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A new class of completely generalized quasi-variational inclusions in Banach spaces ☆

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Abstract

In this paper, a new notion of J-proximal mapping for a nonconvex lower semicontinuous subdifferentiable proper functional on Banach space is introduced. The existence and Lipschitz continuity of J-proximal mapping of a lower semicontinuous subdifferentiable proper functional are proved. By applying the concept, we introduce and study a new class of completely generalized quasi-variational inclusions in reflexive Banach spaces. A novel and innovative iterative algorithm for finding the approximate solutions is suggested and analyzed. The convergence criteria of the iterative sequences generated by the new iterative algorithm is also given. These algorithm and existence result generalize many known results under Hilbert space setting in recent literature to reflexive Banach spaces. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Completely generalized quasi-variational inclusion; Subdifferentiable; J-proximal mapping; Iterative algorithm; Reflexive Banach space

1. Introduction

In recent years, variational inequality theory has been become very effective and powerful tools for studying a wide class of nonlinear problems arising in many diverse fields of pure mathematics and applied sciences, such as mathematical programming, optimization theory, engineering, elasticity theory, and equilibrium theory of mathematical economy and game theory etc, for example, see ([1–19,21–28,30–38]). Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques both for own sake and for its applications. Some useful and important generalizations of variational inequalities are the generalized set-valued mixed variational inequalities and generalized quasi-variational inclusions including the nonlinear term.

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One of the most interesting and important problems in the variational inequality theory is the development of an efficient iterative algorithm to compute approximate solutions.

Under Hilbert space setting, there is a substantial number of iterative algorithms for finding the approximate solutions of various variational inequalities, see ([1–4,8–10,12,14–19,21–28,30–38]). The most effective numerical technique is the projection method and its various variants, auxiliary principle technique, Newton and descent framework. The applicability of the projection method is limited due to the fact that it is not easy to find the projection except in a very special case and it strictly depends on the inner product property of Hilbert spaces. Due to the presence of nonlinear term, the project method cannot be applied to suggest any iterative algorithms for generalized mixed variational inequalities and generalized quasi-variational inclusions in Banach spaces.

Recently, Cohen [7] and Ding [11,13] have extended the auxiliary principle technique to suggest and analyse an innovative and novel iterative algorithm for computing the solution of mixed variational inequalities in reflexive Banach space. Chang et al. [6] and Chang [5] have studied some classes of set-valued variational inclusions with m-accretive operator and ϕ-strongly accretive operators in uniformly smooth Banach spaces.

Motivated and inspired by the research work going on in this field, in this paper, we first introduce a new notion of J-proximal mapping for a lower semicontinuous subdifferentiable proper (may not be convex) functional on Banach spaces. Then, the existence and Lipschitz continuity of the J-proximal mapping of the functional are proved under suitable conditions in reflexive Banach spaces. By using the new concept and a similar technique of resolvent operator in Hilbert spaces, we introduce and study a new class of completely generalized quasi-variational inclusions with nonconvex functional in reflexive Banach spaces. A novel and innovative iterative algorithm for finding the approximate solutions is suggested and analysed. The convergence of the iterative sequences generated by the new algorithm is also discussed. The new algorithm and existence result generalize many known results under Hilbert space setting in recent literature to reflexive Banach spaces. We emphasize that there are not any monotonicity and accretive assumptions on the set-valued mappings in our theorem. Our result and method are new and different from those in the known literature.

2. Preliminaries

Let E be a Banach space with the dual space E^*, $\langle u,x\rangle$ be the dual pairing between $u\in E^*$ and $x\in E$ and $\mathrm{CB}(E^*)$ be the family of all nonempty closed bounded subset of E^*. $H(\cdot,\cdot)$ is the Hausdorff metric on $\mathrm{CB}(E^*)$ defined by

$$H(A,B)=\max\left\{\sup_{u\in A} d(u,B),\sup_{v\in B} d(A,v)\right\},\quad \forall A,B\in \mathrm{CB}(E^*),$$

where $d(u,B)=\inf_{v\in B} d(u,v)$ and $d(A,v)=\inf_{u\in A} d(u,v)$.

Definition 2.1. Let $A:E\to \mathrm{CB}(E^*)$ be a set-valued mapping, $J:E\to E^*$ and $g:E\to E$ be two single-valued mappings.

(1) A is said to be λ_A-Lipschitz continuous with Lipschitz constant $\lambda_A\geqslant 0$ if

$$H(Ax,Ay)\leqslant \lambda_A\|x-y\|,\quad \forall x,y\in E.$$

(2) J is said to be α-strongly monotone ($\alpha > 0$) if

$$\langle Jx - Jy, x - y \rangle \geqslant \alpha \|x - y\|^2, \quad \forall x, y \in E.$$

(3) g is said to be k-strongly accretive ($k \in (0,1)$) if for any $x, y \in E$, there exists $j(x-y) \in \mathscr{F}(x-y)$ such that

$$\langle j(x - y), gx - gy \rangle \geqslant k\|x - y\|^2,$$

where $\mathscr{F} : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$\mathscr{F}(x) = \{ f \in E^*: \langle f, x \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\| \}, \quad \forall x \in E.$$

Definition 2.2. Let $\varphi : E \to \mathbf{R} \cup \{+\infty\}$ be a proper functional. φ is said to be subdifferential at a point $x \in E$ if there exists a point $f^* \in E^*$ such that

$$\varphi(y) - \varphi(x) \geqslant \langle f^*, y - x \rangle, \quad \forall y \in E,$$

where f^* is called a subgradient of φ at x. The set of all subgradient of φ at x is denoted by $\partial\varphi(x)$. The mapping $\partial\varphi : E \to 2^{E^*}$ defined by

$$\partial\varphi(x) = \{ f^* \in E^*: \varphi(y) - \varphi(x) \geqslant \langle f^*, y - x \rangle, \ \forall y \in E \}$$

is said to be subdifferential of φ at x.

Definition 2.3. Let E be a Banach space with the dual space E^*, $\varphi : E \to \mathbf{R} \cup \{+\infty\}$ be a proper subdifferentiable (may not convex) functional and $J : E \to E^*$ be a mapping. If for any given point $x^* \in E^*$ and $\rho > 0$, there is a unique point $x \in E$ satisfying

$$\langle Jx - x^*, y - x \rangle + \rho\varphi(y) - \rho\varphi(x) \geqslant 0, \quad \forall y \in E. \tag{2.1}$$

The mapping $x^* \to x$, denoted by $J_\rho^{\partial\varphi}(x^*)$, is said to be J-proximal mapping of φ. We have $x^* - Jx \in \rho\partial\varphi(x)$, it follows that $J_\rho^{\partial\varphi}(x^*) = (J + \rho\partial\varphi)^{-1}(x^*)$.

Remark 2.1. If E is Hilbert space, φ is a convex lower semicontinuous proper functional on E and J is the identity mapping on E, then the J-proximal mapping of φ reduces to the resolvent operator of φ on Hilbert space.

Let $A, B : E \to \mathrm{CB}(E^*)$ and $C, D, F : E \to \mathrm{CB}(E)$ be set-valued mappings. Let $N : E^* \times E^* \to E^*$, $f : E \to E^*$ and $g, m : E \to E$ be single-valued mappings. Let $\varphi : E \times E \to \mathbf{R} \cup \{+\infty\}$ be such that for each fixed $z \in E$, $\varphi(\cdot, z)$ is a lower semicontinuous subdifferentiable (may not be convex) proper functional on E satisfying $g(x) - m(y) \in \mathrm{dom}(\partial\varphi(\cdot, z))$ for all $x, y \in E$ where $\partial\varphi(\cdot, z)$ is the subdifferential of $\varphi(\cdot, z)$. We consider the following completely generalized quasi-variational inclusion (CGQVI): find $x \in E$, $u \in Ax$, $v \in Bx$, $w \in C(x)$, $z \in D(x)$ and $y \in F(x)$ such that

$$\langle f(w) - N(u, v), h - (g(x) - m(y)) \rangle \geqslant \varphi(g(x) - m(y), z) - \varphi(h, z), \quad \forall h \in E. \tag{2.2}$$

Special cases:

(I) If $C(x)=\{x\}$ and $f(x)=-f^*$ for all $x\in E$ where $f^*\in E^*$ is a given element, and $N_1(u,v)=-N(u,v)$ for all $u,v\in E^*$, then the CGQVI (2.2) reduces to the following generalized set-valued quasi-variational inclusion (GSVQVI): find $x\in E$, $u\in A(x)$, $v\in B(x)$ $z\in D(x)$ and $y\in F(x)$ such that

$$\langle N_1(u,v)-f^*,h-(g(x)-m(y))\rangle \geqslant \varphi(g(x)-m(y),z)-\varphi(h,z), \quad \forall h\in E. \tag{2.3}$$

The GSVQVI (2.3) is new which, includes the generalized set-valued mixed variational inequalities introduced and studied in [32,33,35,36] and many authors under Hilbert space setting and in [5,6] under Banach space setting as very special cases. Most classes of quasi-variational inclusions and generalized quasi-variational inequalities studied by many authors in Hilbert spaces are all very spacial cases of the GSVQVI (2.3), see [1,2,4–18,21–28,30–36].

(II) If $f^*=0$, $D(x)=\{x\}$ for all $x\in E$ and $\varphi(x,z)=\varphi(x)$ for all $x,z\in E$, the GSVQVI (2.3) reduces to the following generalized mixed implicit quasi-variational inequalities (GMIQVI): find $x\in E$, $u\in A(x)$, $v\in B(x)$ and $y\in F(x)$ such that

$$\langle N(u,v),h-(g(x)-m(y))\rangle \geqslant \varphi(g(x)-m(y))-\varphi(h), \ \forall h\in H. \tag{2.4}$$

(III) If $E=H$ is a Hilbert space, $F(x)=\{x\}$ and $m(x)=0$ for all $x\in H$ and $N(u,v)=u+G(v)$ for all $u,v\in H$ where $G:H\to H$ is a given single-valued mapping, then the GMIQVI (2.4) reduces to the following set-valued mixed variational inequality (SVMVI): find $x\in H$, $u\in A(x)$ and $v\in B(x)$ such that

$$\langle u+G(v),h-g(x)\rangle+\varphi(h)-\varphi(g(x))\geqslant 0, \ \forall h\in H. \tag{2.5}$$

The SVMVI (2.5) and its special cases were introduced and studied in [32,33] and many other authors. The problems has many important and significant applications in pure and applied sciences, see, for example, [8,9,21,27,35,36].

(IV) If $E=H$ is a Hilbert space, $K:H\to 2^H$ such that $K(x)$ is a nonempty closed convex subset of H for each $x\in H$, and

$$\varphi(x)=I_{K(x)}=\begin{cases}0, & \text{if } x\in K(x),\\ +\infty, & \text{if } x\notin K(x),\end{cases}$$

then the GMIQVl (2,4) reduces to the following general generalized quasi-variational inequalities (GGQVI): find $x\in H$, $u\in A(x)$, $v\in B(x)$ and $y\in F(x)$ such that

$$g(x)\in m(y)+K(x), \quad \langle N(u,v),\ h-(g(x)-m(y))\rangle\geqslant 0, \quad \forall h\in K(x). \tag{2.6}$$

(V) If $F(x)=\{x\}$ and $m(x)=0$ for all $x\in X$, then the GGQVI (2.6) reduces to the following generalized quasi-variational inequality (GQVI): find $x\in H$, $u\in A(x)$ and $v\in B(x)$ such that

$$g(x)\in K(x) \quad \text{and} \quad \langle N(u,v),h-g(x)\rangle\geqslant 0, \ \forall h\in K(x). \tag{2.7}$$

In brief, for appropriate and suitable choices of $N(\cdot,\cdot)$, A, B, C, D, F, g, f, m and the underlying space E, we can obtain many known and new classes of generalized variational inequalities and generalized set-valued quasi-variational inclusions as special cases of the CGQVI (2.2), for example, see [1–19,21–28,30–38]. Furthermore, these types of generalized quasi-variational inclusions can enable us to study many important nonlinear problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional structure, transportation, elasticity and various applied sciences in a general and unified framework.

Lemma 2.1 (Ding and Tan [20]). *Let D be a nonempty convex subset of a topological vector space and $f : D \times D \to \mathbf{R} \cup \{\pm\infty\}$ be such that*

(i) *for each $x \in D$, $y \to f(x, y)$ is lower semicontinuous on each compact subset of D,*
(ii) *for each finite set $\{x_1, \ldots, x_m\} \in D$ and for each $y = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \geqslant 0$ and $\sum_{i=1}^m \lambda_i = 1$, $\min_{1 \leqslant i \leqslant m} f(x_i, y) \leqslant 0$,*
(iii) *there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \backslash K$, there is an $x \in \mathrm{co}(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$.*

Then there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leqslant 0$, $\forall x \in D$.

Now we give some sufficient conditions which guarantee the existence and Lipschitz continuity of the J-proximal mapping of a proper functional on reflexive Banach space.

Theorem 2.1. *Let E be a reflexive Banach space with the dual space E^* and $\varphi : E \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional which may not be convex. Let $J : E \to E^*$ be an α-strongly monotone continuous mapping. Then for any $\rho > 0$, and any $x^* \in E^*$, there exists a unique $x \in E$ such that*

$$\langle Jx - x^*, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \geqslant 0, \quad \forall y \in E. \tag{2.3}$$

That is $x = J_\rho^{\partial\varphi}(x^*)$ and so the J-proximal mapping of φ is well defined:

Proof. For any given $J : E \to E^*$, $\rho > 0$ and $x^* \in E^*$, define a functional $f : E \times E \to \mathbf{R} \cup \{+\infty\}$ by

$$f(y,x) = \langle x^* - Jx, y - x\rangle + \rho\varphi(x) - \rho\varphi(y), \quad \forall x, y \in E.$$

Since J is continuous and φ is lower semicontinuous, we have that for any $y \in E$, $x \to f(y,x)$ is lower semicontinuous on E, then $f(y,x)$ satisfies the condition (i) of Lemma 2.1. We claim that $f(y,x)$ satisfies the condition (ii) of Lemma 2.1. If it is false, then there exists a finite set $\{y_1, \ldots, y_m\} \in E$ and $x_0 = \sum_{i=1}^m \lambda_i y_i$ with $\lambda_i \geqslant 0$, $\sum_{i=1}^m \lambda_i = 1$ such that

$$\langle x^* - Jx_0, y_i - x_0\rangle + \rho\varphi(x_0) - \rho\varphi(y_i) > 0, \quad \forall i = 1, \ldots, m.$$

Since φ is subdifferentiable at x_0, there exists a point $f^* \in E^*$ such that

$$\rho\varphi(y_i) - \rho\varphi(x_0) \geqslant \rho\langle f^*, y_i - x_0\rangle, \quad \forall i = 1, \ldots, m.$$

It follows that

$$\langle x^* - Jx_0 - \rho f^*, y_i - x_0\rangle > 0, \quad \forall i = 1, \ldots, m.$$

Then we have

$$0 < \sum_{i=1}^{m} \lambda_i \langle x^* - Jx_0 - \rho f^*, y_i - x_0 \rangle = \langle x^* - Jx_0 - \rho f^*,\ x_0 - x_0 \rangle = 0,$$

which is a contradiction. Hence $f(y,x)$ satisfies the condition (ii) of Lemma 2.1. Now take a fixed $\hat{y} \in \operatorname{dom} \varphi$. Since φ is subdifferentiable at $\hat{y}$, there exists a point $f^* \in E^*$ such that

$$\varphi(x) - \varphi(\hat{y} \geqslant \langle f^*, x - \hat{y} \rangle, \quad \forall x \in E.$$

Hence we have

$$\begin{aligned} f(\hat{y},x) &= \langle x^* - Jx, \hat{y} - x \rangle + \rho\varphi(x) - \rho\varphi(\hat{y}) \\ &\geqslant \langle J\hat{y} - Jx, \hat{y} - x \rangle + \langle x^* - J\hat{y}, \hat{y} - x \rangle + \rho \langle f^*, x - \hat{y} \rangle \\ &\geqslant \alpha \|\hat{y} - x\|^2 - (\|x^*\| + \|J\hat{y}\| + \rho\|f^*\|)\|\hat{y} - x\| \\ &= \|\hat{y} - x\| \ [\alpha\|\hat{y} - x\| - (\|x^*\| + \|J\hat{y}\| + \rho\|f^*\|)]. \end{aligned}$$

Let $r = \frac{1}{\alpha}(\|x^*\| + \|J\hat{y}\| + \rho\|f^*\|)$, $K = \{u \in E\colon \|\hat{y} - u\| \leqslant \tau\}$. Then $D_0 = \{\hat{y}\}$ and K are both weakly compact convex subset of E and for each $x \in E \setminus K$, there exists a $\hat{y} \in \operatorname{co}(\{D_0 \cup \{\hat{y}\})$ such that $f(\hat{y},x) > 0$. Hence all conditions of Lemma 2.1 are satisfied. By the Lemma 2.1, there exists an $\hat{x} \in E$ such that $f(y,\hat{x}) \leqslant 0$, $\forall y \in E$, that is

$$\langle J\hat{x} - x^*, y - \hat{x} \rangle + \rho\varphi(y) - \rho\varphi(\hat{x}) \geqslant 0, \quad \forall y \in E.$$

Now we show that $\hat{x}$ is a unique solution of the auxiliary variational inequality (2.3). Suppose that $x_1, x_2 \in E$ are arbitrary two solutions of the auxiliary variational inequality (2.3). Then we have

$$\langle Jx_1 - x^*, y - x_1 \rangle + \rho\varphi(y) - \rho\varphi(x_1) \geqslant 0, \quad \forall y \in E, \tag{2.9}$$

$$\langle Jx_2 - x^*, y - x_2 \rangle + \rho\varphi(y) - \rho\varphi(x_2) \geqslant 0, \quad \forall y \in E. \tag{2.10}$$

Taking $y = x_2$ in (2.4) and $y = x_1$ in (2.5), adding these inequalities, we have

$$\langle Jx_1 - Jx_2,\ x_1 - x_2 \rangle \leqslant 0.$$

Since J is α-strongly monotone, it follows that

$$\alpha\|x_1 - x_2\|^2 \leqslant \langle Jx_1 - Jx_2, x_1 - x_2 \rangle \leqslant 0.$$

Hence we must have $x_1 = x_2$. This completes the proof. □

Remark 2.2. Theorem 2.1 shows that for any strongly monotone and continuous mapping $J : E \to E^*$, and $\rho > 0$, the J-proximal mapping $J_\rho^{\partial\varphi} : E^* \to E$ of a lower semicontinuous subdifferentiable proper functional φ is well defined and for each $x^* \in E^*$, $x = J_\rho^{\varphi}(x^*)$ is the unique solution of the auxiliary variational inequality (2.3). We emphasize that the functional φ may not be convex in Theorem 2.1.

Theorem 2.2. *Let E be a reflexive Banach space with the dual space E^*, $J: E \to E^*$ be a α-strongly monotone continuous mapping, $\varphi: E \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional, and $\rho > 0$ be a arbitrary constant. Then the J-proximal mapping $J_\rho^{\partial\varphi}$ of φ is $1/\alpha$-Lipschitz continuous.*

Proof. For any given $x^*, y^* \in E^*$, let $x = J_\rho^{\partial\varphi}(x^*)$, $y = J_\rho^{\partial\varphi}(y^*)$, then $x^* - Jx \in \rho\partial\varphi(x)$, $y^* - Jy \in \rho\partial\varphi(y)$. Hence

$$\rho\varphi(u) - \rho\varphi(x) \geqslant \langle x^* - Jx, u - x\rangle, \quad \forall u \in E, \tag{2.11}$$

$$\rho\varphi(u) - \rho\varphi(y) \geqslant \langle y^* - Jy, u - y\rangle, \quad \forall u \in E. \tag{2.12}$$

Taking $u = y$ in (2.11) and $u = x$ in (2.12), adding these inequalities, we obtain

$$\langle Jy - Jx, y - x\rangle \leqslant \langle y - x, y^* - x^*\rangle.$$

Since J is α-strongly monotone, we have

$$\alpha\|y - x\|^2 \leqslant \langle y - x, y^* - x^*\rangle \leqslant \|y - x\|\,\|y^* - x^*\|$$

which implies $J_\rho^{\partial\varphi}$ is $1/\alpha$-Lipschitz continuous.

Remark 2.3. If E is Hilbert space, φ is a lower semicontinuous convex proper functional on E and J is the identity mapping, then the Lipschitz continuity of J-proximal mapping is the nonexpansive property of the resolvent operator of φ on Hilbert space.

3. Main result

In order to obtain the main results, we need the following lemma which is Lemma 2.1 of Chang et al. [6].

Lemma 3.1. *Let E be a real Banach space and $\mathscr{F}: E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x + y) \in \mathscr{F}(x + y)$.

We first transfer the CGQVI (2.2) into a fixed point problem.

Theorem 3.1. (x, u, v, w, z, y) *is a solution of the* CGQVI (2.2) *if and only if* (x, u, v, w, z, y) *satisfies the following relation*:

$$g(x) = m(y) + J_\rho^{\partial\varphi(\cdot,z)}(J(g(x) - m(y)) - \rho f(w) + \rho N(u, v)), \tag{3.1}$$

where $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$, $y \in Fx$, $\rho > 0$ and $J_\rho^{\partial\varphi(\cdot,u)} = (J + \rho\partial\varphi(\cdot, z))^{-1}$ is the J-proximal mapping of $\varphi(\cdot, z)$.

Proof. Assume that (x,u,v,w,z,y) satisfies relation (3.1), i.e.,

$$g(x)-m(y)=J_\rho^{\partial\varphi(\cdot,z)}(J(g(x)-m(y))-\rho f(w)+\rho N(u,v)).$$

Since $J_\rho^{\partial\varphi(\cdot,z)}=(J+\rho\partial\varphi(\cdot,z))^{-1}$, the above equality holds if and only if

$$J(g(x)-m(y))-\rho f(w)+\rho N(u,v)\in J(g(u)-m(y))+\rho\partial\varphi(g(x)-m(y),z).$$

By the definition of the subdifferential of $\varphi(\cdot,z)$, the above relation holds if and only if

$$\varphi(h,z)-\varphi(g(x)-m(y),z)\geqslant\langle N(u,v)-f(w),h-(g(x)-m(y))\rangle,\quad\forall h\in E.$$

Hence we have

$$\langle f(w)-N(u,v),h-(g(x)-m(y))\rangle\geqslant\varphi(g(x)-m(y),z)-\varphi(h,z),\quad\forall h\in E,$$

i.e., (x,u,v,w,z,y) is a solution of the CGQVI (2.2).

Remark 3.1. By Theorem 2.1, for any given α-strongly monotone continuous mapping $J:E\to E^*$, the J-proximal mapping $J_\rho^{\partial\varphi}$ of a lower semicontinuous subdifferential proper functional φ is well defined. So, we can transfer the CGQVI (2.2) to the fixed point problem (3.1). We can choose some special J such that it is easy to find the solution of the fixed point problem (3.1).

Algorithm 3.1. Let $A,B:E\to\mathrm{CB}(E^*)$ and $C,D,F:E\to\mathrm{CB}(E)$ be set-valued mappings, $f:E\to E^*$ and $g,m:E\to E$ be single-valued mappings with $g(E)=E$, $J:E\to E^*$ be α-strongly monotone and continuous, and $\varphi:E\times E\to\mathbf{R}\cup\{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional in the first argument such that for each $x,y,z\in E, g(x)-m(y)\in\operatorname{dom}\partial\varphi(\cdot,z)$. For any given $x_0\in E$, $u_0\in A(x_0)$, $v_0\in B(x_0)$, $w_0\in Cx_0$, $z_0\in Dx_0$ and $y_0\in Fx_0$. By $g(E)=E$, there exists a point $x_1\in E$ such that

$$g(x_1)=m(y_0)+J_\rho^{\partial\varphi(\cdot,z_0)}(J(g(x_0)-m(y_0))-\rho f(w_0)+\rho N(u_0,v_0)).$$

By a theorem of Nadler [29], there exists $u_1\in Ax_1$, $v_1\in Bx_1$, $w_1\in Cx_1$, $z_1\in Dx_1$ and $y_1\in Fx_1$ such that

$$\|u_1-u_0\|\leqslant(1+1)H(Ax_1,Ax_0),\quad\|v_1-v_0\|\leqslant(1+1)H(Bx_1,Bx_0),$$

$$\|w_1-w_0\|\leqslant(1+1)H(Cx_1,Cx_0),\quad\|z_1-z_0\|\leqslant(1+1)H(Dx_1,Dx_0),$$

$$\|y_1-y_0\|\leqslant(1+1)H(Fx_1,Fx_0).$$

By induction, we can define the following iterative sequence $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ for solving the CGQVI (2.2) as follows:

$$g(x_{n+1})=m(y_n)+J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_n)-m(y_n))-\rho f(w_n)+\rho N(u_n,v_n)],$$

$$u_n\in Ax_n,\ \|u_{n+1}-u_n\|\leqslant\left(1+\frac{1}{n+1}\right)H(Ax_{n+1},Ax_n),$$

$$v_n \in Bx_n,\ \|v_{n+1} - v_n\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Bx_{n+1}, Bx_n),$$

$$w_n \in Cx_n,\ \|w_{n+1} - w_n\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Cx_{n+1}, Cx_n),$$

$$z_n \in Dx_n,\ \|z_{n+1} - z_n\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Dx_{n+1}, Dx_n),$$

$$y_n \in Fx_n,\ \|y_{n+1} - y_n\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Fx_{n+1}, Fx_n), \quad n = 0, 1, 2, \ldots, \tag{3.2}$$

where $\rho > 0$ is a constant.

Theorem 3.2. *Let $A, B : E \to \mathrm{CB}(E^*)$ and $C, D, F : E \to \mathrm{CB}(E)$ be Lipschitz continuous with the Lipschitz constant λ_A, λ_B, λ_C, λ_D and λ_F, respectively. Let $g, m : E \to E$ and $f : E \to E^*$ be Lipschitz continuous with the Lipschitz constants λ_g, λ_m and λ_f, respectively and g is k-strongly accretive ($k \in (0,1)$) satisfying $g(E) = E$. Let $N : E^* \times E^* \to E^*$ be λ_{N_1}-Lipschitz continuous in the first argument and λ_{N_2}-Lipschitz continuous in the second argument. Let $\varphi : E \times E \to \mathbf{R} \cup \{+\infty\}$ be such that for each fixed $z \in E$, $\varphi(\cdot, z)$ is a lower semicontinuous subdifferentiable proper functional satisfying $g(x) - m(y) \in \mathrm{dom}\, \partial\varphi(\cdot, z)$ for all $x, y, z \in E$. Let $J : E \to E^*$ be α-strongly monotone and λ_J-Lipschitz continuous. Suppose that there exists a constant $\rho > 0$ such that for each $x, y \in E$, $x^* \in E^*$*

$$\|J_\rho^{\partial\varphi(\cdot,x)}(x^*) - J_\rho^{\partial\varphi(\cdot,y)}(x^*)\| \leqslant \mu\|x - y\| \tag{3.3}$$

and the following conditions are satisfied:

$$\frac{2[\lambda_J^2(\lambda_g + \lambda_m\lambda_F)^2]}{\alpha^2} - \frac{3 - 2(\lambda_m^2\lambda_F^2 + \mu^2\lambda_D^2)}{2} < k < 1,$$

$$0 < \rho < \sqrt{\frac{(2k+3)\alpha^2 - 4[\lambda_J^2(\lambda_g + \lambda_m\lambda_f)^2 + \alpha^2(\lambda_m^2\lambda_F^2 + \mu^2\lambda_D^2)]}{8[\lambda_f^2\lambda_C^2 + (\lambda_{N_1}\lambda_A + \lambda_{N_2}\lambda_B)^2]}}. \tag{3.4}$$

Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 converge strongly to $\hat{x}$, $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{z}$ and $\hat{y}$, respectively and $(\hat{x}, \hat{u}, \hat{v}, \hat{w}, \hat{z}, \hat{y})$ is a solution of the CGQVI (2.2).

Proof. Obviously, we have

$$\|x_{n+1} - x_n\|^2 = \|g(x_{n+1}) - g(x_n) - g(x_{n+1}) + g(x_n) - x_{n+1} + x_n\|^2.$$

By Lemma 3.1, we obtain

$$\|x_{n+1} - x_n\|^2 \leqslant \|g(x_{n+1}) - g(x_n)\|^2 - 2\langle g(x_{n+1}) - g(x_n) + x_{n+1} - x_n, j(x_{n+1} - x_n)\rangle. \tag{3.5}$$

By Algorithm (3.1), we have

$$g(x_{n+1}) = m(y_n) + J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_n) - m(y_n)) - \rho f(w_n) + \rho N(u_n, v_n)].$$

Hence, we have

$$\begin{aligned}\|g(x_{n+1}) - g(x_n)\|^2 = {} & \|m(y_n) + J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_n) - m(y_n)) - \rho f(w_n) + \rho N(u_n, v_n)] \\ & -m(y_{n-1}) - J_\rho^{\partial\varphi(\cdot,z_{n-1})}[J(g(x_{n-1}) - m(y_{n-1})) \\ & -\rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1})]\|^2.\end{aligned}$$

Since $\|x + y\|^2 \leqslant 2(\|x\|^2 + \|y\|^2)$, by the assumptions and Theorem 2.2 we have

$$\begin{aligned}&\tfrac{1}{2}\|g(x_{n+1}) - g(x_n)\|^2 \\ &\quad\leqslant \|J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_n) - m(y_n)) - \rho f(w_n) + \rho N(u_n, v_n)] \\ &\qquad - J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_{n-1}) - m(y_{n-1})) - \rho f(w_{n-1}) + N(u_{n-1}, v_{n-1})]\|^2 \\ &\qquad + \|m(y_n) - m(y_{n-1}) + J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_{n-1}) - m(y_n - 1)) - \rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1})] \\ &\qquad - J_\rho^{\partial\varphi(\cdot,z_{n-1})}[J(g(u_{n-1}) - m(y_{n-1})) - \rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1})]\|^2 \\ &\quad\leqslant \frac{1}{\alpha^2}\|J(g(x_n) - m(y_n)) - \rho f(w_n) + \rho N(u_n, v_n) \\ &\qquad - J(g(x_{n-1}) - m(y_{n-1})) + \rho f(w_{n-1}) - \rho N(u_{n-1}, v_{n-1})\|^2 + 2\|m(y_n) - m(y_{n-1})\|^2 \\ &\qquad + 2\|J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_{n-1}) - m(y_{n-1})) - \rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1})] \\ &\qquad - J_\rho^{\partial\varphi(\cdot,z_{n-1})}[J(g(x_{n-1}) - m(y_{n-1})) - \rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1})]\|^2 \\ &\quad\leqslant \frac{2}{\alpha^2}\|J(g(x_n) - m(y_n)) - J(g(x_{n-1}) - m(y_{n-1}))\|^2 + \frac{4\rho^2}{\alpha^2}(\|f(w_n) - f(w_{n-1})\|^2 \\ &\qquad + \frac{4\rho^2}{\alpha^2}\|N(u_n, v_n) - N(u_n, v_{n-1}) + N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|^2 \\ &\qquad + 2\|m(y_n) - m(y_{n-1})\|^2 + 2\mu^2\|z_n - z_{n-1}\|^2. \end{aligned} \tag{3.6}$$

By the Lipschitz continuity of J, g, m, F and (3.2), we have

$$\begin{aligned}&\|J(g(x_n) - m(x_n)) - J(g(x_{n-1}) - m(y_{n-1}))\| \\ &\quad\leqslant \lambda_J(\|g(x_n) - g(x_{n-1})\| + \|m(y_n) - m(y_{n-1})\|) \\ &\quad\leqslant \lambda_J(\lambda_g\|x_n - x_{n-1}\| + \lambda_m\|y_n - y_{n-1}\|)\end{aligned}$$

$$\leqslant \lambda_J \left(\lambda_g \|x_n - x_{n-1}\| + \lambda_m \left(1+\frac{1}{n}\right) H(Fx_n, Fx_{n-1})\right)$$

$$\leqslant \lambda_J \left(\lambda_g + \lambda_m\lambda_F \left(1+\frac{1}{n}\right)\right) \|x_n - x_{n-1}\|. \tag{3.7}$$

By the Lipschitz continuity of f and C, we have

$$\|f(w_n) - f(w_{n-1})\| \leqslant \lambda_f \|w_n - w_{n-1}\| \leqslant \lambda_f \left(1+\frac{1}{n}\right) H(Cx_n, Cx_{n-1})$$

$$\leqslant \lambda_f\lambda_C \left(1+\frac{1}{n}\right) \|x_n - x_{n-1}\|. \tag{3.8}$$

By the Lipschitz continuity of $N(\cdot,\cdot)$ in the first and second arguments, we have

$$\|N(u_n,v_n) - N(u_n,v_{n-1}) + N(u_n,v_{n-1}) - N(u_{n-1},v_{n-1})\|$$

$$\leqslant \|N(u_n,v_n) - N(u_n,v_{n-1})\| + \|N(u_n,v_{n-1}) - N(u_{n-1},v_{n-1})\|$$

$$\leqslant \lambda_{N_2}\|v_n - v_{n-1}\| + \lambda_{N_1}\|u_n - u_{n-1}\|$$

$$\leqslant \lambda_{N_2}\left(1+\frac{1}{n}\right) H(Bx_n, Bx_{n-1}) + \lambda_{N_1}\left(1+\frac{1}{n}\right) H(Ax_n, Ax_{n-1})$$

$$\leqslant (\lambda_{N_1}\lambda_A + \lambda_{N_2}\lambda_B)\left(1+\frac{1}{n}\right) \|x_n - x_{n-1}\|. \tag{3.9}$$

By the Lipschitz continuity of m, D and F, we have

$$\|m(y_n) - m(y_{n-1})\| \leqslant \lambda_m \left(1+\frac{1}{n}\right) H(Fx_n, Fx_{n-1}) \leqslant \lambda_m\lambda_F \left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|, \tag{3.10}$$

$$\|z_n - z_{n-1}\| \leqslant \left(1+\frac{1}{n}\right) H(Dx_n, Dx_{n-1}) \leqslant \lambda_D \left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|, \tag{3.11}$$

By (3.6)–(3.11), we obtain

$$\|g(x_{n+1}) - g(x_n)\|^2 \leqslant \left[\frac{4}{\alpha^2}\lambda_J^2\left(\lambda_g + \lambda_m\lambda_F\left(1+\frac{1}{n}\right)\right)^2 + \frac{8\rho^2}{\alpha^2}\lambda_f^2\lambda_C^2\left(1+\frac{1}{n}\right)^2\right.$$

$$+\frac{8\rho^2}{\alpha^2}(\lambda_{N_1}\lambda_A + \lambda_{N_2}\lambda_B)^2\left(1+\frac{1}{n}\right)^2 + 4\lambda_m^2\lambda_F^2\left(1+\frac{1}{n}\right)^2$$

$$\left. + 4\mu^2\lambda_D^2\left(1+\frac{1}{n}\right)^2\right]\|x_n - x_{n-1}\|$$

$$= \left[\frac{8\rho^2}{\alpha^2}(\lambda_f^2\lambda_C^2 + (\lambda_{N_1}\lambda_A + \lambda_{N_2}\lambda_B)^2)\left(1+\frac{1}{n}\right)^2\right.$$

$$+\frac{4}{\alpha^2}\lambda_J^2\left(\lambda_g+\lambda_m\lambda_F\left(1+\frac{1}{n}\right)\right)^2+4\lambda_m^2\lambda_F^2\left(1+\frac{1}{n}\right)^2$$

$$\left.+4\mu^2\lambda_D^2\left(1+\frac{1}{n}\right)^2\right]\|x_n-x_{n-1}\|. \tag{3.12}$$

Since $g:E\to E$ is k-strongly accretive, by (3.5), we have

$$\begin{aligned}\|x_{n-1}-x_n\|^2 &\leqslant \|g(x_{n+1})-g(x_n)\|^2-2\langle g(x_{n+1})-g(x_n)+x_{n+1}-x_n,\ j(x_{n+1}-x_n)\rangle\\ &\leqslant \left[\frac{8\rho^2}{\alpha^2}(\lambda_f^2\lambda_C^2+(\lambda_{N_1}\lambda_A+\lambda_{N_2}\lambda_B)^2)\left(1+\frac{1}{n}\right)^2\right.\\ &\quad +\frac{4}{\alpha^2}\lambda_J^2\left(\lambda_g+\lambda_m\lambda_F\left(1+\frac{1}{n}\right)\right)^2+4\lambda_m^2\lambda_F^2\left(1+\frac{1}{n}\right)^2\\ &\quad \left.+4\mu^2\lambda_D^2\left(1+\frac{1}{n}\right)^2\right]\|x_n-x_{n-1}\|-(2k+2)\|x_{n+1}-x_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &\leqslant \left\{\frac{8\rho^2(\lambda_f^2\lambda_C^2+(\lambda_{N_1}\lambda_A+\lambda_{N_2}\lambda_B)^2(1+\frac{1}{n})^2)}{(2k+3)\alpha^2}\right.\\ &\quad \left.+\frac{4[\lambda_j^2(\lambda_g+\lambda_m\lambda_F(1+\frac{1}{n}))^2+\alpha^2(\lambda_m^2\lambda_F^2+\mu^2\lambda_D^2)(1+\frac{1}{n})^2]}{(2k+3)\alpha^2}\right\}\|x_n-x_{n-1}\|^2\\ &= \theta_n^2\|x_n-x_{n-1}\|^2,\end{aligned}$$

where

$$\theta_n = \sqrt{\frac{8\rho^2(\lambda_f^2\lambda_C^2+(\lambda_{N_1}\lambda_A+\lambda_{N_2}\lambda_B)^2)(1+\frac{1}{n})^2+4[\lambda_J^2(\lambda_g+\lambda_m\lambda_F(1+\frac{1}{n}))^2+\alpha^2(\lambda_m^2\lambda_F^2+\mu^2\lambda_D^2)(1+\frac{1}{n})^2]}{(2k+3)\alpha^2}}.$$

Let

$$\theta=\sqrt{\frac{8\rho^2(\lambda_f^2\lambda_C^2+(\lambda_{N_1}\lambda_A+\lambda_{N_2}\lambda_B)^2)+4[\lambda_J^2(\lambda_g+\lambda_m\lambda_F)^2+\alpha^2(\lambda_m^2\lambda_F^2+\mu^2\lambda_D^2)]}{(2k+3)\alpha^2}}.$$

Clearly, $\theta_n\to\theta$ as $n\to\infty$. Condition (3.4) implies that $0<\theta<1$ and so $0<\theta_n<1$, when n is sufficiently large. It follows from (3.13) that $\{x_n\}$ is a Cauchy sequence in E. Let $x_n\to\hat{x}$. Since the mappings A,B,C,D,F are Lipschitz continuous, it follows from (3.2) that $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ are also Cauchy sequences, we can assume $u_n\to\hat{u}$, $v_n\to\hat{v}$, $w_n\to\hat{w}$, $z_n\to\hat{z}$ and $y_n\to\hat{y}$, respectively. By Algorithm 3.1, we have

$$g(x_{n+1}) = m(y_n) + J_\rho^{\partial\varphi(\cdot,z_n)}[J(g(x_n) - m(y_n)) - \rho f(w_n) + \rho N(u_n, v_n)]. \tag{3.14}$$

By the Lipschitz continuity of $f, g, J, N(\cdot,\cdot)$, the condition (3.3) and Theorem 2.2, we have

$$\begin{aligned}
&\|J_\rho^{\partial\varphi(\cdot,\, z_n)}[J(g(x_n) - m(y_n)) - \rho f(w_n) + \rho N(u_n, v_n)] \\
&\quad - J_\rho^{\partial\varphi(\cdot,\, \hat{z})}[J(g(\hat{x}) - m(\hat{y})) - \rho f(\hat{w}) + \rho N(\hat{u}, \hat{v})]\| \\
&\leqslant \|J_\rho^{\partial\varphi(\cdot,\, z_n)}[J(g(x_n) - m(y_n)) - \rho f(w_n) + \rho N(u_n, v_n)] \\
&\quad - J_\rho^{\partial\varphi(\cdot,\, z_n)}[J(g(\hat{x}) - m(\hat{y})) - \rho f(\hat{w}) + \rho N(\hat{u}, \hat{v})]\| \\
&\quad + \|J_\rho^{\partial\varphi(\cdot,\, \hat{z})}[J(g(\hat{x}) - m(\hat{y})) - \rho f(\hat{w}) + \rho N(\hat{u}, \hat{v})] \\
&\quad - J^{\partial\varphi(\cdot,\, \hat{z})}[J(g(\hat{x}) - m(\hat{y})) - \rho f(\hat{w}) + \rho N(\hat{u}, \hat{v})]\| \\
&\leqslant \frac{1}{\alpha}[\|J(g(x_n) - m(y_n)) - J(g(\hat{x} - m(\hat{y}))\| + \rho\|f(w_n) - f(\hat{w})\| \\
&\quad + \rho\|(N(u_n, v_n) - N(\hat{u}, \hat{v})\| + \mu\|z_n - \hat{z}\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, by letting $n \to +\infty$ in equality (3.14), we obtain

$$g(\hat{x}) = m(\hat{y}) + J_\rho^{\partial\varphi(\cdot,\hat{z})}[J(g(\hat{x}) - m(\hat{y})) - \rho f(\hat{w}) + \rho N(\hat{u}, \hat{v})].$$

Now we prove that $\hat{u} \in A\hat{x}$, $\hat{v} \in B\hat{x}$, $\hat{w} \in C\hat{x}$, $\hat{z} \in D\hat{x}$ and $\hat{y} \in F\hat{x}$, respectively. Since $u_n \in Ax_n$, we have

$$\begin{aligned}
d(\hat{u}, A\hat{x}) &\leqslant \|\hat{u} - u_n\| + d(u_n, Ax_n) + H(Ax_n, A\hat{x}) \\
&\leqslant \|\hat{u} - u_n\| + \lambda_A\|x_n - \hat{x}\| \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Hence $d(\hat{u}, A\hat{x}) \to 0$ and so $\hat{u} \in A\hat{x}$ since $A\hat{x} \in \mathrm{CB}(E^*)$. In a similar way, we can also prove that $\hat{v} \in B\hat{x}$, $\hat{w} \in C\hat{x}$, $\hat{z} \in D\hat{x}$ and $\hat{y} \in F\hat{x}$. By Theorem 3.1, $(\hat{x}, \hat{u}, \hat{v}, \hat{w}, \hat{y})$ is a solution of the CGQVI (2.2). This completes the proof.

Remark 3.2. In Theorem 3.2, by suitable choices of $N(\cdot,\cdot)$, A, B, C, D, F, f, g, m and the space E, we can obtain many new results and generalizations of some known results in ([1–19,21–28,30 –38]). We emphasize that the existence result and algorithm of solutions for the CGQVI (2.2) are given in reflexive Banach spaces. Furthermore, the methods given in this paper are quite different from the methods given in the previous known literature ([1–19,21–28,30–38]). We emphasize that there are no any monotonicity and accretive assumptions on set-valued mappings in Theorem 3.2.

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