Extension of the operational Tau method for solving 1-D nonlinear transient heat conduction equations

S.A. Hosseini a, S. Shahmorad a,*, H. Masoumi b

a Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
b Niroo Research Institute, Tehran, Iran

Received 29 September 2012; accepted 8 January 2013
Available online 22 January 2013

KEYWORDS
Nonlinear transient heat conduction equation; Operational Tau method

Abstract In this paper, we consider a class of nonlinear transient heat conduction equations with some supplementary conditions. We apply the operational Tau method with arbitrary polynomial bases to approximate the solution of these equations. In addition, some theoretical results are given to simplify and reduce the computational cost. Finally some numerical examples are given to clarify the efficiency and accuracy of the proposed method.

1. Introduction

A parabolic partial differential equation has extensive applications in engineering and applied sciences such as phenomena of dispersion, diffusion, conduction, convection, reaction and dissipation (Su et al., 2009; Helal, 2012; Khan et al., 2012; Ozisik, 1993; Rahman, 2002; Wazwaz, 2009). Most of these equations are usually difficult to solve analytically, therefore approximate or numerical techniques must be used. We are interested in presenting an approximate scheme based on the operational Tau method to solve nonlinear transient heat conduction equations with variable thermo-physical properties which can involve heat generation terms.

The Tau method can be described as a spectral method with arbitrary polynomial bases. In fact, the Tau method is a modification of the spectral Galerkin method that is applicable to problems with non-periodic boundary conditions. The main difference between them are the test functions which are not required individually to satisfy the boundary conditions in the Tau method (Canuto et al., 2006; Gottlieb and Orszag, 1977).

Ortiz and Samara (1981) proposed an operational approach to the Tau method as an approximation technique for solving nonlinear ordinary differential equations with supplementary conditions. The advantages of this technique are using simple operational matrices which reduce the computational costs remarkably, since only non zero elements of these matrices are needed to save.

During recent years, much work has been done for solving various types of ordinary differential equations, partial differential equations, and integral and integro-differential equations (Ebbadi et al., 2007; Ghoreishi and Hadizadeh, 2009; Hosseini, 2009; Hosseini Aliabadi and Shahmorad, 2002; Liu and Ortiz, 1989; Liu and Pan, 1999) by the Tau method.

In this work, we state the required preliminaries of the operational Tau method to apply on nonlinear second-order partial differential equations. Then, we present a numerical
scheme for solving 1-D nonlinear transient heat conduction equation of the form
\[
\frac{\partial}{\partial x} \left( k(u) \frac{\partial u}{\partial x} \right) + f(x, t) = \rho C_e \frac{\partial u}{\partial t}, \quad x \in I = [a, b], \quad t > 0, \quad (1)
\]
with the supplementary conditions
\[
B_1 u(x, t) = \varphi_1(t),
\]
\[
B_2 u(x, t) = \varphi_2(t),
\]
\[
I, u(x, t) = \psi(x),
\]
where \( f(x, t) \), \( \varphi_1(t) \), \( \varphi_2(t) \) and \( \psi(x) \) are given smooth real valued functions and the linear operators \( B_1, B_2 \) and \( I \) define the left-hand side of the supplementary conditions. Note that (2) and (3) denote the boundary conditions and (4) denotes the initial condition. For example, these conditions can be expressed as follows:
\[
e_{11} u(a, t) + e_{12} \frac{\partial}{\partial x} (a, t) = \varphi_1(t),
\]
\[
e_{21} u(b, t) + e_{22} \frac{\partial}{\partial x} (b, t) = \varphi_2(t),
\]
\[
u(x, t_0) = \psi(x),
\]
where \( e_{ij} \) are constants.

The parameters \( \rho \) and \( C_e \) in Eq. (1) denote density and specific heat, respectively. Also, we assume that the thermal conductivity varies with temperature in the form
\[
k(u) = k_0(1 + z_1 u + z_2 u^2),
\]
where \( k_0 \) is the value of \( k(u) \) at reference temperature and \( z_1, z_2 \) are known coefficients.

This paper is organized as follows: In the next section, we briefly review some preliminaries of the Tau method. Also, we present some theorems and lemmas to formulate nonlinear transient heat conduction equation with some given supplementary conditions. In Section 3, we recall an efficient Tau error estimator. Numerical results of some problems are given in Section 4 to clarify the efficiency of the method. Finally, Section 5 contains the conclusion.

2. Some theoretical results

The operational approach to the Tau method proposed by Ortiz and Samara (1981) based on the use of following simple matrices
\[
\begin{align*}
\mu &= [\mu_{ij}]_{i<j} = \delta_{i+j}, \\
\eta &= [\eta_{ij}]_{i<j} = (i+1) \delta_{j+i+1}
\end{align*}
\]
having the following properties:

**Lemma 2.1** Ortiz and Samara, 1981. If \( y_N(x) = g_N X_x \), where \( g_N = (a_0, a_1, \ldots, a_N, 0, \ldots) \) and \( X_x = (1, x, x^2, \ldots)^T \), then
\[
\begin{align*}
& \text{(a)} \quad x y_N(x) = g_N x X_x, \\
& \text{(b)} \quad \frac{\partial}{\partial x} y_N(x) = g_N \eta X_x.
\end{align*}
\]

**Corollary 2.2** Ortiz and Samara, 1981. Generally, under assumptions of Lemma 2.1, we have
\[
\begin{align*}
& \text{(a)} \quad x^k X_x = \mu^k X_x, \\
& \text{(b)} \quad \frac{\partial}{\partial x^p} u_N(x, t) = \frac{\partial}{\partial x^p} \left( u_N \Phi_x \right) = \mu \Phi \frac{\partial}{\partial x^p} (X_x \otimes X), \\
& \quad = u \Phi \Phi_x \Phi^{-1} \Phi_x = \mu \Phi \Phi_x.
\end{align*}
\]

Let \( \phi_i = \{ \phi_i(x) : i \in N = \{0, 1, \ldots, N\} \} \) be a polynomial basis given by \( \phi_i = \Phi X_x \), where \( \Phi \) is a nonsingular lower triangular coefficients matrix and \( X_x = (1, x, x^2, \ldots)^T \) is the standard basis. In this work, we assume the approximate solution has the truncated series form
\[
\begin{align*}
& \text{(a)} \quad x u_N(x, t) = \frac{\partial}{\partial x} u_N(x, t) = \frac{\partial}{\partial x^p} u_N(x, t) = 0, \\
& \text{(b)} \quad \frac{\partial}{\partial x^p} u_N(x, t) = \frac{\partial}{\partial x^p} \left( u_N \Phi_x \right) = \mu \Phi \frac{\partial}{\partial x^p} (X_x \otimes X), \\
& \quad = u \Phi \Phi_x \Phi^{-1} \Phi_x = \mu \Phi \Phi_x.
\end{align*}
\]

**Theorem 2.3**. If \( u_N(x, t) = \mu \Phi \), then
\[
\begin{align*}
& \text{(a)} \quad x u_N(x, t) = \frac{\partial}{\partial x} u_N(x, t) = \frac{\partial}{\partial x^p} u_N(x, t) = 0, \\
& \text{(b)} \quad \frac{\partial}{\partial x^p} u_N(x, t) = \frac{\partial}{\partial x^p} \left( u_N \Phi_x \right) = \mu \Phi.
\end{align*}
\]

**Proof.**
\[
\begin{align*}
& \text{(a)} \quad \frac{\partial}{\partial x^p} u_N(x, t) = \frac{\partial}{\partial x^p} \left( u_N \Phi_x \right) = \mu \Phi, \\
& \text{(b)} \quad \frac{\partial}{\partial x^p} u_N(x, t) = \frac{\partial}{\partial x^p} \left( u_N \Phi_x \right) = \mu \Phi
\end{align*}
\]
where \( \mu \equiv (j + 1) \delta_{ij} I, j = 0, 1, \ldots, N - 1. \)

(d) The proof is similar to the proof of part (c). \( \Box \)

**Remark 2.4.** The effect of \( x^p, p \in \mathbb{N} \) on the coefficients of \( u_h(x,t) = \psi \phi_{x,t} \) is equivalent to post multiplication of \( \mu \) by \( \mu \phi_{x,t} \), i.e.,

\[
x^p \psi \phi_{x,t} = \psi \mu \phi_{x,t} \quad \text{or} \quad x^p \phi_{x,t} = \mu \phi_{x,t},
\]

where \( \mu \phi_{x,t} = \Phi \mu \phi_{x,t} \Phi^{-1} \) and the elements of the matrices \( \mu \phi_{x,t} \) and \( \mu \phi_{x,t} \) are determined by

\[
(\mu \phi)_{ij} = \begin{cases} 1, & \text{if } i = j, \quad i = 0, 1, \ldots, N; \\
0, & \text{otherwise.}
\end{cases}
\]

And

\[
(\mu \phi)_{ij} = \begin{cases} \mu, & \text{if } i = j, \quad i = 0, 1, \ldots, N; \\
0, & \text{otherwise.}
\end{cases}
\]

with \((\mu \phi)_{ij} = \delta_{ij} + \mu, j = 1, \ldots, N - q\).

In addition, the matrix \( \mu \phi_{x,t} \) has the following simple form

\[
\mu \phi_{x,t} = \begin{pmatrix} 0 & B \\ \bar{0} & \bar{0} \end{pmatrix},
\]

where \( \bar{0} \) is an \( \mu \times \mu \) zero matrix and \( B \) is an \((N - p + 1) \times (N - p + 1)\) diagonal matrix with the elements \( \mu \).

**Remark 2.5.** The matrices \( \phi_{x,t} \) and \( \phi_{x,t} \) have the following simple forms

\[
\phi_{x,t} = \begin{pmatrix} 0 & C \\ \bar{0} & \bar{0} \end{pmatrix},
\]

where

\[
C = \text{diag}((p+1) \times (N+1) + \cdots + 2) I, \quad \text{and}
\]

\[
\phi_{x,t} = \text{diag}(\eta^0, \eta^0, \ldots, \eta^0).
\]

To obtain a matrix representation for the nonlinear part of Eq. (1), we state the following lemma and corollaries.

**Lemma 2.6.** Let \( u_h(x,t) = \psi \phi_{x,t} = \psi \Phi \psi \phi_{x,t} \) and \( v_h(x,t) = \psi \phi_{x,t} = \psi \Phi \psi \phi_{x,t} \), where \( \psi \) is a vector similar to \( u \) with \( \psi \) elements \( v \). Then

\[
u_h(x,t) = \psi \Phi \psi \phi_{x,t} = \psi \psi \Phi \psi \phi_{x,t},
\]

where \( \Phi \psi \Phi^{-1} \) and \( \Phi \)'s are the columns of matrix \( \Phi \) with

\[
V = \begin{pmatrix} \psi \Phi_0 & \psi \Phi_q & \cdots & \psi \Phi_{(N+1)^2-1} \\ 0 & \psi \Phi_0 & \cdots & \psi \Phi_{(N+1)^2-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi \Phi_0 \end{pmatrix},
\]

Proof. By assumption of lemma, we have

\[
u_h(x,t) = \psi \Phi \psi \phi_{x,t} = \psi \Phi \phi_{x,t} \times \psi \Phi \phi_{x,t},
\]

thus it suffices to show \( \nu_h(x,t) = \nu \Phi \psi \phi_{x,t} \). By a simple computation, we can write

\[
\nu_h(x,t) = \begin{pmatrix} \psi \Phi_0 & \psi \Phi_q & \cdots & \psi \Phi_{(N+1)^2-1} \\ 0 & \psi \Phi_0 & \cdots & \psi \Phi_{(N+1)^2-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi \Phi_0 \end{pmatrix} \psi \Phi \phi_{x,t}.
\]

**Corollary 2.7.** If \( u_h(x,t) = \psi \phi_{x,t} \), then

\[
u_h(x,t) = \psi \Phi \psi \phi_{x,t}, \quad k \in \mathbb{N},
\]

where \( \tilde{U} = \Phi \psi \phi_{x,t} \) and \( U \) is an upper triangular matrix with elements

\[
U_q = \begin{pmatrix} \sum_{j=0}^{N+1} u \phi_{x,t} \phi_{x,t}, i \leq j, i = 0, 1, \ldots, N+1 \end{pmatrix},
\]

\[
i \neq j.
\]

**Corollary 2.8.** If \( u_h(x,t) = \psi \phi_{x,t} \), then

(a) \( \phi \psi \psi \phi_{x,t} = \phi \psi \psi \phi_{x,t} \psi \phi_{x,t} \psi \phi_{x,t} \), for \( p, q \in \mathbb{N} \), where

\[
M_p = \begin{pmatrix} \psi \psi \phi_{x,t} & \psi \phi_{x,t} & \cdots & \psi \phi_{x,t} \\ 0 & \psi \psi \phi_{x,t} & \cdots & \psi \phi_{x,t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi \psi \phi_{x,t} \end{pmatrix},
\]

(b) \( \phi \psi \phi_{x,t} = \phi \psi \phi_{x,t} \psi \phi_{x,t} \), for \( q, s \in \mathbb{N} \), where

\[
N_q = \begin{pmatrix} \phi \phi \psi \psi \phi_{x,t} & \phi \phi \psi \psi \phi_{x,t} & \cdots & \phi \phi \psi \psi \phi_{x,t} \\ 0 & \phi \phi \psi \psi \phi_{x,t} & \cdots & \phi \phi \psi \psi \phi_{x,t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi \phi \psi \psi \phi_{x,t} \end{pmatrix},
\]

**Corollary 2.9.** If \( u_h(x,t) = \psi \phi_{x,t} \), then

(a) \( u_h(x,t) = \psi \phi_{x,t} \), \( k, p \in \mathbb{N}; \)

(b) \( u_h(x,t) = \psi \phi_{x,t} \), \( k, q \in \mathbb{N} \),

where \( \tilde{U} = \Phi \psi \phi_{x,t} \phi_{x,t} \), \( \tilde{M} = \phi \phi \psi \psi \phi_{x,t} \phi_{x,t} \) and \( \tilde{N} = \phi \phi \psi \psi \phi_{x,t} \phi_{x,t} \).

To convert Eq. (1) to a matrix form, we assume that the right-hand side of Eq. (1) has the following form
where \( f = (f_0, f_1, \ldots, f_{(N+1)^2-1}) \).

Using the above results, we have provided all requirements to convert Eq. (1) and supplementary conditions (2)–(4) to the corresponding matrix representation.

For simplicity, we write Eq. (1) in the operator form

\[ Du(x, t) = f(x, t), \] (11)

where

\[ D = \rho c_p \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \left( k \frac{\partial}{\partial x} \right), \] (12)

is a differential operator.

Replacing the approximate solution (9) in 2,3,4 and (11) and using the above lemmas and theorem gives

\[
\begin{align*}
\hat{\eta}_D &= f, \\
\hat{B}_1 &= \varphi_1, \\
\hat{B}_2 &= \varphi_2, \\
\hat{I} &= \varphi,
\end{align*}
\]

Example 4.1. Consider the following nonlinear transient heat conduction equation

\[
\frac{\partial}{\partial t} \left( k(u) \frac{\partial u}{\partial x} \right) + f(x, t) = \frac{\partial u}{\partial t}, \quad x \in [0, 1], \quad t > 0,
\]

where

\( k(u) = 1 + u^2 \)

and

\[ f(x, t) = 1 - 2x - 2t, \]

with the supplementary conditions

\[ u(0, t) = 1, \quad u(1, t) = 1 + t, \quad u(x, 0) = x. \]

The exact solution of this problem is \( u(x, t) = x + t. \) For \( N = 2 \) and using the standard basis, the proposed method gives the nonlinear system

\[
\begin{align*}
u_0 &= u_z = u_6 = 0, \\
u_1 &= u_3 = 1, \\
u_4 + u_7 &= 0, \\
u_5 + u_8 &= 0,
\end{align*}
\]

\[
\begin{align*}
u_0(2u_0u_7 + 2u_1u_6 + 4u_3u_4) + u_1(2u_0u_8 + 2u_1u_6 + 2u_7u_5) - 2u_2 + 2u_7 &= 2, \\
u_0(2u_0u_7 + 2u_1u_6 + 2u_2u_6 + 4u_3u_4 + 2u_7u_5) + u_1(2u_0u_8 + 2u_1u_6 + 4u_3u_4) + u_2(2u_0u_7 + 2u_1u_6 + 2u_7u_5) + 2u_8 &= 0.
\end{align*}
\]
with the solution \( \{u_i = u_1 = 1, u_i = 0 \text{ for } i \neq 1,3\} \) which leads to the exact solution. Indeed, as mentioned in Ortiz and Samara (1981), the operational Tau method for equations with polynomial solution is exact, whenever the degree of the Tau approximation is at least equal to the degree of solution.

**Example 4.2.** Consider a plane wall having variable thermo-physical properties (thermal conductivity, specific heat and density) in which its surface temperatures are related to the time. We assume that a heat sink or heat source is presented in the wall in which the magnitude of released or dissipated energy is a nonlinear function of space and temperature (for example chemical reaction or electrical resistance). Also, the ambient temperature is zero. The following nonlinear heat conduction equation can be obtained using the first law of thermodynamics

\[
\frac{\partial}{\partial x} \left( k(x) \frac{\partial u}{\partial x} \right) + f(x,t) = u \frac{\partial u}{\partial t}, \quad x \in [0, 1], \quad t > 0.
\]

In a special case we assume that thermo-physical properties and sum of heat flux are

\[
k(u) = 1 + u + \frac{1}{2} u^2,
\]

and

\[
f(x,t) = -\sin^2 t(1 + x \sin t - x^3 \cos t).
\]

The supplementary conditions are assumed to be of the Dirichlet kind, namely

\[
u(0, t) = 0, \quad u(1, t) = \sin t, \quad u(x, 0) = 0.
\]

The exact solution of this problem is \( u(x, t) = x \sin t \).

Table 1 shows the absolute errors with respect to shifted Chebyshev basis functions at the selected points for various choices of \( N \).

**Example 4.3.** Consider the nonlinear transient heat conduction equation of the form

\[
\frac{\partial}{\partial x} \left( k(u) \frac{\partial u}{\partial x} \right) + f(x,t) = u \frac{\partial u}{\partial t}, \quad x \in [0, 1], \quad t > 0.
\]

with supplementary conditions

\[
u(0, t) = 0, \quad u(1, t) + \frac{\partial u(1, t)}{\partial x} = 2e^t, \quad u(x, 0) = x,
\]

where thermo-physical properties and sum of heat flux are

\[
k(u) = 1 + u^2,
\]

### Table 1 Absolute errors of Example 4.2.

<table>
<thead>
<tr>
<th>(x,t)</th>
<th>N = 4</th>
<th>N = 8</th>
<th>N = 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1,0.1)</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>(0.2,0.3)</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>(0.4,0.3)</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>(0.6,0.8)</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>(1,1)</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>Cpu time (s)</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
</tr>
</tbody>
</table>

The exact solution of the problem is \( u(x, t) = x e^t \).

The absolute and estimate errors for this example with respect to shifted Chebyshev basis for various choices of \( N \), reported in Tables 2 and 3, show the accuracy and efficiency of the proposed method. These results confirm that the absolute and estimate errors are in good agreement.

To check the stability of the proposed method, we perturb the coefficients of approximate solution by \( \varepsilon = 10^{-5}, 10^{-3}, 10^{-5} \) and \( 10^{-7} \). Then, we solve the perturbed problem by the method and find out that there are no total changes in the final results. Table 4 shows the maximum absolute errors of the perturbed problem with respect to shifted Chebyshev basis for various choices of \( N \).

### Table 2 Absolute errors of Example 4.3.

<table>
<thead>
<tr>
<th>(x,t)</th>
<th>N = 4</th>
<th>N = 8</th>
<th>N = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1,0.1)</td>
<td>8.72</td>
<td>8.72</td>
<td>8.72</td>
</tr>
<tr>
<td>(0.2,0.3)</td>
<td>8.72</td>
<td>8.72</td>
<td>8.72</td>
</tr>
<tr>
<td>(0.4,0.3)</td>
<td>8.72</td>
<td>8.72</td>
<td>8.72</td>
</tr>
<tr>
<td>(0.6,0.8)</td>
<td>8.72</td>
<td>8.72</td>
<td>8.72</td>
</tr>
<tr>
<td>(1,1)</td>
<td>8.72</td>
<td>8.72</td>
<td>8.72</td>
</tr>
<tr>
<td>Cpu time (s)</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
</tr>
</tbody>
</table>

The absolute and estimate errors for this example with respect to shifted Chebyshev basis for various choices of \( N \), reported in Tables 2 and 3, show the accuracy and efficiency of the proposed method. These results confirm that the absolute and estimate errors are in good agreement.

To check the stability of the proposed method, we perturb the coefficients of approximate solution by \( \varepsilon = 10^{-5}, 10^{-3}, 10^{-5} \) and \( 10^{-7} \). Then, we solve the perturbed problem by the method and find out that there are no total changes in the final results. Table 4 shows the maximum absolute errors of the perturbed problem with respect to shifted Chebyshev basis for various choices of \( N \).

### Table 3 Estimate errors of Example 4.3.

<table>
<thead>
<tr>
<th>(x,t)</th>
<th>N = 4</th>
<th>N = 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1,0.1)</td>
<td>7.52</td>
<td>7.52</td>
</tr>
<tr>
<td>(0.2,0.3)</td>
<td>5.47</td>
<td>5.47</td>
</tr>
<tr>
<td>(0.4,0.4)</td>
<td>3.65</td>
<td>3.65</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>1.41</td>
<td>1.41</td>
</tr>
<tr>
<td>(0.6,0.6)</td>
<td>4.31</td>
<td>4.31</td>
</tr>
<tr>
<td>(0.7,0.7)</td>
<td>1.10</td>
<td>1.10</td>
</tr>
<tr>
<td>(0.8,0.8)</td>
<td>2.51</td>
<td>2.51</td>
</tr>
<tr>
<td>(0.9,0.9)</td>
<td>5.18</td>
<td>5.18</td>
</tr>
<tr>
<td>(1,1)</td>
<td>9.94</td>
<td>9.94</td>
</tr>
<tr>
<td>Cpu time (s)</td>
<td>3.15</td>
<td>3.15</td>
</tr>
</tbody>
</table>

### Table 4 Maximum absolute errors of perturb problem of Example 4.3.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>N = 4</th>
<th>N = 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-3} )</td>
<td>3.32</td>
<td>3.32</td>
</tr>
<tr>
<td>( 10^{-5} )</td>
<td>3.21</td>
<td>3.21</td>
</tr>
<tr>
<td>( 10^{-7} )</td>
<td>3.14</td>
<td>3.14</td>
</tr>
</tbody>
</table>

5. Conclusion

In this work, a computational method based on the operational Tau method is present for solving 1-D nonlinear...
transient heat conduction equations by converting it and necessary supplementary conditions to a nonlinear system of equations. Our results indicate the proposed algorithm can be regarded as a structurally simple algorithm and high superior performance that is conventionally applicable to the numerical solution of these type of equations. The accuracy of the method is improved as the degree of approximation is increased.

Acknowledgment

The authors would like to thank the reviewers for their relevant and useful comments that improved the structure of this paper.

References


