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Some properties of log-convex function and applications for the exponential function

Xiaoming Zhang [a,∗], Weidong Jiang [b]

[a] Zhejiang Broadcast & Radio University Haining College, Zhejiang 314400, PR China
[b] Department of Information Engineering, Weihai Vocational College, Weihai, Shandong province, 264200, PR China

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ABSTRACT

In this paper, some properties of log-convex function are researched, and integral inequalities of log-convex functions are proved. As an application, an estimation formula of remainder terms in Taylor series expansion is given.

1. Introduction

Throughout the paper we assume that $\mathbb{R}$, $\mathbb{R}_{++}$ and $\mathbb{N}$ respectively stands for real number set, positive real number set and natural number set.

Recall that the definition of a log-convex function.

Definition 1. Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{++}$. Then f is called a log-convex(concave) function, if

$$f(\alpha x + (1-\alpha)y) \leq (\geq)(f(x))^{\alpha}(f(y))^{1-\alpha}$$

holds for any $x, y \in [a, b]$, $\alpha \in [0, 1]$.

The authors of [1] proved the following results on the log-convex functions.

Theorem 1. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{++}$ be log-convex (concave), denote*

$$M = \begin{cases} \dfrac{(b-a)(f(b)-f(a))}{\ln f(b) - \ln f(a)}, & \text{if } f(a) \neq f(b); \\ (b-a)f(a), & \text{if } f(a) = f(b). \end{cases}$$

Then

$$\int_a^b f(t)dt \leq (\geq)M.$$

∗ Corresponding author. Tel.: +86 13957359895.
E-mail addresses: zjzxm79@126.com (X. Zhang), jackjwd@163.com (W. Jiang).

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The aim of this paper is to show some results on the log-convex functions. In Section 2, we give some integral properties of the log-convex function, including a lower bound of its integral inequality. As an application, in Section 3, an estimation formula of remainder terms in Taylor series expansion of $e^x (x > 0)$ is given. This result is better than the result in [2,3] or [4].

2. Some properties of the log-convex function

Theorem 2. (i) *Let $f : [a, b] \to [0, +\infty)$ be a strictly decreasing and differentiable function, $f(x) > 0$ for $x \in (a, b]$. Define $F(x) = \int_a^x f(t)dt$ with $x \in (a, b]$. Then F is a log-concave function.*

(ii) *Let $f : [a, b] \to [0, +\infty)$ be a twice differentiable log-concave function, $f(x) > 0$ and $f'(x) > 0$ for $x \in (a, b]$. Define $F(x) = \int_a^x f(t)dt$ with $x \in (a, b]$. Then F is a log-concave function.*

(iii) *Let $f : [a, b] \to [0, +\infty)$ be a twice differentiable log-convex function, $f(x) > 0$ and $f'(x) > 0$ for $x \in (a, b]$, $\lim_{x\to a+} f^2(x)/f'(x) = 0$. Define $F(x) = \int_a^x f(t)dt$ with $x \in (a, b]$. Then F is log-convex function.*

Proof. We only prove (ii), the other proof is similar.

Let

$$G(x) := \frac{F''(x)F(x) - (F'(x))^2}{f'(x)}, \quad x \in (a, b].$$

Then

$$\begin{aligned} G(x) &= \frac{\left(\int_a^x f(t)dt\right)'' \cdot \int_a^x f(t)dt - f^2(x)}{f'(x)} \\ &= \int_a^x f(t)dt - \frac{f^2(x)}{f'(x)}. \end{aligned}$$

and

$$\begin{aligned} G'(x) &= f(x) - \frac{2f(x)(f'(x))^2 - f^2(x)f''(x)}{(f'(x))^2} \\ &= f(x) \cdot \frac{f(x)f''(x) - (f'(x))^2}{(f'(x))^2} \le 0. \end{aligned}$$

Then G is decreasing. We have

$$G(x) \le \lim_{x\to a+} G(x) = -\lim_{x\to a+} \frac{f^2(x)}{f'(x)} \le 0,$$

and

$$F''(x)F(x) - (F'(x))^2 \le 0.$$

The proof of (ii) is completed. □

Theorem 3. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{++}$ be log-convex (concave), $c \in (a, b)$, $f'_-(c) \neq 0$ and $f'_+(c) \neq 0$. Then*

$$\int_a^b f(t)dt \ge (\le) \frac{(f(c))^2}{f'_-(c)} \left[1 - \exp\left(-(c-a)\frac{f'_-(c)}{f(c)}\right)\right] + \frac{(f(c))^2}{f'_+(c)} \left[\exp\left((b-c)\frac{f'_+(c)}{f(c)}\right) - 1\right], \tag{1}$$

the equality holds if and only if f is a pe^{qx}-type function, where $p > 0$, $q \in \mathbb{R}$.

Proof. Since $f'_+(c) \neq 0$, we can choose $d \in (c, b)$ such that $f(d) \neq f(c)$. For any $t \in (d, b)$ and $\alpha = (d-c)/(t-c)$, $d = (1-\alpha)c + \alpha t$ hold. Then

$$\begin{aligned} &f(d) = f((1-\alpha)c + \alpha t) \le (\ge) (f(c))^{1-\alpha}(f(t))^{\alpha}, \\ &f(t) \ge (\le) (f(d))^{(t-c)/(d-c)}(f(c))^{-(t-d)/(d-c)}. \end{aligned} \tag{2}$$

Therefore,

$$\begin{aligned} \int_d^b f(t)dt &\ge (\le) \int_d^b (f(d))^{(t-c)/(d-c)} \cdot (f(c))^{-(t-d)/(d-c)} dt \\ &= \frac{(f(c))^{d/(d-c)}}{(f(d))^{c/(d-c)}} \int_d^b \left(\frac{f(d)}{f(c)}\right)^{t/(d-c)} dt \end{aligned}$$

$$= \frac{(d-c)(f(c))^{d/(d-c)}}{(f(d))^{c/(d-c)}(\log f(d)-\log f(c))}\left[\left(\frac{f(d)}{f(c)}\right)^{b/(d-c)}-\left(\frac{f(d)}{f(c)}\right)^{d/(d-c)}\right]$$

$$= \frac{(d-c)f(d)\left[\left(\frac{f(d)}{f(c)}\right)^{(b-d)/(d-c)}-1\right]}{\log f(d)-\log f(c)}.$$

Let $d \to c+$, we have

$$\int_c^b f(t)dt \geq (\leq) \lim_{d\to c+} \frac{f(d)\left[\exp\{(b-d)(\log f(d)-\log f(c))/(d-c)\}-1\right]}{(\log f(d)-\log f(c))/(d-c)}$$

$$= f(c)\lim_{d\to c+}\frac{\left[\exp\left\{(b-c)\frac{\log f(d)-\log f(c)}{f(d)-f(c)}\cdot\frac{f(d)-f(c)}{d-c}\right\}-1\right]}{\frac{\log f(d)-\log f(c)}{f(d)-f(c)}\cdot\frac{f(d)-f(c)}{d-c}}$$

$$= \frac{(f(c))^2}{f'_+(c)}\left\{\exp\left((b-c)\frac{f'_+(c)}{f(c)}\right)-1\right\}.$$

Similarly, we get

$$\int_a^c f(t)dt \geq (\leq)\frac{(f(c))^2}{f'_-(c)}\left(1-\exp\left\{-(c-a)\frac{f'(c)}{f(c)}\right\}\right).$$

Hence

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

$$\geq (\leq)\frac{(f(c))^2}{f'_-(c)}\left[1-\exp\left(-(c-a)\frac{f'(c)}{f(c)}\right)\right]+\frac{(f(c))^2}{f'_+(c)}\left[\exp\left((b-c)\frac{f'_+(c)}{f(c)}\right)-1\right].$$

Because of (2), the above equality holds if and only if

$$f(\alpha x+(1-\alpha)y) = (f(x))^\alpha\cdot(f(y))^{1-\alpha}$$

holds for any $x, y \in [a, b]$, $\alpha \in (0, 1)$. Then there exists $q, m \in \mathbb{R}$, such that

$$\log f(x) = qx+m, \qquad f(x) = e^m(e^q)^x.$$

The proof of Theorem 3 is completed. □

Corollary 1. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{++}$ be a log-convex(concave) function, $c \in [a, b]$ and $f'(c) \neq 0$. Then*

$$\int_a^b f(t)dt \geq (\leq)\frac{(f(c))^2}{f'(c)}\left[\exp\left((b-c)\frac{f'(c)}{f(c)}\right)-\exp\left(-(c-a)\frac{f'(c)}{f(c)}\right)\right].$$

Particularly, if $f'(a) \neq 0, f'(b) \neq 0$ or $f'((a+b)/2) \neq 0$, we have

$$\int_a^b f(t)dt \geq (\leq)\frac{(f(a))^2}{f'(a)}\left[\exp\left((b-a)\frac{f'(a)}{f(a)}\right)-1\right],$$

$$\int_a^b f(t)dt \geq (\leq)\frac{(f(b))^2}{f'(b)}\left[1-\exp\left(-(b-a)\frac{f'(b)}{f(b)}\right)\right], \tag{3}$$

or

$$\int_a^b f(t)dt \geq (\leq)\frac{\left(f\left(\frac{a+b}{2}\right)\right)^2}{f'\left(\frac{a+b}{2}\right)}\left[\exp\left(\frac{b-a}{2}\cdot\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right)-\exp\left(-\frac{b-a}{2}\cdot\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right)\right],$$

with equality holding if and only if f is a pe^{qx}-type function, where $p > 0$, $q \in \mathbb{R}$.

3. Some applications

In this section, we firstly introduce the Definition 2 and some lemmas as follows.

Definition 2 (*[5–7]*)**.** Let interval $I \subseteq \mathbb{R}_{++}, f : I \to \mathbb{R}_{++}$. Then f is called a geometrically convex function, if

$$f(x^\alpha y^{1-\alpha}) \le (f(x))^\alpha (f(y))^{1-\alpha}$$

holds for any $x, y \in I, \alpha \in [0, 1]$.

Lemma 1. *Let interval $I \subseteq \mathbb{R}_{++}, f : I \to \mathbb{R}_{++}$ be a differentiable function, then the following assertions are equivalent:*

(i) *f is geometrically convex.*
(ii) *The function $xf'(x)/f(x)$ is increasing.*

If moreover f is twice differentiable, then f is geometrically convex if and only if

$$x[f(x)f''(x) - (f'(x))^2] + f(x)f'(x) \ge 0 \tag{4}$$

holds for any $x \in I$.

Lemma 2 (*[6,7]*)**.** *Let $f : [0, a) \to [0, \infty)$ be a continuous function, which is geometrically convex on $(0, a)$. Then $F(x) = \int_0^x f(t)dt$ is also geometrically convex on $(0, a)$.*

Recall equality $e^x = \sum_{i=0}^{+\infty} x^i/i!$, let $f_n(x) = e^x - \sum_{i=0}^{n} x^i/i!$, $n = 0, 1, 2, \ldots, x > 0$. According to [2,3] or [4],

$$f_n(x) \le \frac{x^{n+1}}{n!(n+1-x)} \tag{5}$$

with $n + 1 - x > 0$. We can improve the result as the following Theorem 4. In Ref. [8], Alzer proves

$$\frac{f_{n-1}(x)f_{n+1}(x)}{(f_n(x))^2} > \frac{n+1}{n+2}.$$

We get the upper bound of $f_{n-1}(x)f_{n+1}(x)/(f_n(x))^2$ in the the following Theorem 5.

Theorem 4. *Let $f_n(x) = e^x - \sum_{i=0}^{n} x^i/i!$, $n = 0, 1, 2, \ldots, x > 0$. Then*

$$f_n(x) < \frac{2}{\sqrt{(n+1-x)^2 + 4x(n+1)e^{-x-n-1}} + (n+1-x)} \cdot \frac{x^{n+1}}{n!}. \tag{6}$$

If $0 < x \le c$, then

$$f_n(x) \le \frac{2}{\sqrt{(n+1-x)^2 + 4x(n+1)\exp\left(-\frac{cf_{n-1}(c)}{f_n(c)}\right)} + (n+1-x)} \cdot \frac{x^{n+1}}{n!}. \tag{7}$$

Proof. It is easy to see that $f_{-1} : x \in \mathbb{R}_{++} \to e^x$ satisfies the conditions of (ii) in Theorem 2, where $a = 0$. So $f_0(x) = \int_0^x f_{-1}(t)dt = e^x - 1$ $(x \in \mathbb{R}_{++})$ is the log-concave function, and it is easily proved to satisfy the conditions of (ii) in Theorem 2. Repeating this, we find $f_n(x) = \int_0^x f_{n-1}(t)dt$ $(x \in \mathbb{R}_{++})$ are the log-concave functions.

Let $b = x, a = 0, f(x) = f_n(x), n \ge 0$ in (3), we have

$$\int_0^x f_n(t)dt \le \frac{(f_n(x))^2}{f_n'(x)}\left[1 - \exp\left(-x\frac{f_n'(x)}{f_n(x)}\right)\right],$$

$$f_{n+1}(x) \le \frac{(f_n(x))^2}{f_{n-1}(x)}\left[1 - \exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right)\right]. \tag{8}$$

Further,

$$f_{n+1}(x)f_{n-1}(x) \le (f_n(x))^2\left[1 - \exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right)\right],$$

$$\left(f_n(x) - \frac{x^{n+1}}{(n+1)!}\right)\left(f_n(x) + \frac{x^n}{n!}\right) \le (f_n(x))^2\left[1 - \exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right)\right],$$

$$\exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right) \cdot (f_n(x))^2 + \left(\frac{x^n}{n!} - \frac{x^{n+1}}{(n+1)!}\right)f_n(x) - \frac{x^{2n+1}}{n!(n+1)!} \le 0.$$

Then

$$f_n(x) \le \frac{-\left(\frac{x^n}{n!}-\frac{x^{n+1}}{(n+1)!}\right)+\sqrt{\left(\frac{x^n}{n!}-\frac{x^{n+1}}{(n+1)!}\right)^2+4\exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right)\cdot\frac{x^{2n+1}}{n!(n+1)!}}}{2\exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right)},$$

$$\le \frac{\frac{2x^{2n+1}}{n!(n+1)!}}{\sqrt{\left(\frac{x^n}{n!}-\frac{x^{n+1}}{(n+1)!}\right)^2+4\exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right)\cdot\frac{x^{2n+1}}{n!(n+1)!}}+\left(\frac{x^n}{n!}-\frac{x^{n+1}}{(n+1)!}\right)},$$

$$\le \frac{2x^{n+1}}{n!\sqrt{(n+1-x)^2+4x(n+1)\exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right)}+n!(n+1-x)}. \tag{9}$$

On the other hand, taking into account that $e^x = \sum_{i=0}^{+\infty} x^i/i!$, we find

$$\frac{xf_{n-1}(x)}{f_n(x)} = \frac{x\left(e^x-\sum_{i=0}^{n-1}\frac{x^i}{i!}\right)}{e^x-\sum_{i=0}^{n}\frac{x^i}{i!}} = x+\frac{x\cdot\frac{x^n}{n!}}{e^x-\sum_{i=0}^{n}\frac{x^i}{i!}}$$

$$< x+\frac{x\cdot\frac{x^n}{n!}}{\frac{x^{n+1}}{(n+1)!}} = x+n+1. \tag{10}$$

Taking (10) into (9), we complete the proof of (6).

According to (4), f_{-1} is a geometrically convex function. By Lemma 2, $f_0(x) = \int_0^x f_{-1}(t)dt = e^x - 1$ is a geometrically convex function. Repeating this, $f_n(x) = \int_0^x f_{n-1}(t)dt$ are a geometrically convex. Owing to (ii) in Lemma 1, we know $xf_n'(x)/f_n(x) = xf_{n-1}(x)/f_n(x)$ is increasing for $x \in \mathbb{R}_{++}$. If $0 < x < c$, then $xf_{n-1}(x)/f_n(x) \le cf_{n-1}(c)/f_n(c)$. According to (9), we have

$$f_n(x) \le \frac{2x^{n+1}}{n!\sqrt{(n+1-x)^2+4x(n+1)\exp\left(-\frac{cf_{n-1}(c)}{f_n(c)}\right)}+n!(n+1-x)}.$$

This completes the proof of the (7). □

Remark 1. (i) It is clear that (6) and (7) are better than (5).
(ii) For the $f_n(x)$ in Theorem 4, according to $e^x = \sum_{i=0}^{+\infty} x^i/i!$, we know that $x^{n+1}/(n+1)!$ is the lower bound of $f_n(x)$.

Corollary 2. *Let $n \ge 1, n \in \mathbb{N}$*

$$e-\sum_{i=0}^{n}\frac{1}{i!} < \frac{2}{\sqrt{n^2+4(n+1)e^{-n-2}}+n}\cdot\frac{1}{n!} < \frac{1}{n\cdot n!}.$$

Let $x = 1$ in (6), we easily prove Corollary 2.

Owing to inequalities (8) and (10), the following Theorem 5 holds.

Theorem 5. *Let $f_n(x) = e^x - \sum_{i=0}^{n} x^i/i!$, $n = 0, 1, 2, \ldots, x > 0$. Then*

$$\frac{f_{n-1}(x)f_{n+1}(x)}{(f_n(x))^2} \le 1-\exp\left(-\frac{xf_{n-1}(x)}{f_n(x)}\right) < 1-\exp(-x-n-1).$$

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