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Real projective structures on a real curve

Indranil Biswas^{a,*}, Jacques Hurtubise^b^a *School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India*^b *Department of Mathematics, McGill University, Burnside Hall, 805 Sherbrooke St. W., Montreal, Que. H3A 2K6, Canada*

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Abstract

Given a compact connected Riemann surface X equipped with an antiholomorphic involution τ , we consider the projective structures on X satisfying a compatibility condition with respect to τ . For a projective structure P on X , there are holomorphic connections and holomorphic differential operators on X that are constructed using P . When the projective structure P is compatible with τ , the relationships between τ and the holomorphic connections, or the differential operators, associated to P are investigated. The moduli space of projective structures on a compact oriented C^∞ surface of genus $g \geq 2$ has a natural holomorphic symplectic structure. It is known that this holomorphic symplectic manifold is isomorphic to the holomorphic symplectic manifold defined by the total space of the holomorphic cotangent bundle of the Teichmüller space \mathcal{T}_g equipped with the Liouville symplectic form. We show that there is an isomorphism between these two holomorphic symplectic manifolds that is compatible with τ .

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1. Introduction

A projective structure on a compact Riemann surface X is defined by giving a covering of X by holomorphic coordinate charts such that all the transition functions are Möbius transformations.

* Corresponding author.

E-mail addresses: indranil@math.tifr.res.in (I. Biswas), jacques.hurtubise@mcgill.ca (J. Hurtubise).

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Projective structures have other equivalent formulations using projective connections, differential operators etc.

Assume that there is an antiholomorphic involution

$$\tau : X \longrightarrow X.$$

Just as compact Riemann surfaces are same as smooth projective curves defined over \mathbb{C} , pairs of the form (X, τ) are same as geometrically irreducible smooth projective curves over \mathbb{R} . We consider projective structures on X compatible with τ ; a projective structure P on X is compatible with τ if τ takes any holomorphic coordinate function associated to P to the conjugate of another holomorphic coordinate function associated to P . So a projective structure on X compatible with τ can be called a *real* projective structure.

This “reality” of projective structures becomes more clear if we consider the equivalent formulations using projective connections or differential operators. Associated to each projective structure is a holomorphic (equivalently, algebraic) projective connection. A projective structure is real if and only if the corresponding projective connection is defined over \mathbb{R} . Also, associated to each projective structure is a holomorphic (equivalently, algebraic) differential operator of order three from TX to $((TX)^*)^{\otimes 2}$, where TX is the holomorphic tangent bundle. A projective structure is real if and only if the corresponding differential operator is defined over \mathbb{R} . We investigate these interrelations.

To define a projective structure on a C^∞ oriented surface, we do not need to fix a complex structure on the surface. On the contrary, given a projective structure, there is a underlying complex structure on the surface.

Let Y be a compact connected oriented C^∞ surface. Let $\text{Diffeo}_0(Y)$ denote the group of diffeomorphisms of Y homotopic to the identity map. This group acts on the space of all projective structures on Y compatible with the orientation of Y . The corresponding quotient space will be denoted by $\mathcal{P}_0(Y)$, which is a complex manifold equipped with a natural holomorphic symplectic form. The Teichmüller space $\mathcal{T}(Y)$ for Y is the quotient by $\text{Diffeo}_0(Y)$ of the space of all complex structures on Y compatible with its orientation. There is a natural projection of

$$\varphi : \mathcal{P}_0(Y) \longrightarrow \mathcal{T}(Y)$$

making $\mathcal{P}_0(Y)$ a torsor for the holomorphic cotangent bundle $\Omega_{\mathcal{T}(Y)}^1 \longrightarrow \mathcal{T}(Y)$. Let

$$\tau : Y \longrightarrow Y$$

be an orientation reversing diffeomorphism of order two. Let τ_P (respectively, τ_T) be the involution of $\mathcal{P}_0(Y)$ (respectively, $\mathcal{T}(Y)$) constructed using τ . We prove that there is a holomorphic section of the projection φ that

- intertwines τ_P and τ_T , and
- the corresponding biholomorphism of $\mathcal{P}_0(Y)$ with $\Omega_{\mathcal{T}(Y)}^1$ takes the natural symplectic form on $\mathcal{P}_0(Y)$ to the Liouville symplectic form on $\Omega_{\mathcal{T}(Y)}^1$.

In a work with Huisman, [4], the representations associated to stable real vector bundles are introduced. The PGL_2 analog of these representations arise in real projective structures.

2. Real projective structures

Let X be a compact connected Riemann surface. Let

$$J : T^{\mathbb{R}}X \longrightarrow T^{\mathbb{R}}X$$

be the almost complex structure of X . Assume that X is equipped with an anti-holomorphic involution

$$\tau : X \longrightarrow X. \tag{2.1}$$

This means that τ is a smooth self-map of X such that $\tau \circ \tau = \text{Id}_X$, and the equality

$$d\tau(J(x)(v)) = -J(\tau(x))(d\tau(v)) \tag{2.2}$$

holds for all $x \in X$ and $v \in T_x^{\mathbb{R}}X$, where $d\tau : T^{\mathbb{R}}X \longrightarrow T^{\mathbb{R}}X$ is the differential of τ . As mentioned in the introduction, such a pair (X, τ) corresponds to a geometrically irreducible smooth projective curve defined over \mathbb{R} . (See [7,14,8,13] for curves defined over \mathbb{R} .)

We will define projective structures on X compatible with τ . Before that, let us recall the definition of a projective structure.

The standard action of $\text{SL}(2, \mathbb{C})$ on \mathbb{C}^2 induces an action of the Möbius group $\text{PSL}(2, \mathbb{C})$ on \mathbb{CP}^1 , and furthermore, $\text{PSL}(2, \mathbb{C}) = \text{Aut}(\mathbb{CP}^1)$ (the group of holomorphic automorphisms). A *holomorphic coordinate function* on an open subset $U \subset X$ is an injective holomorphic map $\phi : U \longrightarrow \mathbb{CP}^1$.

A projective structure on X is defined by giving a collection $\{(U_i, \phi_i)\}_{i \in I}$ of holomorphic coordinate functions such that

- $\bigcup_{i \in I} U_i = X$, and
- for each pair $i, i' \in I$, the composition

$$\phi_{i'} \circ \phi_i^{-1} : \phi_i(U_i \cap U_{i'}) \longrightarrow \phi_{i'}(U_i \cap U_{i'})$$

is the restriction of some Möbius transformation.

Two such data $\{(U_i, \phi_i)\}_{i \in I}$ and $\{(U_{i'}, \phi_{i'})\}_{i' \in I'}$ satisfying the above conditions are called *equivalent* if their union $\{(U_i, \phi_i)\}_{i \in I \cup I'}$ also satisfies the two conditions. A *projective structure* on X is an equivalence class of such data. Given a projective structure on X , the holomorphic coordinate functions compatible with it will be called *projective coordinates*.

Consider the involution τ in Eq. (2.1). Note that for any holomorphic coordinate function (U, ϕ) , the composition $(\tau^{-1}(U), \overline{\phi \circ \tau})$ is also a holomorphic coordinate function.

Definition 2.1. A projective structure on X is said to be *compatible* with τ if for each projective coordinate (U, ϕ) , the composition $(\tau^{-1}(U), \overline{\phi \circ \tau})$ is also a projective coordinate.

For convenience, a projective structure on X compatible with τ will also be called a *real* projective structure.

Remark 2.2. The complex projective line \mathbb{CP}^1 has the real structure defined by $z \longmapsto \bar{z}$. The condition that a projective structure is real means that the projective structure is compatible with τ and this real structure on \mathbb{CP}^1 . We note that \mathbb{CP}^1 has another real structure defined by $z \longmapsto -1/\bar{z}$. However, the real projective structures on X defined using this real structure are not different from those defined above because the two real structures on \mathbb{CP}^1 differ by an element of the Möbius group.

Given a projective structure P on X , we can construct another projective structure P' on X which is uniquely determined by the following condition. Take any holomorphic coordinate function (U, ϕ) compatible with P , then the composition $(\tau(U), \overline{\phi \circ \tau})$ is compatible with P' .

Clearly this construction defines an involution on the space of all projective structures on X . The following lemma is obvious.

Lemma 2.3. *A projective structure P on X is real if and only if P is fixed by the involution constructed above.*

We will describe the space of all projective structures on X compatible with τ . We begin with the following simple lemma.

Lemma 2.4. *There exists a projective structure on X compatible with τ .*

Proof. If $X = \mathbb{C}\mathbb{P}^1$, the unique projective structure on $\mathbb{C}\mathbb{P}^1$, which is defined by taking the identity map of $\mathbb{C}\mathbb{P}^1$ to be a projective coordinate, is clearly compatible with τ .

Assume that $\text{genus}(X) \geq 1$. Let

$$\gamma : \tilde{X} \longrightarrow X$$

be a universal cover of X , where \tilde{X} is either the complex line \mathbb{C} or the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. Fix a biholomorphism of \tilde{X} with \mathbb{C} or \mathbb{H} . The holomorphic coordinates on X given by the inclusion of \tilde{X} in $\mathbb{C}\mathbb{P}^1$ define a projective structure on X , which we will denote by P . We will show that P is compatible with τ Eq. (2.1).

The involution τ lifts to an anti-holomorphic automorphism of \tilde{X} . Any lift

$$\tilde{\tau} : \tilde{X} \longrightarrow \tilde{X}$$

of τ is of the form $\tilde{\tau}(z) = \overline{T(z)}$ (respectively, $\tilde{\tau}(z) = -\overline{T(z)}$) if $\tilde{X} = \mathbb{C}$ (respectively, $\tilde{X} = \mathbb{H}$), where T is a holomorphic automorphism of \tilde{X} . Since \tilde{X} is either \mathbb{C} or \mathbb{H} , the map defined by $z \mapsto T(z)$ is a Möbius transformation. Hence the map defined by $z \mapsto \tilde{\tau}(z)$ is a Möbius transformation. Therefore, the projective structure P is compatible with τ . This completes the proof of the lemma. \square

Let V be a holomorphic vector bundle over X . Let \overline{V} denote the C^∞ complex vector bundle over X which is identified with V as a real C^∞ vector bundle, while the complex structure of the fiber \overline{V}_x , $x \in X$, is the conjugate of the complex structure of V_x . The smooth complex vector bundle $\tau^*\overline{V}$ has a natural holomorphic structure. A C^∞ section of $\tau^*\overline{V}$ defined over an open subset $U \subset X$ is holomorphic if the corresponding section of V over $\tau(U)$ is holomorphic; it is easy to check that this condition defines a holomorphic structure on $\tau^*\overline{V}$.

Let TX denote the holomorphic tangent bundle of X . Since the differential $d\tau$ takes the almost complex structure J to $-J$ (see Eq. (2.2)), it follows immediately that $d\tau$ gives a C^∞ isomorphism of TX with $\tau^*\overline{TX}$. It is easy to check that this is a holomorphic isomorphism. Consequently, $(TX)^{\otimes m} = \tau^*(\overline{TX})^{\otimes m}$ for all $m \in \mathbb{Z}$.

Fix a holomorphic line bundle $L \rightarrow X$ such that $L^{\otimes 2}$ is holomorphically isomorphic to TX . So, L is a theta characteristic on X . Fix an isomorphism of $L^{\otimes 2}$ with TX . We have the following short exact sequence of vector bundles on X

$$0 \longrightarrow L^* = L \otimes K_X \longrightarrow J^1(L) \longrightarrow L \longrightarrow 0, \tag{2.3}$$

where $J^1(L)$ is the jet bundle, and K_X is the holomorphic cotangent bundle. Let

$$\mathbb{P}_L \longrightarrow X \tag{2.4}$$

be the principal $\text{PGL}(2, \mathbb{C})$ -bundle defined by the projective bundle $\mathbb{P}(J^1(L))$.

If $\text{genus}(X) > 1$, then $J^1(L)$ is indecomposable. If $\text{genus}(X) = 1$, then $J^1(L) = L \oplus L^*$, and if $\text{genus}(X) = 0$, then $J^1(L)$ is a trivial vector bundle. Hence by a criterion of Atiyah [2], Weil [16], the vector bundle $J^1(L)$ in Eq. (2.4) admits a holomorphic connection. Consequently,

the projective bundle \mathbb{P}_L admits a holomorphic connection. For a holomorphic connection ∇ on \mathbb{P}_L , the second fundamental form of the holomorphic section of \mathbb{P}_L defined by the subbundle $L \otimes K_X$ in Eq. (2.3) is a section

$$S(\nabla, L \otimes K_X) \in H^0(X, \text{Hom}(L \otimes K_X, L) \otimes K_X) = H^0(X, \mathcal{O}_X). \tag{2.5}$$

We recall that the projective structures on X are in bijective correspondence with the holomorphic connections ∇ on \mathbb{P}_L such that the second fundamental form $S(\nabla, L \otimes K_X)$ in Eq. (2.5) is the constant function 1 (see [9]).

Lemma 2.5. *The involution τ in Eq. (2.1) has a canonical lift to a C^∞ involution of \mathbb{P}_L .*

A projective structure on X is real if and only if the corresponding holomorphic connection on \mathbb{P}_L is preserved by the above involution of \mathbb{P}_L .

Proof. Let $\xi \rightarrow X$ be a holomorphic line bundle. Let $\xi_0 \rightarrow X$ be a holomorphic line bundle of finite order. We will show that there is a canonical isomorphism

$$J^m(\xi \otimes \xi_0) = J^m(\xi) \otimes \xi_0 \tag{2.6}$$

for all $m \geq 0$.

Let k be a positive integer such that $\xi_0^{\otimes k}$ is holomorphically trivial. There is a unique connection ∇_0 on ξ_0 such that the connection on $\xi_0^{\otimes k} = \mathcal{O}_X$ induced by ∇_0 is the trivial one (it has trivial monodromy). This connection ∇_0 is independent of the choice of k . Take any $v \in J^m(\xi)_x$ and $\alpha \in (\xi_0)_x$, where $x \in X$. Take a holomorphic section \tilde{v} of ξ defined around x such that v represents \tilde{v} . Let $\tilde{\alpha}$ be the unique flat section, with respect to the connection ∇_0 , of ξ_0 defined around x such that $\tilde{\alpha}(x) = \alpha$. Now sending $v \otimes \alpha$ to the element of $J^m(\xi \otimes \xi_0)_x$ representing the section $\tilde{v} \otimes \tilde{\alpha}$ we get a homomorphism from $J^m(\xi) \otimes \xi_0$ to $J^m(\xi \otimes \xi_0)$. It is easy to see that this homomorphism is a holomorphic isomorphism.

Let $L \rightarrow X$ be a holomorphic line bundle such that $L^{\otimes 2}$ is isomorphic to TX . Consider the line bundle $L_1 := \tau^* \bar{L}$. Since $L_1^{\otimes 2} = \tau^* \overline{TX} = TX$, there is a line bundle ξ_0 of order two such that $L_1 = L \otimes \xi_0$. Hence from Eq. (2.6),

$$J^1(L_1) = J^1(L) \otimes \xi_0.$$

Consequently, there is a canonical isomorphism

$$\mathbb{P}_{L_1} := \mathbb{P}(J^1(L_1)) \xrightarrow{\sim} \mathbb{P}_L.$$

This isomorphism gives a C^∞ involution

$$\widehat{\tau} : \mathbb{P}_L \rightarrow \mathbb{P}_L \tag{2.7}$$

that lifts τ .

For any point $x \in X$, the isomorphism $\widehat{\tau}(x)$ of the fiber $(\mathbb{P}_L)_x$ with $(\mathbb{P}_L)_{\tau(x)}$ is anti-holomorphic. In fact, the pulled back fiber bundle $\tau^* \mathbb{P}_L$ has a natural holomorphic structure which is uniquely determined by the following condition: a section of $\tau^* \mathbb{P}_L$ over an open subset $U \subset X$ is holomorphic if and only if the corresponding section of \mathbb{P}_L over $\tau(U)$ is holomorphic. The involution $\widehat{\tau}$ in Eq. (2.7) is a holomorphic isomorphism of \mathbb{P}_L with $\tau^* \mathbb{P}_L$ equipped with the above holomorphic structure.

Let P be a projective structure on X . Let ∇ be the holomorphic connection on \mathbb{P}_L associated to P . Note that any holomorphic connection on a Riemann surface is flat, because there are no nonzero holomorphic two-forms on it (the curvature of a holomorphic connection is a

holomorphic two-form with values in the adjoint bundle). Let \tilde{P} be the projective structure on X given by P using the involution in Lemma 2.3. The flat connection on \mathbb{P}_L corresponding to \tilde{P} coincides with $\widehat{\tau}^*\nabla$, where $\widehat{\tau}$ is the involution in Eq. (2.7). Therefore, Lemma 2.3 completes the proof. \square

3. Theta characteristics and projective structure

3.1. First jet bundle of a theta characteristic

A theta characteristic of (X, τ) is a holomorphic line bundle $\theta \rightarrow X$ such that

- $\theta^{\otimes 2} = K_X$, and
- the line bundle $\tau^*\bar{\theta}$ is holomorphically isomorphic to θ .

A theta characteristic θ of (X, τ) is called *real* if there is a holomorphic isomorphism

$$f : \theta \rightarrow \tau^*\bar{\theta}$$

such that the composition

$$\theta \xrightarrow{f} \tau^*\bar{\theta} \xrightarrow{\tau^*\bar{f}} \tau^*\overline{\tau^*\bar{\theta}} = \theta$$

is the identity map.

A theta characteristic θ of (X, τ) is called *quaternionic* if there is a holomorphic isomorphism

$$f : \theta \rightarrow \tau^*\bar{\theta}$$

such that the composition

$$\theta \xrightarrow{f} \tau^*\bar{\theta} \xrightarrow{\tau^*\bar{f}} \tau^*\overline{\tau^*\bar{\theta}} = \theta$$

is $-\text{Id}_\theta$.

The set of theta characteristics on (X, τ) decomposes into a disjoint union of real and quaternionic theta characteristics. It is known that (X, τ) admits a theta characteristic (see [3, pp. 61–62]). We note that if τ does not have any fixed points, then there are no real holomorphic line bundles of odd degree on X . Hence in that case there are no real theta characteristics provided the genus of X is even; see [4] for detailed discussions on the existence of real and quaternionic vector bundles, in particular, line bundles.

Let θ be a theta characteristic on (X, τ) . Fix a holomorphic isomorphism

$$f : \theta \rightarrow \tau^*\bar{\theta} \tag{3.1}$$

such that the composition

$$\theta \xrightarrow{f} \tau^*\bar{\theta} \xrightarrow{\tau^*\bar{f}} \tau^*\overline{\tau^*\bar{\theta}} = \theta$$

is either Id_θ or $-\text{Id}_\theta$.

We have $\bigwedge^2 J^1(\theta^*) = \theta^* \otimes \theta^* \otimes K_X = \mathcal{O}_X$ (see Eq. (2.3)). We noted in Section 2 that $J^1(\theta^*)$ admits a holomorphic connection.

The isomorphism in Eq. (3.1) induces an isomorphism

$$\widehat{f} : J^1(\theta^*) \rightarrow \tau^*\overline{J^1(\theta^*)}. \tag{3.2}$$

We note that $\tau^* \widehat{f} \circ \widehat{f}$ is $\text{Id}_{J^1(\theta^*)}$ (respectively, $-\text{Id}_{J^1(\theta^*)}$) if $\tau^* \widehat{f} \circ f$ is Id_θ (respectively, $-\text{Id}_\theta$). A holomorphic connection ∇ on $J^1(\theta^*)$ induces a holomorphic connection on $\tau^* \overline{J^1(\theta^*)}$; this induced connection will be denoted by $\tau^* \overline{\nabla}$.

Let \mathcal{C} denote the space of all holomorphic connections ∇ on $J^1(\theta^*)$ such that

- the isomorphism \widehat{f} is Eq. (3.2) takes ∇ to $\tau^* \overline{\nabla}$, and
- the connection on $\bigwedge^2 J^1(\theta^*)$ induced by ∇ has trivial monodromy.

Proposition 3.1. *The space of connections \mathcal{C} defined above is in bijective correspondence with the space of all projective structures on X compatible with τ .*

Proof. Take any $\nabla \in \mathcal{C}$. The holomorphic connection on $\mathbb{P}(J^1(\theta^*))$ induced by ∇ will be denoted by ∇' . From Lemma 2.5 it follows that ∇' defines a real projective structure on (X, τ) .

Let P be a real projective structure on (X, τ) . Let ∇^P be the corresponding holomorphic connection on the projective bundle $\mathbb{P}(J^1(\theta^*))$. From Lemma 2.5 we know that ∇^P is preserved by the involution of $\mathbb{P}(J^1(\theta^*))$. Since $\text{Lie}(\text{SL}(2, \mathbb{C})) = \text{Lie}(\text{PGL}(2, \mathbb{C}))$, and $\bigwedge^2 J^1(\theta^*) = \mathcal{O}_X$, the connection ∇^P defines a holomorphic connection on $J^1(\theta^*)$ such that the induced connection on $\bigwedge^2 J^1(\theta^*)$ has trivial monodromy. \square

See [15] for a detailed study of projective structures.

3.2. Trivializations on nonreduced diagonal

Let X be a compact connected Riemann surface. Let

$$\Delta := \{(x, x) \mid x \in X\} \subset X \times X$$

be the (reduced) diagonal. For any integer $n \geq 2$, the nonreduced diagonal with multiplicity n will be denoted by $n\Delta$. Consider the holomorphic line bundle

$$\mathcal{L} := p_1^* K_X \otimes p_2^* K_X \otimes \mathcal{O}_{X \times X}(2\Delta)$$

on $X \times X$, where $p_i, i = 1, 2$, is the projection of $X \times X$ to the i -th factor.

The Poincaré adjunction formula identifies the holomorphic tangent bundle TX with the restriction of the line bundle $\mathcal{O}_{X \times X}(\Delta)$ to Δ (the Riemann surface X is identified with Δ by sending any $x \in X$ to $(x, x) \in \Delta$). In view of this identification of TX with $\mathcal{O}_{X \times X}(\Delta)|_\Delta$, the restriction $\mathcal{L}|_\Delta$ gets identified with the trivial line bundle \mathcal{O}_Δ over Δ . Let

$$\psi_0 \in H^0(\Delta, \mathcal{L}|_\Delta) \tag{3.3}$$

be the section given by the constant function 1 through the identification of $\mathcal{L}|_\Delta$ with \mathcal{O}_Δ .

Consider the holomorphic involution σ of $X \times X$ defined by $(x, y) \mapsto (y, x)$. This involution clearly has a natural lift to an involution of the line bundle \mathcal{L} . Let

$$\tilde{\sigma} : \mathcal{L} \longrightarrow \mathcal{L}$$

be this involution over σ .

The following two statements hold:

- There is a unique section

$$\psi_1 \in H^0(2\Delta, \mathcal{L}|_{2\Delta}) \tag{3.4}$$

invariant under $\tilde{\sigma}$ such that the restriction of ψ_1 to $\Delta \subset 2\Delta$ coincides with ψ_0 in Eq. (3.3).

- The space of all projective structures on X is in bijective correspondence with the space of sections

$$\psi \in H^0(3\Delta, \mathcal{L}|_{3\Delta})$$

satisfying the condition that the restriction of ψ to $2\Delta \subset 3\Delta$ coincides with the above section ψ_1 .

(See [5] for the details.)

Now let τ be an antiholomorphic involution of X . It produces a holomorphic isomorphism

$$\tau'' : K_X \longrightarrow \tau^* \overline{K}_X$$

that sends any locally defined holomorphic one-form w to $\sigma^* \omega$ (note that the line bundle $\tau^* \overline{K}_X$ has a natural holomorphic structure). Since \overline{K}_X is identified, as a C^∞ line bundle, with K_X , this isomorphism τ'' produces a C^∞ isomorphism

$$\tau' : K_X \longrightarrow X_K$$

over τ . The isomorphism τ' is fiberwise conjugate linear. This τ' produces a C^∞ isomorphism

$$\tau_2 : \mathcal{L} \longrightarrow \mathcal{L}$$

over the involution $\tau \times \tau$ of $X \times X$. The isomorphism τ_2 is fiberwise conjugate linear. This τ_2 is given by the holomorphic isomorphism of line bundles

$$p_1^* K_X \otimes p_2^* K_X \xrightarrow{p_1^* \tau'' \otimes p_2^* \tau''} p_1^* \tau^* \overline{K}_X \otimes p_2^* \tau^* \overline{K}_X = (\tau \times \tau)^*(p_1^* \overline{K}_X \otimes p_2^* \overline{K}_X).$$

Let P be a projective structure on X given by a section

$$\psi \in H^0(3\Delta, \mathcal{L}|_{3\Delta})$$

satisfying the condition that the restriction of ψ to $2\Delta \subset 3\Delta$ coincides with the section ψ_1 in Eq. (3.4). The projective structure P is compatible with τ if and only if

$$\tau_2(\psi) = \psi.$$

This is straight-forward to check.

3.3. Differential operators of order two

As before, θ is a theta characteristic of (X, τ) . Let $J^2(\theta^*) \longrightarrow X$ be the second order jet bundle. It fits in the following short exact sequence of holomorphic vector bundles on X :

$$0 \longrightarrow K_X^{\otimes 2} \otimes \theta^* = \theta^{\otimes 3} \xrightarrow{\iota} J^2(\theta^*) \xrightarrow{q} J^1(\theta^*) \longrightarrow 0. \tag{3.5}$$

A holomorphic differential operator D of order two from θ^* to $K_X^{\otimes 2} \otimes \theta^*$ is, by definition, an \mathcal{O}_X -linear homomorphism from $J^2(\theta^*)$ to $K_X^{\otimes 2} \otimes \theta^*$. In other words, it is a holomorphic section

$$D \in H^0(X, K_X^{\otimes 2} \otimes \theta^* \otimes J^2(\theta^*)^*). \tag{3.6}$$

Since $D \circ \iota$ is a holomorphic endomorphism of the line bundle $K_X^{\otimes 2} \otimes \theta^*$, where ι is the homomorphism in Eq. (3.5), it follows that $D \circ \iota$ is multiplication by a scalar $\sigma(D) \in \mathbb{C}$. This complex number $\sigma(D)$ is called the *symbol* of D . So the symbol is a homomorphism

$$\sigma : H^0(X, \text{Diff}^2(\theta^*, K_X^{\otimes 2} \otimes \theta^*)) = H^0(X, K_X^{\otimes 2} \otimes \theta^* \otimes J^2(\theta^*)^*) \longrightarrow \mathbb{C}, \tag{3.7}$$

where $\text{Diff}^2(\theta^*, K_X^{\otimes 2} \otimes \theta^*) = \mathcal{H}om(J^2(\theta^*), K_X^{\otimes 2} \otimes \theta^*)$ is the holomorphic vector bundle on X associated to the sheaf of differential operators of order two from θ^* to $K_X^{\otimes 2} \otimes \theta^*$. Note that any D as in Eq. (3.6) with $\sigma(D) = 1$ gives a holomorphic splitting of the short exact sequence in Eq. (3.5).

From the properties of the jet bundles it follows that for any holomorphic vector bundle W , there is a natural injective homomorphism

$$J^{p+q}(W) \longrightarrow J^p(J^q(W))$$

for all $p, q \geq 0$. This fits in a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K_X^{\otimes 2} \otimes \theta^* & \longrightarrow & J^2(\theta^*) & \longrightarrow & J^1(\theta^*) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \beta & & \parallel \\
 0 & \longrightarrow & K_X \otimes J^1(\theta^*) & \longrightarrow & J^1(J^1(\theta^*)) & \xrightarrow{q_0} & J^1(\theta^*) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & K_X \otimes \theta^* & = & K_X \otimes \theta^* & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array} \tag{3.8}$$

where the top horizontal short exact sequence is the one in Eq. (3.5), and the bottom horizontal short exact sequence is the jet sequence for the vector bundle $J^1(\theta^*)$. Consider the homomorphism

$$J^1(J^1(\theta^*)) \longrightarrow J^1(\theta^*)$$

induced by the projection $J^1(\theta^*) \longrightarrow \theta^*$. The vertical homomorphism $J^1(J^1(\theta^*)) \longrightarrow K_X \otimes \theta^*$ in Eq. (3.8) is the difference between this homomorphism and the projection $J^1(J^1(\theta^*)) \longrightarrow J^1(\theta^*)$ in Eq. (3.8).

Take a holomorphic differential operator D as in Eq. (3.6) such that $\sigma(D) = 1$. We noted above that D gives a holomorphic splitting of Eq. (3.5). Let

$$\rho_D : J^1(\theta^*) \longrightarrow J^2(\theta^*) \tag{3.9}$$

be the holomorphic homomorphism of vector bundles corresponding to the splitting of Eq. (3.5) given by D . The composition

$$\beta \circ \rho_D : J^1(\theta^*) \longrightarrow J^1(J^1(\theta^*)), \tag{3.10}$$

where β is the homomorphism in Eq. (3.8), gives a holomorphic splitting of the bottom short exact sequence in Eq. (3.8). In other words,

$$q_0 \circ \beta \circ \rho_D = \text{Id}_{J^1(\theta^*)},$$

where q_0 is the projection in Eq. (3.8).

Therefore, $\beta \circ \rho_D$ defines a holomorphic connection on the vector bundle $J^1(\theta^*)$ (see [2]). Let $\nabla(D)$ be this holomorphic connection on $J^1(\theta^*)$ constructed from D .

Let

$$\mathcal{D}_X \subset H^0(X, \text{Diff}^2(\theta^*, K_X^{\otimes 2} \otimes \theta^*)) \tag{3.11}$$

be the space of all second order differential operators D such that

- $\sigma(D) = 1$, and
- the corresponding connection $\nabla(D)$ has the property that the connection on $\wedge^2 J^1(\theta^*)$ induced by $\nabla(D)$ has trivial monodromy.

There is a bijective correspondence between the space of all projective structures on X and \mathcal{D}_X defined in Eq. (3.11). We recall below the construction of a differential operator lying in \mathcal{D}_X from a projective structure on X .

Let P be a projective structure on X . Let

$$\phi : U \longrightarrow \mathbb{C} \subset \mathbb{C}\mathbb{P}^1$$

be a holomorphic coordinate function compatible with P . Fix a holomorphic section ω of θ^* over U such that the section $\omega \otimes \omega$ of $\theta^* \otimes \theta^* = TX$ coincides with the vector field $\frac{\partial}{\partial z}$ on U with defined by the coordinate function ϕ . Let D_U be the second order holomorphic differential operator

$$\theta^*|_U \longrightarrow (K_X^{\otimes 2} \otimes \theta^*)|_U$$

defined by

$$D_U(h \cdot \omega) = \frac{d^2h}{dz^2} \cdot (dz)^{\otimes 2} \otimes \omega,$$

where h is any holomorphic function on U . It is straight-forward to check that the differential operator D_U is independent of the choice of ϕ compatible with P (it depends only on P). Therefore, these locally defined differential operators D_U patch together compatibly to define a global differential operator from θ^* to $K_X^{\otimes 2} \otimes \theta^*$.

Fix a holomorphic isomorphism f as in Eq. (3.1). Let

$$f_0 : \theta^* \longrightarrow \tau^*\overline{\theta^*} = \tau^*\overline{\theta^*}$$

be the isomorphism induced by f . This isomorphism f_0 induces an isomorphism

$$\widehat{f}_2 : J^2(\theta^*) \longrightarrow J^2(\tau^*\overline{\theta^*}) = \tau^*\overline{J^2(\theta^*)}. \tag{3.12}$$

We note that $\tau^*\overline{\widehat{f}_2} \circ \widehat{f}_2$ is $\text{Id}_{J^2(\theta^*)}$ (respectively, $-\text{Id}_{J^2(\theta^*)}$) if $\tau^*\overline{f} \circ f$ is Id_θ (respectively, $-\text{Id}_\theta$).

The isomorphisms \widehat{f}_2 and f_0 together define a conjugate linear automorphism of the vector space $H^0(X, \text{Diff}^2(\theta^*, K_X^{\otimes 2} \otimes \theta^*))$, which we will describe.

For any homomorphism of vector bundles

$$F : J^2(\theta^*) \longrightarrow K_X^{\otimes 2} \otimes \theta^* = \theta^{\otimes 3},$$

define $F' := \tau^*\overline{f^{\otimes 3}} \circ \tau^*\overline{F} \circ \widehat{f}_2$. It is straight-forward to check that

$$F' : J^2(\theta^*) \longrightarrow \theta^{\otimes 3}$$

is a holomorphic homomorphism of vector bundles. Hence we have a conjugate linear homomorphism

$$\tau_D : H^0(X, \text{Diff}^2(\theta^*, K_X^{\otimes 2} \otimes \theta^*)) \longrightarrow H^0(X, \text{Diff}^2(\theta^*, K_X^{\otimes 2} \otimes \theta^*)) \tag{3.13}$$

defined by $F \mapsto F'$. Since $\tau^* \widehat{f_2} \circ \widehat{f_2}$ and $\tau^* \bar{f} \circ f$ are both Id (respectively, $-\text{Id}$) if $\tau^* \bar{f} \circ f$ is Id (respectively, $-\text{Id}$), it follows that τ_D in Eq. (3.13) is an involution.

The involution τ_D clearly preserves the subset \mathcal{D}_X defined in Eq. (3.11).

Lemma 3.2. *The bijection between the projective structures on X and \mathcal{D}_X takes the projective structures on X compatible with τ surjectively to the fixed point set $(\mathcal{D}_X)^{\tau_D}$.*

Proof. The bijection between the projective structures on X and \mathcal{D}_X takes the involution on the space of projective structures given by τ (see Lemma 2.3) to the involution τ_D of \mathcal{D}_X . In view of this, the lemma follows from Lemma 2.3. \square

4. Projective structure and representation of fundamental group

So far we considered projective structures on a fixed Riemann surface. We will now consider all projective structures without fixing the underlying complex structure. So the definitions in Section 2 have to be modified accordingly, which we do below.

Let Y be a compact connected oriented C^∞ surface.

A C^∞ coordinate function on Y is a pair (U, ϕ) , where U is an open subset of Y , and $\phi : U \rightarrow \mathbb{C}\mathbb{P}^1$ is an orientation preserving C^∞ embedding.

A projective structure on Y is defined by giving a collection $\{(U_i, \phi_i)\}_{i \in I}$ of C^∞ coordinate functions such that

- $\bigcup_{i \in I} U_i = Y$, and
- for each pair $i, i' \in I$, the composition

$$\phi_{i'} \circ \phi_i^{-1} : \phi_i(U_i \cap U_{i'}) \rightarrow \phi_{i'}(U_i \cap U_{i'})$$

is the restriction of some Möbius transformation.

Two such data $\{(U_i, \phi_i)\}_{i \in I}$ and $\{(U_{i'}, \phi_{i'})\}_{i' \in I'}$ satisfying the above conditions are called *equivalent* if their union $\{(U_i, \phi_i)\}_{i \in I \cup I'}$ also satisfies the two conditions. A *projective structure* on Y is an equivalence class of such data.

Therefore, a projective structure on Y gives a complex structure on Y and a projective structure on the corresponding Riemann surface.

Given a projective structure on Y , the coordinate functions compatible with it will be called *projective coordinates*.

Let $\text{Diffeo}_0(Y)$ denote the group of all diffeomorphisms of Y homotopic to the identity map. The group $\text{Diffeo}_0(Y)$ has a natural action on the space of all projective structures on Y .

Definition 4.1. The quotient by $\text{Diffeo}_0(Y)$ of the space of all projective structures on Y will be denoted by $\mathcal{P}_0(Y)$.

The isomorphism classes of flat principal $\text{PSL}(2, \mathbb{C})$ -bundle on Y are identified with the equivalence classes of representations

$$\mathcal{R} := \text{Hom}(\pi_1(Y), \text{PSL}(2, \mathbb{C})) / \text{PSL}(2, \mathbb{C}), \tag{4.1}$$

which, using complex structure of the group $\text{PSL}(2, \mathbb{C})$, has a natural structure of a complex analytic space; the irreducible representations form an open subset contained in the smooth locus, and this open subset has a natural holomorphic symplectic structure [6]. Note that the equivalence classes of homomorphisms from $\pi_1(Y)$ are independent of the choice of the base point in Y needed to define the fundamental group; hence we omit the base point from the notation.

A projective structure P on Y gives a flat principal $\mathrm{PSL}(2, \mathbb{C})$ -bundle on Y . We will briefly recall the construction of this flat principal bundle. Take a $\{(U_i, \phi_i)\}_{i \in I}$ giving a projective structure on Y . On each U_i , consider the trivial principal $\mathrm{PSL}(2, \mathbb{C})$ -bundle

$$E^i_{\mathrm{PSL}(2, \mathbb{C})} := U_i \times \mathrm{PSL}(2, \mathbb{C}).$$

Note that this trivial bundle has a natural flat connection given by the constant sections of the principal bundle. For any ordered pair $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, glue the two principal $\mathrm{PSL}(2, \mathbb{C})$ -bundles $E^i_{\mathrm{PSL}(2, \mathbb{C})}$ and $E^j_{\mathrm{PSL}(2, \mathbb{C})}$ over the open subset $U_i \cap U_j$ using the element of $\mathrm{PSL}(2, \mathbb{C})$ given by $\phi_j \circ \phi_i^{-1}$. (Recall that $\phi_j \circ \phi_i^{-1}$ is the restriction of a Möbius transformation, and the group of Möbius transformations is identified with $\mathrm{PSL}(2, \mathbb{C})$; hence $\phi_j \circ \phi_i^{-1}$ gives an element of $\mathrm{PSL}(2, \mathbb{C})$.) This way we get a principal $\mathrm{PSL}(2, \mathbb{C})$ -bundle on Y . Since the transition functions are constants, the natural connection on the trivial principal bundles $\{E^i_{\mathrm{PSL}(2, \mathbb{C})}\}_{i \in I}$ patch together compatibly to define a flat connection on the principal $\mathrm{PSL}(2, \mathbb{C})$ -bundle over Y .

Sending a flat $\mathrm{PSL}(2, \mathbb{C})$ -connection to its monodromy homomorphism, we get a bijection between \mathcal{RR} (defined in (4.1)) and the isomorphism classes of flat $\mathrm{PSL}(2, \mathbb{C})$ -bundles. Consider $\mathcal{P}_0(Y)$ (see Definition 4.1). Let

$$\mu : \mathcal{P}_0(Y) \longrightarrow \mathcal{R} \tag{4.2}$$

be the map that sends any projective structure to the monodromy of the corresponding flat $\mathrm{PSL}(2, \mathbb{C})$ connection. It is known that the map μ is injective, and furthermore, its image is an open subset of \mathcal{R} contained in the locus of irreducible representations [10,11]. Therefore, $\mathcal{P}_0(Y)$ is a complex manifold; the complex structure is uniquely determined by the condition that the map μ is holomorphic.

The holomorphic symplectic form on the locus of irreducible representations defines a holomorphic symplectic form on $\mathcal{P}_0(Y)$.

Let

$$\mathcal{R}_P \subset \mathcal{R} \tag{4.3}$$

be the open subset defined by the projective structures on Y .

Let

$$\tau : Y \longrightarrow Y \tag{4.4}$$

be an orientation reversing diffeomorphism such that

$$\tau \circ \tau = \mathrm{Id}_Y.$$

Imitating Definition 2.1, we define the following:

Definition 4.2. A projective structure on Y is said to be *compatible* with τ if for each projective coordinate (U, ϕ) , the composition $(\tau(U), \phi \circ \tau)$ is also a projective coordinate.

Our aim in this section is to identify the subset of \mathcal{R}_P (see Eq. (4.3)) that corresponds to the projective structures compatible with τ .

Fix a base point $y_0 \in Y$ such that $\tau(y_0) \neq y_0$. We recall from [4] an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\pi_1(Y, y_0)$.

Let Γ denote the space of all homotopy classes, with fixed end points, of continuous paths $\gamma : [0, 1] \longrightarrow Y$ such that

- $\gamma(0) = y_0$, and
- $\gamma(1) \in \{y_0, \tau(y_0)\}$.

So Γ is a disjoint union of $\pi_1(Y, y_0)$ and $\text{Path}(Y, y_0)$ (the homotopy classes of paths from y_0 to $\tau(y_0)$). We recall below the group structure of Γ . For $\gamma_1 \in \pi_1(Y, y_0)$, we have the usual composition $\gamma_2\gamma_1 = \gamma_2 \circ \gamma_1$ of paths. If $\gamma_1 \in \text{Path}(Y, y_0)$, then define

$$\gamma_2\gamma_1 := \sigma(\gamma_2) \circ \gamma_1.$$

Therefore, we have a short exact sequence of groups

$$e \longrightarrow \pi_1(Y, y_0) \longrightarrow \Gamma \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow e. \tag{4.5}$$

Let \mathcal{G} denote the set of all diffeomorphisms of $\mathbb{C}\mathbb{P}^1$ that are either holomorphic or anti-holomorphic. Note that \mathcal{G} is a group under the composition of maps. Therefore, we have a short exact sequence of groups

$$e \longrightarrow \text{PSL}(2, \mathbb{C}) \longrightarrow \mathcal{G} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow e. \tag{4.6}$$

Proposition 4.3. *The subset of \mathcal{R}_P defined by the projective structures on Y compatible with τ consists of those homomorphisms*

$$\rho : \pi_1(Y, y_0) \longrightarrow \text{PSL}(2, \mathbb{C})$$

in \mathcal{R}_P (see Eq. (4.3)) that extend to a homomorphism

$$\tilde{\rho} : \Gamma \longrightarrow \mathcal{G}$$

fitting in the following commutative diagram

$$\begin{array}{ccccccccc} e & \longrightarrow & \pi_1(Y, y_0) & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & e \\ & & \downarrow \rho & & \downarrow \tilde{\rho} & & \parallel & & \\ e & \longrightarrow & \text{PSL}(2, \mathbb{C}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & e \end{array}$$

(see Eqs. (4.5) and (4.6)).

Proof. Let P be a projective structure on Y compatible with τ . Let

$$(\mathbb{P}_P, \nabla) \longrightarrow Y$$

be the flat projective bundle of relative (complex) dimension one associated to P . We also have a lift of the automorphism τ to a C^∞ involution

$$\widehat{\tau} : \mathbb{P}_P \longrightarrow \mathbb{P}_P \tag{4.7}$$

that preserves the connection ∇ (see Lemma 2.5 and Eq. (2.7)).

Fix a holomorphic isomorphism

$$\zeta_0 : \mathbb{C}\mathbb{P}^1 \longrightarrow (\mathbb{P}_P)_{y_0} \tag{4.8}$$

of the fiber $(\mathbb{P}_P)_{y_0}$ with the projective line.

For any $\gamma \in \pi_1(Y, y_0)$, consider the parallel translation

$$T(\nabla, \gamma) : (\mathbb{P}_P)_{y_0} \longrightarrow (\mathbb{P}_P)_{y_0}$$

along γ for the connection ∇ . We have the monodromy homomorphism for the connection ∇

$$\rho : \pi_1(Y, y_0) \longrightarrow \mathrm{PSL}(2, \mathbb{C}) \tag{4.9}$$

defined by $\gamma \mapsto \zeta_0^{-1} \circ T(\nabla, \gamma) \circ \zeta_0$, where ζ_0 is the fixed isomorphism in Eq. (4.8).

Now, take any element $\gamma \in \mathrm{Path}(Y, y_0)$, so γ is a homotopy class of paths from y_0 to $\tau(y_0)$. Let

$$T(\nabla, \gamma) : (\mathbb{P}^P)_{y_0} \longrightarrow (\mathbb{P}^P)_{\tau(y_0)}$$

be the isomorphism of fibers obtained by taking parallel translation along γ for the connection ∇ . Consider the diffeomorphism

$$g_\gamma := \zeta_0^{-1} \circ \widehat{\tau}_{\tau(y_0)} \circ T(\nabla, \gamma) \circ \zeta_0 \tag{4.10}$$

of $\mathbb{C}\mathbb{P}^1$, where

$$\widehat{\tau}_{\tau(y_0)} : (\mathbb{P}^P)_{\tau(y_0)} \longrightarrow (\mathbb{P}^P)_{y_0}$$

is the restriction of the diffeomorphism $\widehat{\tau}$ in Eq. (4.7). Note that the diffeomorphism g_γ in Eq. (4.10) is anti-holomorphic, because $\widehat{\tau}$ is anti-holomorphic, while ζ_0 and $T(\nabla, \gamma)$ are both holomorphic isomorphisms.

Let

$$\rho' : \mathrm{Path}(Y, y_0) \longrightarrow \mathcal{G} \tag{4.11}$$

be the map defined by $\gamma \mapsto g_\gamma$, where \mathcal{G} is defined in Eq. (4.6), and g_γ is constructed in Eq. (4.10). The two maps ρ and ρ' constructed in Eqs. (4.9) and (4.11) respectively together define a homomorphism

$$\widetilde{\rho} : \Gamma \longrightarrow \mathcal{G}$$

that fits in the commutative diagram in the statement of the proposition.

To prove the converse, let

$$\rho : \pi_1(Y, y_0) \longrightarrow \mathrm{PSL}(2, \mathbb{C}) \tag{4.12}$$

be a homomorphism that lies in \mathcal{R}_P (defined in Eq. (4.3)). Let P be the projective structure on Y defined by ρ .

The self-map τ in Eq. (4.4) defines an isomorphism

$$\tau_* : \pi_1(Y, y_0) \longrightarrow \pi_1(Y, \tau(y_0))$$

that sends any loop to its image by τ . Let

$$\overline{\rho}_\tau : \pi_1(Y, \tau(y_0)) \longrightarrow \mathrm{PSL}(2, \mathbb{C}) \tag{4.13}$$

be the homomorphism defined by $g \mapsto \overline{\rho}(\tau_*^{-1}(g))$.

Let \overline{P} be the projective structure on Y corresponding to the projective structure P by the involution in Lemma 2.3. This projective structure \overline{P} is given by the homomorphism $\overline{\rho}_\tau$ constructed in Eq. (4.13).

Assume that there is a homomorphism

$$\widetilde{\rho} : \Gamma \longrightarrow \mathcal{G} \tag{4.14}$$

that fits in the commutative diagram

$$\begin{array}{ccccccc}
 e & \longrightarrow & \pi_1(Y, y_0) & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow e \\
 & & \downarrow \rho & & \downarrow \tilde{\rho} & & \parallel \\
 e & \longrightarrow & \mathrm{PSL}(2, \mathbb{C}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow e
 \end{array}$$

(as in the statement of the proposition).

Take any element $g_0 \in \mathcal{G}$ that projects to the generator of $\mathbb{Z}/2\mathbb{Z}$. Let

$$\rho(g_0) : \pi_1(Y, y_0) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$$

be the homomorphism defined by $\gamma \mapsto \gamma_0^{-1} \rho(\gamma) g_0$. One can check that the element in \mathcal{R} (see Eq. (4.1)) given by $\rho(g_0)$ coincides with the element given by $\bar{\rho}_\tau$ constructed in Eq. (4.13).

Now from the injectivity of the map μ in Eq. (4.2) we conclude that the projective structure P coincides with the projective structure \bar{P} . This completes the proof of the proposition. \square

5. Projective structure and symplectic form

We continue with the notation of the previous section. Let $\mathcal{C}(Y)$ denote the space of all complex structures on Y compatible with the orientation of Y . The group $\mathrm{Diffeo}_0(Y)$ has a natural action on $\mathcal{C}(Y)$. The quotient

$$\mathcal{T}(Y) := \mathcal{C}(Y)/\mathrm{Diffeo}_0(Y)$$

is the Teichmüller space, which has a natural complex structure. The involution τ (see Eq. (4.4)) gives an involution of $\mathcal{T}(Y)$ that sends any almost complex structure J on Y to $(d\tau)^{-1} \circ J \circ d\tau$, where $d\tau$ is the differential of τ . This involution of $\mathcal{T}(Y)$ is anti-holomorphic.

Consider $\mathcal{P}_0(Y)$ introduced in Definition 4.1. Let

$$\varphi : \mathcal{P}_0(Y) \longrightarrow \mathcal{T}(Y) \tag{5.1}$$

be the forgetful map that sends a projective structure to the underlying complex structure. It is known that φ makes $\mathcal{P}_0(Y)$ a holomorphic fiber bundle over $\mathcal{T}(Y)$. More precisely, $\mathcal{P}_0(Y)$ is a holomorphic torsor for the holomorphic cotangent bundle $\Omega^1_{\mathcal{T}(Y)} \longrightarrow \mathcal{T}(Y)$; this means that there is a natural holomorphic map

$$\mathcal{A} : \mathcal{P}_0(Y) \times_{\mathcal{T}(Y)} \Omega^1_{\mathcal{T}(Y)} \longrightarrow \mathcal{P}_0(Y), \tag{5.2}$$

from the fiber product, with the property that the restriction of \mathcal{A} over any point $t \in \mathcal{T}(Y)$ is a free transitive action of the cotangent space $(\Omega^1_{\mathcal{T}(Y)})_t$ on the fiber $(\mathcal{P}_0(Y))_t$. Therefore, given any holomorphic section of the map φ

$$s : \mathcal{T}(Y) \longrightarrow \mathcal{P}_0(Y), \tag{5.3}$$

we get a holomorphic isomorphism

$$I_s : \Omega^1_{\mathcal{T}(Y)} \longrightarrow \mathcal{P}_0(Y) \tag{5.4}$$

that sends any $v \in (\Omega^1_{\mathcal{T}(Y)})_t$ to $\mathcal{A}(s(t), v)$, where \mathcal{A} is the map in Eq. (5.2).

Recall that $\mathcal{P}_0(Y)$ has a natural symplectic form which is obtained by pulling back, by the map μ in Eq. (4.2), the natural symplectic form on the smooth locus of \mathcal{R} . It is known that there are holomorphic sections s as in Eq. (5.3) such that the corresponding map I_s in Eq. (5.4) takes the Liouville symplectic form on $\Omega^1_{\mathcal{T}(Y)}$ to the natural symplectic form on $\mathcal{P}_0(Y)$ (see [12,1]).

Recall that both $\mathcal{T}(Y)$ and $\mathcal{P}_0(Y)$ are equipped with anti-holomorphic involutions constructed using τ . Let τ_T (respectively, τ_P) be the anti-holomorphic involution of $\mathcal{T}(Y)$ (respectively, $\mathcal{P}_0(Y)$) given by τ .

Lemma 5.1. *There is a holomorphic section $s : \mathcal{T}(Y) \longrightarrow \mathcal{P}_0(Y)$ of the projection φ in Eq. (5.1) such that*

- $s \circ \tau_T = \tau_P \circ s$, and
- the map I_s in Eq. (5.4) takes the Liouville symplectic form on $\Omega_{\mathcal{T}(Y)}^1$ to the natural symplectic form on $\mathcal{P}_0(Y)$.

Proof. Fix a holomorphic section $s_0 : \mathcal{T}(Y) \longrightarrow \mathcal{P}_0(Y)$ such that the map I_{s_0} in Eq. (5.4) takes the Liouville symplectic form on $\Omega_{\mathcal{T}(Y)}^1$ to the natural symplectic form on $\mathcal{P}_0(Y)$; from [12,1] we know that such sections exist. Let

$$s'_0 : \mathcal{T}(Y) \longrightarrow \mathcal{P}_0(Y)$$

be the section of φ defined by $z \longmapsto (\tau_P \circ s_0 \circ \tau_T)(z)$. Since both τ_P and τ_T are anti-holomorphic, and s is holomorphic, it follows that s'_0 is holomorphic. Let

$$\omega : H^0(\mathcal{T}(Y), \Omega_{\mathcal{T}(Y)}^1)$$

be the holomorphic section uniquely determined by the condition that

$$\mathcal{A}(s_0(z), \omega(z)) = s'_0(z)$$

for all $z \in \mathcal{T}(Y)$, where \mathcal{A} is the map in Eq. (5.2). The involution τ_P takes the symplectic form on $\mathcal{P}_0(Y)$ to its negative. From this it can be deduced that $I_{s'_0}$ also takes the Liouville symplectic form on $\Omega_{\mathcal{T}(Y)}^1$ to the symplectic form on $\mathcal{P}_0(Y)$. Indeed, it suffices to show that the image of s'_0 is Lagrangian. But this is a consequence of the fact that the image of s_0 is Lagrangian.

Let

$$s : \mathcal{T}(Y) \longrightarrow \mathcal{P}_0(Y)$$

be the holomorphic section of φ defined by $z \longmapsto \mathcal{A}(s_0(z), \omega(z)/2)$. We will show that s satisfies all the conditions in the lemma.

From the construction of s it follows that $s \circ \tau_T = \tau_P \circ s$.

Using s_0 , identify $\mathcal{P}_0(Y)$ with $\Omega_{\mathcal{T}(Y)}^1$. Using this identification of $\mathcal{P}_0(Y)$ with $\Omega_{\mathcal{T}(Y)}^1$, both s'_0 and s are holomorphic one-forms on $\mathcal{T}(Y)$. These one-forms will be denoted by \widehat{s}'_0 and \widehat{s} respectively. From the construction of s it follows that

$$\widehat{s} = \frac{1}{2} \widehat{s}'_0. \tag{5.5}$$

Since $I_{s'_0}$ takes the Liouville symplectic form on $\Omega_{\mathcal{T}(Y)}^1$ to the symplectic form on $\mathcal{P}_0(Y)$, we conclude that $d\widehat{s}'_0 = 0$. Hence from Eq. (5.5),

$$d\widehat{s} = 0.$$

This immediately implies that I_s in Eq. (5.4) takes the Liouville symplectic form on $\Omega_{\mathcal{T}(Y)}^1$ to the natural symplectic form on $\mathcal{P}_0(Y)$. This completes the proof of the lemma. \square

Let M be a manifold. The total space of the cotangent bundle T^*M is equipped with the Liouville symplectic form. Let $S \subset M$ be a smooth submanifold. Let

$$N_S^* \subset (T^*M)|_S \subset T^*M$$

be the co-normal bundle which is given by the dual of the projection of $(TM)|_S$ to the normal bundle N_S to S . The submanifold $N_S^* \subset T^*M$ is Lagrangian.

Take any component \mathcal{S} of the fixed point locus of the involution τ_P of $\mathcal{P}_0(Y)$. Fix a section s as in Lemma 5.1; identify $\mathcal{P}_0(Y)$ with $\Omega_{\mathcal{T}(Y)}^1$ using s . In terms of this identification, the submanifold \mathcal{S} coincides with the co-normal bundle of the submanifold $\varphi(\mathcal{S}) \subset \mathcal{T}(Y)$.

6. Symplectic structure on moduli space of projective structures

Let V be a complex vector space of dimension two. For any nonnegative integer i , consider the homomorphism

$$\Phi_i : H^0(\mathbb{P}(V), T\mathbb{P}(V)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)} \longrightarrow J^i(T\mathbb{P}(V)) \tag{6.1}$$

that sends any pair (x, s) , where $x \in \mathbb{P}(V)$ and $s \in H^0(\mathbb{P}(V), T\mathbb{P}(V))$, to the element of $J^2(T\mathbb{P}(V))_x$ obtained by restricting s to the i -th order infinitesimal neighborhood of x . It is easy to see that Φ_2 is an isomorphism. The composition

$$\Phi_3 \circ \Phi_2^{-1} : J^2(T\mathbb{P}(V)) \longrightarrow J^3(T\mathbb{P}(V)) \tag{6.2}$$

is a splitting of the jet sequence

$$0 \longrightarrow K_{\mathbb{P}(V)}^{\otimes 3} \otimes T\mathbb{P}(V) = K_{\mathbb{P}(V)}^{\otimes 2} \xrightarrow{\iota_0} J^3(T\mathbb{P}(V)) \longrightarrow J^2(T\mathbb{P}(V)) \longrightarrow 0. \tag{6.3}$$

Let

$$\mathcal{D}(\mathbb{P}(V)) : J^3(T\mathbb{P}(V)) \longrightarrow K_{\mathbb{P}(V)}^{\otimes 2} \tag{6.4}$$

be the unique homomorphism such that

$$\text{kernel}(\mathcal{D}(\mathbb{P}(V))) = \text{image}(\Phi_3 \circ \Phi_2^{-1}),$$

and

$$\mathcal{D}(\mathbb{P}(V)) \circ \iota_0 = \text{Id}_{K_{\mathbb{P}(V)}^{\otimes 2}}, \tag{6.5}$$

where ι_0 is the homomorphism in Eq. (6.3). Therefore, $\mathcal{D}(\mathbb{P}(V))$ is a global holomorphic differential operator of third order, more precisely,

$$\mathcal{D}(\mathbb{P}(V)) \in H^0(\mathbb{P}(V), \text{Diff}^3(T\mathbb{P}(V), K_{\mathbb{P}(V)}^{\otimes 2})).$$

The symbol of the differential operator $\mathcal{D}(\mathbb{P}(V))$ is a holomorphic section of

$$(T\mathbb{P}(V))^{\otimes 3} \otimes \text{Hom}(T\mathbb{P}(V), K_{\mathbb{P}(V)}^{\otimes 2}) = \mathcal{O}_X,$$

and it coincides with $\mathcal{D}(\mathbb{P}(V)) \circ \iota_0$, where ι_0 is the homomorphism in Eq. (6.3). From Eq. (6.5) it follows that the symbol of the differential operator $\mathcal{D}(\mathbb{P}(V))$ is the constant function 1.

Let

$$\mathcal{S}(\mathbb{P}(V)) \longrightarrow \mathbb{P}(V) \tag{6.6}$$

be the local system defined by the sheaf of solutions of the differential operator $\mathcal{D}(\mathbb{P}(V))$; it coincides with the trivial vector bundle with fiber $H^0(\mathbb{P}(V), T\mathbb{P}(V)) = \mathfrak{sl}(V)$, where $\mathfrak{sl}(V) \subset \text{End}(V)$ is the endomorphisms of trace zero. Hence

$$\mathcal{S}(\mathbb{P}(V)) = \mathbb{P}(V) \times \mathfrak{sl}(V). \tag{6.7}$$

The standard action of $\text{GL}(V)$ on V defines an action of $\text{GL}(V)$ on $T(\mathbb{P}(V))$; hence we have an action of $\text{GL}(V)$ on each tensor power of $T(\mathbb{P}(V))$. The differential operator $\mathcal{D}(\mathbb{P}(V))$ is clearly fixed by the action of $\text{GL}(V)$. Also, the identification in Eq. (6.7) is $\text{GL}(V)$ -equivariant (the action of $\text{GL}(V)$ on $\mathfrak{sl}(V)$ is the adjoint one).

Now, let X be a compact connected Riemann surface equipped with a projective structure \mathcal{P} . Identifying $\mathbb{C}\mathbb{P}^1$ with $\mathbb{P}(V)$, we may consider the projective coordinates as embeddings of open subsets of X into $\mathbb{P}(V)$. With this identification, the transition functions lie in $\text{PGL}(V)$.

Since the differential operator $\mathcal{D}(\mathbb{P}(V))$ in Eq. (6.4) is $\text{GL}(V)$ -invariant, and the transition functions for the projective coordinate functions lie in $\text{PGL}(V)$, the projective structure \mathcal{P} produces a differential operator

$$\mathcal{D}_{\mathcal{P}} \in H^0(X, \text{Diff}^3(TX, K_X^{\otimes 2})) \tag{6.8}$$

which is constructed locally from $\mathcal{D}(\mathbb{P}(V))$. The symbol of $\mathcal{D}_{\mathcal{P}}$ is the constant function 1 because the symbol of $\mathcal{D}(\mathbb{P}(V))$ is so. Let

$$\mathcal{S}(\mathcal{P}) \longrightarrow X \tag{6.9}$$

be the local system defined by the sheaf of solutions of $\mathcal{D}_{\mathcal{P}}$.

Let

$$\mathbb{P}_{\mathcal{P}} \longrightarrow X$$

be the flat $\text{PGL}(V)$ -bundle given by the projective structure \mathcal{P} (see Lemma 2.5); we recall that the holomorphic $\text{PGL}(V)$ -bundle underlying $\mathbb{P}_{\mathcal{P}}$ coincides with \mathbb{P}_L constructed in Eq. (2.4). Let

$$\text{ad}(\mathbb{P}_{\mathcal{P}}) \longrightarrow X \tag{6.10}$$

be the corresponding flat adjoint vector bundle; recall that $\text{ad}(\mathbb{P}_{\mathcal{P}})$ is associated to $\mathbb{P}_{\mathcal{P}}$ for the adjoint action of $\text{PGL}(V)$ on $\mathfrak{sl}(V)$. Let

$$\underline{\text{ad}}(\mathbb{P}_{\mathcal{P}}) \longrightarrow X \tag{6.11}$$

be the local system defined by the sheaf of flat sections of $\text{ad}(\mathbb{P}_{\mathcal{P}})$. Since the identification in Eq. (6.7) is $\text{GL}(V)$ -invariant, we have

$$\underline{\text{ad}}(\mathbb{P}_{\mathcal{P}}) = \mathcal{S}(\mathcal{P}), \tag{6.12}$$

where $\mathcal{S}(\mathcal{P})$ and $\underline{\text{ad}}(\mathbb{P}_{\mathcal{P}})$ are constructed in Eqs. (6.9) and (6.11) respectively.

The space of infinitesimal deformations of a representation of $\pi_1(X)$ is given by the first cohomology of the local system defined by the adjoint representation [6]. In particular, the space of infinitesimal deformations of the representation of $\pi_1(X)$ in $\text{PGL}(V)$ associated to \mathcal{P} is given by $H^1(X, \text{ad}(\mathbb{P}_{\mathcal{P}}))$. This implies that the space of infinitesimal deformations of the projective structure \mathcal{P} (in the isotopy classes of projective structures) on the oriented C^∞ surface X is given by $H^1(X, \underline{\text{ad}}(\mathbb{P}_{\mathcal{P}}))$ (recall that the map μ in Eq. (4.2) is an open embedding).

As in Eq. (4.2), let $\mathcal{P}_0(X)$ denote the space of all projective structures on the oriented C^∞ surface X modulo the group $\text{Diffeo}_0(X)$ of diffeomorphisms of X homotopic to the identity map. For the projective structure \mathcal{P} on X , we have noted above that

$$T_{\mathcal{P}}\mathcal{P}_0(X) = H^1(X, \underline{\text{ad}}(\mathbb{P}_{\mathcal{P}})). \tag{6.13}$$

Let

$$\mathcal{C}^\bullet : \mathcal{C}^0 := TX \xrightarrow{\mathcal{D}_{\mathcal{P}}} \mathcal{C}^1 := K_X^{\otimes 2}$$

be the complex of sheaves of X . From Eqs. (6.12), (6.9) and (6.13),

$$T_{\mathcal{P}}\mathcal{P}_0(X) = \mathbb{H}^1(X, \mathcal{C}^\bullet), \tag{6.14}$$

where \mathbb{H}^1 is the hypercohomology.

Consider the tensor product of the complex of sheaves \mathcal{C}^\bullet with itself

$$\begin{aligned} \mathcal{C}^\bullet \otimes_{\mathbb{C}} \mathcal{C}^\bullet : \mathcal{C}^0 \otimes_{\mathbb{C}} \mathcal{C}^0 &\xrightarrow{\text{Id} \otimes \mathcal{D}_{\mathcal{P}} \oplus \mathcal{D}_{\mathcal{P}} \otimes \text{Id}} (TX \otimes_{\mathbb{C}} K_X^{\otimes 2}) \oplus (K_X^{\otimes 2} \otimes_{\mathbb{C}} TX) \\ &\xrightarrow{\mathcal{D}_{\mathcal{P}} \otimes \text{Id} - \text{Id} \otimes \mathcal{D}_{\mathcal{P}}} \mathcal{C}^1 \otimes_{\mathbb{C}} \mathcal{C}^1. \end{aligned}$$

Also, consider the complex

$$K_X[1] : 0 \longrightarrow K_X.$$

The natural contraction of TX with K_X defines a homomorphism of complexes

$$\mathcal{C}^\bullet \otimes_{\mathbb{C}} \mathcal{C}^\bullet \longrightarrow K_X[1].$$

Using it, we have homomorphisms

$$\mathbb{H}^1(X, \mathcal{C}^\bullet) \otimes_{\mathbb{C}} \mathbb{H}^1(X, \mathcal{C}^\bullet) \longrightarrow \mathbb{H}^2(X, \mathcal{C}^\bullet \otimes_{\mathbb{C}} \mathcal{C}^\bullet) \longrightarrow \mathbb{H}^2(X, K_X[1]) = H^1(X, K_X) = \mathbb{C}.$$

The resulting pairing on $T_{\mathcal{P}}\mathcal{P}_0(X)$ (see Eq. (6.13)) coincides with the natural symplectic form on $\mathcal{P}_0(X)$.

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