect to Lebesgue's measure on R.

1. Fractional Brownian motion

Fix a parameter $\frac{1}{2} < H < 1$. The fractional Brownian motion of the Hurst parameter H is a centered Gaussian process $B = \{B(t), t \in [0, T]\}$ with the covariance function

$$R(t, s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t - s|^{2H}\right).$$

Let us assume that B is defined in a complete probability space $(\Omega, F.P)$. Then

$$R(t, s) = \int_0^{t\wedge s} K(t, r)K(s, r)\mathrm{d}r$$

where $K(t, s)$ is the kernel defined by

$$K(t, s) = C_H s^{\frac{1}{2}-H} \int_0^t (r - s)^{H-\frac{3}{2}} r^{H-\frac{1}{2}}\, \mathrm{d}r$$

for $s < t$, where

$$C_H = \left[\frac{H(2H - 1)}{B(2 - 2H, H - \frac{1}{2})}\right]^{\frac{1}{2}},$$

$B(\alpha, \beta)$ is the Beta function and $K(t, s) = 0$ if $s > t$.

This means that R is non-negative definite and therefore, there exists a Gaussian process with this covariance [1].

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Malliavin calculus and stochastic integrals for the fractional Brownian motion

Let us recall the definition of the derivative and divergence operators and some basic facts of the stochastic calculus of variation, taken mainly from Alós and Nualart [2].

Let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions ϵ on $[0, t]$ with respect to the scalar product

$$\langle 1_{[0,T]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R(t, s).$$

Then

$$R(t, s) = \alpha_{H_0} \int_0^t \int_0^s |r - u|^{2H-2}\, \mathrm{d}u\mathrm{d}r$$

where $\alpha_H = H(2H - 1)$; then

$$\langle \phi, \psi\rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r - u|^{2H-2} \phi(r)\psi(u)\, \mathrm{d}u\, \mathrm{d}r$$

for all ϕ and ψ in ε. The mapping $1_{[0,t]} \longrightarrow B(t)$ can be extended to an isometry between H and the first Chaos H_1 associated with B, and we denote this isometry by $\phi \longrightarrow B(\phi)$. The elements of $\mathcal{H}$ may not be functions but distributions of negative order to introduce the Banach space, $|\mathcal{H}|$ of measurable functions ϕ on $[0, T]$ satisfying:

$$\|\phi\|^2_{|\mathcal{H}|} = \alpha_H \int_0^T \int_0^T |\phi(r)||\phi(u)||r - u|^{2H-2}\, \mathrm{d}r\, \mathrm{d}u < \infty.$$

Given $s = (s_1, \dots, s_k) \in [0, T]^k$, we denote by $|s|$ the length of s, that is k.

For a random variable, $Y \in \mathbb{D}^{k,p}$ and $s \in [0, T]^k$.

We denote by $D^k_s Y$ the iterative derivative $D_{s_k} D_{s_{k-1}} \dots D_{s_1} Y$.

Let $f \in C_b^{0,\infty}(R)$, the space of continuous functions defined on R infinitely differentiable with bounded derivatives.

$$\text{Set} \quad \Gamma_s(f; Y) = \sum_{m=1}^{|s|} \sum f^{(m)}(Y) \prod_{i=1}^{m} D_{p_i}(p_i) Y,$$

where the second sum extends to all partition $P_1, \dots, P_m$ of length m of s [3].

Let S be the set of smooth and cylindrical random variables of the form

$$X = f(B(\phi_1), \dots, B(\phi_n)),$$

where $n \geq 1, f \in C_b^\infty(R^n)$, ($f$ and all its partial derivatives are bounded) and $\phi_i \in \mathcal{H}$; the derivative operator D of a smooth and cylindrical random variable X is defined as

$$DX = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(B(\phi_1), \dots, B(\phi_n))\phi_j.$$

Let D^k be the iteration of the derivative operator D.

For any $p \geq 1$ the Sobolev space $\mathbb{D}^{k,p}$ is the closure of S with respect to the norm

$$\|X\|^p_{k,p} = E|X|^p + E\left(\sum_{i=1}^{k} \|D^i X\|^p_{\mathcal{H}}\right).$$

Lemma 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variable in $\mathbb{D}^{k,p}$, $k \geq 1$, $p \geq 2$. Assume that there exists $X \in \mathbb{D}^{k-1,p}$ such that $\{D^{k-1}X_n, n \geq 1\}$ converges to $D^{k-1}X$ in $L^p(\Omega)$ as n goes to infinity and moreover, the sequence $\{D^k X_n, n \geq 1\}$ is bounded in $L^p(\Omega)$, then $X \in \mathbb{D}^{k,P}$.*

Lemma 1.2. *Let $\{Z(t), t \in [0, T]\}$ be a stochastic process such that for any $t \in [0, T]$, $Z(t) \in \mathbb{D}^{1,2}$ and*

$$\sup_t E(|Z(t)|^2) \leq k_1, \qquad \sup_{r,t} E(|D_r Z(t)|^2) \leq k_2;$$

then given $r > 0$ and deterministic continuous function f, the stochastic process $\{Y(t), t \in [0, T]\}$ defined as

$$Y(t) = \begin{cases} Z(t - r) & \text{if } t > r \\ f(t) & \text{if } t < r \end{cases} \tag{1.1}$$

belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$.

2. The Cauchy problem for a stochastic delay differential equation

In the present paper we shall consider the Cauchy problem for a stochastic delay differential equation:

$$\mathrm{d}u(t) = Au(t) + b(u(t))\mathrm{d}t + \sigma(u(t-r))\,\mathrm{d}B(t), \quad t \in [0, T] \tag{2.1}$$

$$u(s) = \psi(0), \quad s \in [-r, 0] \tag{2.2}$$

where $\psi \in C([-r, 0])$ and the noise process $\{B(t), t \geq 0\}$ represents a fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$. It is supposed that A is a closed linear operator which generates a strongly continuous semigroup $\{Q(t) : t \geq 0\}$ on a real separable Hilbert space $\mathscr{H}$, [4–6].

Theorem 2.1. *As a solution to this problem, we shall define a process $\{u(t), t \in [-r, T]\}$ that satisfies*

$$u(t) = Q(t)\psi(0) + \int_0^t Q(t-s)b(u(s))\mathrm{d}s + \int_0^t Q(t-s)\sigma(u(s-r))\mathrm{d}B(s), \quad t \in [0, T] \tag{2.3}$$
$$u(s) = \psi(0).$$

We shall assume that b and σ are real functions that satisfy the following conditions:

Hypotheses (H):

(i) *b and σ are $\tilde{\lambda}$-times differentiable functions with bounded derivatives up to order $\tilde{\lambda}$.*
(ii) *σ is bounded and $|Qb(0)| \leq C_1$ for some constant C_1.*
(iii) *$\|Q(t)\| \leq q\mathrm{e}^{\alpha t}$ for all $0 \leq t \leq T$ for some $\alpha > 0, q \geq 1$.*
(iv) *$\int_0^T \|Au(t)\|\mathrm{d}s \leq \infty$, almost surely.*

The proof of this theorem is based on the following lemmas.

Lemma 2.3. *Let $\{\lambda(t), t \in [0, T]\}$ be a quadratic integrable stochastic process. Assume that b is a Lipschitz function defined on R such that $|Qb(0)| \leq C_1$ for some constant C_1; then for a fixed $T_1 \leq T$, the stochastic integral equation*

$$u(t) = \psi(0) + \int_0^t Q(t-s)b(u(s))\mathrm{d}s + \lambda(t), \quad t \in [0, T_1] \tag{2.4}$$

$$u(t) = 0 \quad \text{if } t > T_1 \tag{2.5}$$

admits a unique solution $u(t)$ on $[0, T]$.

Proof. Consider the classical Picard–Lindelöf iterations

$$\begin{cases} u^{(n+1)}(t) = \psi(0) + \int_0^t Q(t-s)b(u^{(n)}(s))\mathrm{d}s + \lambda(t) \\ u^{(0)}(t) = \psi(0) + \lambda(t) \end{cases} \tag{2.6}$$

for $t \in [0, T_1]$ and $u^{(n)}(t) = 0$ for any $t \geq T_1$ and all n, we have

$$\begin{aligned} E(|u^{(1)}(t) - u^{(0)}(t)|^2) &= E\left|\int_0^t Q(t-s)b(u^{(0)}(s))\mathrm{d}s\right|^2 \\ &\leq N_2 \quad \text{uniformly in } t. \end{aligned}$$

We thus have

$$\begin{aligned} E\left(|u^{(n+1)}(t) - u^{(n)}(t)|^2\right) &\leq N_2 \int_0^t E(|u^{(n)}(s_1) - u^{(n-1)}(s_1)|^2)\mathrm{d}s_1 \\ &\leq N_1 \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{n_1}} E(|u^{(1)}(s) - u^{(0)}(s)|^2)\mathrm{d}s_n \ldots \mathrm{d}s_2\mathrm{d}s_1 \\ &\leq N.\frac{N_2^{n-1}}{n!}. \end{aligned}$$

From this we can easily prove that the Picard–Lindelöf iterations converge in $L^2(\Omega)$ on $[0, T_1]$ to a solution of (2.4) [3]. □

Definition 2.1. We shall say that the stochastic process $\{Z(t), t \in [0, T]\}$ satisfies condition $(D_{(*,m,p)})$ if $Z(t) \in \mathbb{D}^{m,p}$ for any $t \in [0, T]$, $\sup_t E(|Z(t)|^p) \leq C_1$ and

$$\sup_t \sup_{r,|r|=k} E(|D_r^k|Z(t)\,|^p) \leq C_2$$

when $m \geq 1, p \geq 2, k \leq m$ and for some positive constants C_1, C_2 in this case $Z(t) \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ and

$$E\|Z\|^2_{|\mathcal{H}|} + E\|DZ\|_{|\mathcal{H}|(*)|\mathcal{H}|} < A(c_1 + c_2)$$

proved in [7].

Lemma 2.4. *Let $\{\lambda(t), t \in [0, T]\}$ be a stochastic process satisfying condition $(D_{(*,m,p)})$. Assume that b has bounded derivatives up to order m and that $|Qb(0)| \leq C_1$; then for a fixed $T_1 \leq T$, the stochastic integral equations (2.4) and (2.5) have a solution $\{u(t), t \in [0, T]\}$ satisfying the condition $(D(*, m.p))$.*

Proof. We have to prove that the Picard–Lindelöf iterations converge in $L^2(\Omega)$ to a solution of (2.6):

$$\begin{aligned} E(|u(t)|^p) &\leq N_p\, E\left|\int_0^t Q(t-s)b(u(s))\mathrm{d}s\right|^p + E|\lambda(t)|^p + E\left|\int_0^T \psi(0)\mathrm{d}s\right|^p \\ &\leq NP\left(1 + \int_0^t E(|u(s)|^p)\right)\mathrm{d}s, \end{aligned}$$

and by Gronwall's lemma that $\sup_t E(|u(t)|^p) < \infty, t \in [0, T]$. Moreover for all $k \leq m, u(t) \in D^{k,p}$, we have $\sup_t \sup_{r,|r|=k} E(|D_r^k u(t)|^p) < c_2$. Now consider the hypothesis of induction for $k \leq m$, $(\tilde{H}_k)$:

(a) for all $n \geq 0$, $u^{(n)}(t) \in \mathbb{D}^{k,p}$ for all t.
(b) $D^{k-1}u^{(n)}(t)$ converges to $D^{k-1}u(t)$ in $L^p(\Omega)$ when n tends to ∞.
(c) $\sup_n \sup_t \sup_{r,|r|=k} E(|D_r^k u^{(n)}(t)|^p) < \infty$.

Step (1). We prove $(\tilde{H}_1)$: (b) is true because $u^{(n)}(t)$ converges, when n tends to ∞, to $u(t)$ in $L^p(\Omega)$; for (a) and (c) we will use another induction to check for all $n \geq 0$, the hypothesis $(\hat{H}_n)$:

(i) $u^{(n)}(t) \in \mathbb{D}^{1,p}$ for all $t \in [0, T]$.
(ii) $\sup_t \sup_r E(|D_r u^{(n)}(t)|^p) \leq \infty$.

It is clear that $u^{(0)}(t) \in \mathbb{D}^{1,p}$ for all t and $t \in [0, T_1]$. So our hypothesis of induction $(\hat{H}_0)$ has been proved.

Assume now that the hypothesis $(\hat{H}_n)$ is true. Then from the definition of $u^{(n+1)}(t)$ it follows that $u^{(n+1)}(t) \in \mathbb{D}^{1,p}$ and for any $t \in [0, T]$,

$$D_r u^{(n+1)}(t) = \int_0^t D_r[Q(t-s)b(u^{(n)}(s))]\mathrm{d}s + D_r\psi(0) + D_r\lambda(t). \tag{2.7}$$

Then

$$\sup_r E(|D_r u^{(n+1)}(t)|^p) \leq N_p\left\{\sup_r E|D_r\psi(0)|^p + \int_0^t \sup_r E(|D_r[Q(t-s)b(u^{(n)}(s))]|^p)\mathrm{d}s + \sup_r E(|D_r\lambda(t)|^p)\right\}$$

since

$$\sup_r E(|D_r\psi(0)|^p) = 0,$$

$$\int_0^t \sup_r E(|D_r[Q(t-s)]|^p)\mathrm{d}s \leq \int_0^t \sup_r E(|D_r u^{(n)}(s)|^p)\mathrm{d}s.$$

Then

$$\sup_r E(|D_r u^{(n+1)(t)}|^p) \leq N_p\left\{\int_0^t \sup_r E(|D_r u^{(n)}(s)|^p)\mathrm{d}s\right\} + \sup_r E(|D_r\lambda(t)|^p) < \infty.$$

Hence $(\hat{H}_{n+1})$ can be easily proved.

Finally, we have the relationship

$$\sup_r E(|D_r u^{(n+1)}(t)|^p) \leq N_p \int_0^t \sup_r E(|D_r u^{(n)}(s)|^p)\mathrm{d}s + N_p.$$

Iterating n times we get

$$\begin{aligned} \sup_r E\left(|D_r u^{(n+1)}(t)|^p\right) &\leq N_p\left\{\int_0^t \left[N_p \int_0^s \sup_r E(|D_r u^{(n-1)}(\xi)|^p)\mathrm{d}\xi + N_p\right]\right\}\mathrm{d}s \\ &\leq (N_p)^2 \int_0^t \int_0^s \sup_r E(|D_r u^{(n-1)}(\xi)|^p)\mathrm{d}\xi\,\mathrm{d}s + (N_p)^2 t + (N_p) \end{aligned}$$

$$\vdots$$
$$\leq \sum_{k=0}^{n} \frac{N_p^{k+1}}{k!} t^k \leq N_p \exp(N_p t).$$

Then

$$\sup_n \sup_t \sup_r E(|D_r u^{(n)}(t)|^p) \leq N_p \exp(N_p T) < \infty.$$

This means that $(\tilde{H}_1)$ has been proved.

Step (2). Let us assume that $(\tilde{H}_k)$, $i \leq k \leq m-1$ is true. We want to check $(\tilde{H}_{k+1})$. We will prove (a) by doing another induction over n similar to the step (1).

Let us consider, for all $n \geq 0$, the hypothesis $(\tilde{H}_n)$:

(i) $u^{(n)}(t) \in \mathbb{D}^{k+1,p}$ for all $t \in [0, T]$
(ii) $\sup_t \sup_{r,|r|=k+1} E(|D_r^{k+1} u^{(n)}(t)|^p) < \infty$.

From the definition of $u^{(0)}(t)$ it is clear that $(\hat{H}_0)$ is true. Assuming that it is true for n, from the definition of $u^{(n+1)}$ it follows that for all t, $u^{(n+1)}(t) \in \mathbb{D}^{k+1,p}$ and for r, $|r| = k+1$

$$D_r^{k+1} u^{(n+1)}(t) = \int_0^t \Gamma_r(Qb, u^{(n)}(s))\mathrm{d}s + D_r^{k+1}\lambda(t) \quad \text{for any } t \in [0, T_1].$$

The proof of (b) and (c) is given by repeating the same method as in step (1). Using Lemma 1.1 we have

$$\sup_t \sup_{r,|r|=k+1} E(|D_r^{k+1} u(t)|^p) < \infty. \quad \square$$

Lemma 2.5. *Let $\{Z(t), t \in [0.T]\}$ be a stochastic process satisfying condition $(D_{*,m+1,p})$. Then the stochastic Stratonovich integral*

$$\lambda(t) = \int_0^t Z(s)\mathrm{d}B(s), \quad t \in [0, T]$$

is well defined and the stochastic process $\{\lambda(t), t \in [0, T]\}$ satisfies condition $(D_{*,m,p})$, proved in [7].

3. Proof of the main theorem

To prove that Eq. (2.3) has a unique solution on $[0, T]$, we first prove the result on $t \in [0, r]$. Then by induction, we shall prove that if Eq. (2.3) has a unique solution on $[0, Nr]$, then we can extend this solution to the interval $[0, (N+1)r]$ and this extension is unique.

The hypothesis (H_N): for $N \leq \tilde{\lambda}$, the equation

$$u(t) = Q(t)\psi(0) + \int_0^t Q(t-s)b(u(s))\mathrm{d}s + \int_0^t Q(t-s)\sigma(u(s-r))\mathrm{d}B(s), \quad t \in [0, Nr]$$

and $u(t) = 0$ if $t < N_r$, has a unique solution and $u(t)$ satisfies condition $(D_{(*,\tilde{\lambda}-N,p)})$.

Step (1) check (H_1): Let $t \in [0, r]$ and define the process

$$\lambda(t) = \int_0^t Q(t-s)\sigma(\psi(s-r))1_{\{t<r\}}\,\mathrm{d}B(s) \quad t \in [0, T].$$

It is clear that

$$Q(t-s)\sigma(\psi(s-r)) \in \mathbb{D}^{1,2}(|\mathcal{H}|)$$

and

$$D[Q(t-s)\sigma[\psi(s-r)]] = 0.$$

Since ψ is a deterministic continuous function, then $\lambda(t) \in \mathbb{D}^{k,p}$,

$$D_\nu\lambda(t) = Q(s-t)\sigma(\psi(\nu-r))1_{\{\nu<t<r\}}$$

and

$$D^k\lambda(t) = 0 \quad \text{when } k \geq 2.$$

Moreover for all $k \geq 1, p \geq 2$ we have that

$$\sup_t E(|\lambda(t)|^p) \leq \|Q\sigma(\psi(s-r))\|^p \leq C_3$$

and

$$\sup_t \sup_{\nu,|\nu|=k} E(|D_\nu^k \lambda(t)|^p) \leq C_4.$$

So, $\lambda(t)$ satisfies condition $(D_{(*,k,p)})$.

From Lemmas 2.4 and 2.5, there exist a unique solution $u(t)$ and satisfies condition $(D_{(*,\tilde{\lambda}-1,p)})$.

Step (2): Assuming that (H_N) is true for $N < \tilde{\lambda}$, we want to check (H_{N+1}).

Consider the stochastic process $\{Y(t), t \in [0, T]\}$ defined as

$$y(t) = \begin{cases} \varphi(t-r)t & \leq r \\ u(t-r) & r < t \leq (N+1)r \\ 0 & t > (N+1)r. \end{cases}$$

Set $Z(t) = Q\sigma(Y(t))$. Then for $t \in [0, (N+1)r]$, the problem becomes

$$u(t) = Q(t)\psi(0) + \int_0^t Q(t-s)b(u(s))\mathrm{d}s + \int_0^t Z(t)\, \mathrm{d}B(s).$$

Let us define the process

$$\lambda(t) = \int_0^t Z(s) 1_{\{t<(N+1)r\}} \mathrm{d}B(s) \quad t \in [0, T]$$

from Lemma 1.2

(1) $Z(t) \in \mathbb{D}^{1,2}(|\mathcal{H}|)$

(2) $\int_0^T \int_0^T |D_\nu Z(s)||s-\nu|^{2H-2} \mathrm{d}s\mathrm{d}\nu < \infty$

and

$$D_\nu Z(t) = D_\nu [Q\sigma(Z(t))].$$

Now by using that the stochastic process $Y(t)$ satisfies condition $(D_{(*,\tilde{\lambda}-N,p)})$ and that $Q\sigma$ has a derivative up to order $\tilde{\lambda}$, it is clear that for any $t \in [0, T]$, $Z(t) \in D^{k,p}$ for all $k \leq \tilde{\lambda} - N$ and

$$D_\nu^k Z(t) = \Gamma_\nu(Q\sigma, Y(t)).$$

Furthermore, $Y(t)$ satisfies condition $(D_{(*,\tilde{\lambda}-N,p)})$. From Lemmas 2.4 and 2.5, λ satisfies condition $(D_{(*,\tilde{\lambda}-N-1,p)})$ and as we did in step (1) we can finish the proof of this theorem (Comp. [8–21]).

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