A generalization of the cellular indecomposable property via fiber dimension

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Abstract

The cellular indecomposable property, introduced by Olin and Thomson in 1984 [11], is well known for the Dirichlet space, but it fails trivially for the vector-valued case. The purpose of this paper is to use the fiber dimension to reformulate the property such that it naturally extends the scalar-valued case, yet fix the vector-valued case in a meaningful way. Using the new formulation, we are able to generalize several previous results to the vector-valued setting. In particular, we extend a theorem of Bourdon relating the cellular indecomposable property and the codimension-one property to codimension-$N$. Several of our results appear to be new even for the Hardy space over the unit disc.

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1. Introduction

The cellular indecomposable property (CIP), introduced by R. Olin and J. Thomson in [11], states that any two nontrivial invariant subspaces $M_1, M_2 \subset H$ of a Hilbert space $H$, with respect to an operator $T \in B(H)$, have a nontrivial intersection $M_1 \cap M_2 \neq \{0\}$. It is well known that (CIP) holds for the Dirichlet space $D$ over the unit disk $\mathbb{D}$, see Richter and Shields [14].

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**Theorem.** (See [14].) Any two nonzero invariant subspaces $M_1, M_2$ of the Dirichlet space $D$ with respect to the Dirichlet shift $Mz$ have a nontrivial intersection $M_1 \cap M_2 \neq \{0\}$.

This important property has many applications in operator theory, even to the transitive algebra problem [3,13]. But it fails trivially for the vector-valued case: just consider invariant subspaces $M_1 = D \oplus \{0\}$ and $M_2 = \{0\} \oplus D$ of $H = D \oplus D$. It is desirable to extend (CIP) to the vector-valued case in a meaningful way and the purpose of this paper is to present such an extension.

**Theorem 1.** For any two invariant subspaces $M_1, M_2 \subset D \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, if

$$fd(M_1) + fd(M_2) > N,$$

then

$$M_1 \cap M_2 \neq \{0\}.$$

Here the fiber dimension $fd(M)$ of an invariant subspace $M$ is defined as

$$fd(M) = \sup_{\lambda \in D} \dim M(\lambda)$$

with

$$M(\lambda) = \{ f(\lambda) : f \in M \} \subset \mathbb{C}^N.$$

Theorem 1 clearly generalizes the above result of Richter and Shields [14], since $fd(M_1) = fd(M_2) = 1$ when $M_1, M_2$ are nonzero invariant subspaces of $D$.

Instead of proving Theorem 1 directly, we will show that a quantitative result is indeed true: under the condition of Theorem 1, we have

$$fd(M_1 \cap M_2) \geq fd(M_1) + fd(M_2) - N. \quad (1)$$

Further, we are able to establish a relative version of the above (1); namely, the two invariant subspaces $M_1, M_2$ can be chosen relative to another invariant subspace $M \subset D \otimes \mathbb{C}^N$.

We also point out that it is not hard to extend the result to spaces with complete Nevanlinna–Pick (NP for short) kernels, which certainly cover the Dirichlet space. We state the next theorem in this more general setting. Let $\mathcal{H}(k)$ denote a Hilbert space of analytic functions over a domain $\Omega \subset \mathbb{C}$ containing the origin, determined by a reproducing kernel $k$ with the complete NP kernel property.

**Theorem 2.** Let $k$ be a complete NP kernel. For any multiplier invariant subspace $M \subset \mathcal{H}(k) \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, and two multiplier invariant subspaces $M_1, M_2 \subset M$,

$$fd(M_1 \cap M_2) \geq fd(M_1) + fd(M_2) - fd(M). \quad (2)$$

In particular, one has that

(†) if $fd(M_1) + fd(M_2) > fd(M)$, then $M_1 \cap M_2 \neq \{0\}$. 


Here an analytic function $\varphi$ on $\Omega$ is a multiplier of $\mathcal{H}(k)$ if $\varphi f \in \mathcal{H}(k)$ for every $f \in \mathcal{H}(k)$. The closed graph theorem implies that each multiplier $\varphi$ induces a bounded multiplication operator $M_\varphi : f \mapsto \varphi f$ on $\mathcal{H}(k)$. A subspace $\mathcal{M}$ of $\mathcal{H}(k) \otimes \mathbb{C}^N$ is called multiplier invariant if it is invariant for each $M_\varphi$. When $k$ is a complete NP kernel, any multiplier invariant subspace of $\mathcal{H}(k) \otimes \mathbb{C}^N$ has the following nice property, which follows from Theorem 0.7 in [10].

**Lemma 3.** Suppose that $k$ is a complete NP kernel and $\mathcal{M}$ is a multiplier invariant subspace of $\mathcal{H}(k) \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, then the subset

$$\{f \in \mathcal{H}(k) \otimes \mathbb{C}^N : f \text{ has multiplier entries}\} \cap \mathcal{M}$$

is dense in $\mathcal{M}$. Here $f = (f_1, \ldots, f_N) \in \mathcal{H}(k) \otimes \mathbb{C}^N$ has multiplier entries if each $f_i$ is a multiplier.

The rest of the paper is organized as follows: In Section 2 we gather preliminary facts on the fiber dimension and a notion called “occupy invariant” which is introduced in [8] and will be needed in the proof of Theorem 2 in Section 3. Section 3 is devoted to the proof of Theorem 2. Section 4 contains a direct application of Theorem 2 which yields a subadditivity result (Theorem 8) for Samuel multiplicities on coinvariant subspaces. In Section 5, we introduce the “complete cellular indecomposable property (CCIP)”, which extends the familiar cellular indecomposable property (CIP) of Olin and Thomson [11]. We also introduce a weaker version (CCIP′) and show that it implies the stronger (CCIP) under a natural complementary condition (C), see Theorem 12. Sections 6 and 7 are devoted to generalizations of two results of Bourdon [2]: First, Bourdon showed that the cellular indecomposable property implies the well-known codimension-one property. On the other hand, it is a folklore that for the vector-valued case, the codimension-one property is replaced by the codimension-$N$ property. In Section 6 we show that one can indeed establish a parallel result for codimension-$N$ (Theorem 13) if using the complete cellular indecomposable property introduced in Section 5. The second result of Bourdon which we will generalize in Section 7 is a partial converse of the cellular indecomposable property, see Theorem 18. It is not hard to see that the converse of Bourdon’s result is not true.

2. Preliminaries on fiber dimension and occupy invariant

In this section we gather some basic facts on the fiber dimension and a notion called “occupy invariant” which provides a way to describe the structure of certain invariant subspaces and will be used in the proof of Theorem 2.

**Definition 4.** For any subspace $\mathcal{M} \subset \mathcal{H}(k) \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, define the occupy invariant of $\mathcal{M}$, denoted by $l_{\mathcal{M}}$, to be the maximal dimension of a subspace $E$ of $\mathbb{C}^N$ with the following property: there exists a basis (not necessarily orthonormal) $e_1, \ldots, e_l$ ($l = l_{\mathcal{M}}$) of $E$ and $h_1, \ldots, h_l \in \mathcal{M}$ such that

$$P_{\mathcal{H}(k) \otimes E} h_i \neq 0 \in \mathcal{H}(k) \otimes e_i, \quad i = 1, \ldots, l.$$

When $E$ has the above property we say that $\mathcal{M}$ occupies $\mathcal{H}(k) \otimes E$ in $\mathcal{H}(k) \otimes \mathbb{C}^N$.

The following is probably the most useful fact about $l_{\mathcal{M}}$ for our purpose.
Lemma 5. Let \( k \) be a complete NP kernel. If \( \mathcal{M} \) is a multiplier invariant subspace of \( \mathcal{H}(k) \otimes \mathbb{C}^N \), \( N \in \mathbb{N} \), then \( l_{\mathcal{M}} = f_d(\mathcal{M}) \).

For a proof see Proposition 3.3 of [3], which essentially follows from Lemma 23 of [8].

A subspace \( \mathcal{M} \subset \mathcal{H}(k) \otimes \mathbb{C}^N \), \( N \in \mathbb{N} \), is called a \( d \)-graph subspace (\( d \leq N \)) [3] if there exists a basis of \( \mathbb{C}^N \) such that with respect to this basis, \( \mathcal{M} \) has the form

\[
\mathcal{M} = \{(f_1, \ldots, f_d, T_1 f, \ldots, T_{N-d} f) : f = (f_1, \ldots, f_d) \in \mathcal{L}\},
\]

where \( \mathcal{L} \) is the linear manifold of the first \( d \) entries of elements in \( \mathcal{M} \) and we assume

\[
f_d(\mathcal{L}) = d.
\]

Moreover, each \( T_i \) is a linear transform from \( \mathcal{L} \) to \( \mathcal{H}(k) \). When \( \mathcal{M} \) is multiplier invariant, one has that \( T_i M_\varphi = M_\varphi T_i \) for any multiplier \( \varphi \).

If \( k \) is a complete NP kernel and \( \mathcal{M} \) occupies \( \mathcal{H}(k) \otimes E \) for some \( E \subset \mathbb{C}^N \) with \( \dim(E) = f_d(\mathcal{M}) \), then we can extend the basis of \( E \) (as in Definition 4) to a basis of \( \mathbb{C}^N \). With respect to this basis, it is easy to check that \( \mathcal{M} \) is a \( d \)-graph subspace with \( d = f_d(\mathcal{M}) \). For the details, see Theorem 3.6 of [3].

Lemma 6. Let \( k \) be a complete NP kernel. Suppose that \( \mathcal{M} \subset \mathcal{H}(k) \otimes \mathbb{C}^N \), \( N \in \mathbb{N} \), is a multiplier invariant subspace with \( f_d(\mathcal{M}) = d \), then \( \mathcal{M} \) is a \( d \)-graph subspace.

The following lemma plays a key role in the proof of Theorem 2 and is of independent interests.

Lemma 7. Let \( k \) be a complete NP kernel. If \( \mathcal{M} \subset \mathcal{H}(k) \otimes \mathbb{C}^N \), \( N \in \mathbb{N} \), is a multiplier invariant subspace, then it occupies \( \mathcal{H}(k) \otimes \mathcal{M}(\lambda) \) for any point \( \lambda \in \Omega \).

Proof. Assume \( \dim \mathcal{M}(\lambda) = d \). Then we take \( f_1, \ldots, f_d \in \mathcal{M} \) such that

\[
\{f_1(\lambda), \ldots, f_d(\lambda)\}
\]

form a basis for \( \mathcal{M}(\lambda) \). Moreover, by Lemma 3 we can require that each \( f_i \) has multiplier entries. Extend \( \{f_i(\lambda)\}_{i=1}^d \) to a basis of \( \mathbb{C}^N \) and with respect to this basis, we write

\[
f_i = (f_{i1}, \ldots, f_{iN}), \quad 1 \leq i \leq d.
\]

By our choice of \( f_i \), the determinant of matrix

\[
\Theta = (f_{ij})_{i,j=1}^d,
\]

denoted by \( \det(\Theta) \), is a nonzero analytic function and is nonzero at \( \lambda \) in particular. Moreover, note that \( \det(\Theta) \) is a multiplier on \( \mathcal{H}(k) \).

Recall that the inverse matrix of \( \Theta \) is given by

\[
\frac{1}{\det(\Theta)} (A_{ij})_{i,j=1}^d,
\]

where \( A_{ij} \) is the \( (d - 1) \times (d - 1) \) minor of \( \Theta \) associated with \( f_{ji} \). The useful fact here is that \( A_{ij} \) is still a multiplier on \( \mathcal{H}(k) \). It follows that
\[(A_{ij})_{d \times d} \cdot (f_{ij})_{d \times N} = \begin{pmatrix} \det(\Theta) & 0 & \cdots & 0 & g_{11} & \cdots & g_{1,N-d} \\ 0 & \det(\Theta) & \cdots & 0 & g_{21} & \cdots & g_{2,N-d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(\Theta) & g_{d1} & \cdots & g_{d,N-d} \end{pmatrix}.\]

So the following vectors are in \(M\) because \(A_{ij}\) are multipliers,

\[g_{i} = (0, \ldots, 0, \det(\Theta), 0, \ldots, 0, g_{i1}, \ldots, g_{i,N-d}), \quad 1 \leq i \leq d.\]

They show that \(M\) occupies \(\mathcal{H}(k) \otimes \mathcal{M}(\lambda)\).

Lastly we observe that the fiber dimension \(fd(M)\) of a subspace \(M \subset \mathcal{H}(k) \otimes \mathbb{C}^N\) is achieved at almost all points \(\lambda \in \Omega\); namely,

\[fd(M) = \dim M(\lambda), \quad \text{a.e. } \lambda \in \Omega.\]  

Here \(M(\lambda) = \{f(\lambda): f \in M\} \subset \mathbb{C}^N\). We say that \(\lambda\) is a maximal fiber point if (3) holds for \(\lambda\). Any point \(\mu \in \Omega\) in the complement of maximal fiber points is called a degenerate point. The set of degenerate points is denoted by \(Z_{dg}(M)\).

Let \(\lambda\) be a maximal fiber point of \(M\), and let \(\{e_1, \ldots, e_d\}\) be an orthonormal basis for \(M(\lambda) \subset \mathbb{C}^N\), and extend it to an orthonormal basis \(\{e_1, \ldots, e_N\}\) for \(\mathbb{C}^N\). Let \(f_1, \ldots, f_d\) be such that \(\{f_1(\lambda), \ldots, f_d(\lambda)\}\) form a basis for \(M(\lambda)\). Write

\[f_i = (f_{i1}, \ldots, f_{iN})\]

according to the orthonormal basis \(\{e_1, \ldots, e_N\}\). Then the determinant

\[F(z) = \det(f_{ij})_{i,j=1}^{d}\]

is a nonzero function and is nonzero at \(\lambda\) in particular. It follows that

\[Z_{dg}(M) \subset Z(F(z)),\]

the zero set of \(F(z)\). So, in general, \(Z_{dg}(M)\) is a discrete subset of the domain \(\Omega\).

3. Proof of Theorem 2

**Proof.** Without loss of generality, we assume that \(M = M_1 \cup M_2\), the closed subspace spanned by \(M_1\) and \(M_2\). Let \(\lambda_0\) be a maximal fiber point for \(M, M_1\) and \(M_2\). By Lemma 7, \(M\) and \(M_i\) occupy \(\mathcal{H}(k) \otimes M(\lambda_0)\) and \(\mathcal{H}(k) \otimes M_i(\lambda_0)\), respectively, \(i = 1, 2\). Let

\[E' = M_1(\lambda_0) \cap M_2(\lambda_0),\]

\[E_1 = M_1(\lambda_0) \ominus E', \quad \text{and} \quad E_2 = M_2(\lambda_0) \ominus E'.\]

Then assume
\[ \dim(E_i) = d_i, \quad i = 1, 2, \]

and

\[ \dim(E') = d'. \]

We take bases, not necessarily orthonormal,

\[ \{e_1, \ldots, e_{d_1}\}, \quad \{e_{d_1+1}, \ldots, e_{d_1+d_2}\}, \quad \{e_{d_1+d_2+1}, \ldots, e_{d_1+d_2+d'}\} \]

for \( E_1, E_2, E' \), respectively. Obviously,

\[ E = \{e_1, \ldots, e_{d_1}, e_{d_1+1}, \ldots, e_{d_1+d_2}, e_{d_1+d_2+1}, \ldots, e_{d_1+d_2+d'}\} \]

is a basis for \( M(\lambda_0) \). Let

\[ d = d_1 + d_2 + d' = fd(M), \]

the fiber dimension of \( M \). Extend \( E \) to a basis of \( \mathbb{C}^N \), denoted by \( E' \).

Under the basis \( E' \), by Lemma 6, \( M \) is a \( d \)-graph subspace. As in Lemma 6, let \( L \) be the linear manifold of the first \( d \)-components of elements in \( M \), so there are \( N - d \) linear transformations \( T_j : L \to \mathcal{H}(k) \) such that \( M \) is of the form

\[ M = \{(f_1, \ldots, f_d, T_1 f, \ldots, T_{N-d} f) : f = (f_1, \ldots, f_d) \in L\}. \]

The rest of the proof is divided into three steps.

**Step I:** In this step, we will choose functions of a particular form in \( M_1 \) and \( M_2 \) to represent the fiber dimensions in a way suitable for considering the fiber dimension of \( M_1 \cap M_2 \).

Since by Lemma 7, \( M_1 \) occupies \( \mathcal{H}(k) \otimes (E_1 + E') \), we can find the following \( d_1 + d' \) elements, all with multiplier entries, in \( M_1 \):

\[ \mathcal{F}_i = (F_i, T_1 F_i, \ldots, T_{N-d} F_i), \quad 1 \leq i \leq d_1 + d', \]  \( (4) \)

where for \( i = 1, \ldots, d_1, \)

\[ F_i = \left(0, \ldots, 0, f_i, 0, \ldots, 0, h_{i1}, \ldots, h_{i,d_2}, 0, \ldots, 0\right), \]

and for \( i = d_1 + 1, \ldots, d_1 + d', \)

\[ F_i = \left(0, \ldots, 0, k_{i1}, \ldots, k_{i,d_2}, 0, \ldots, 0, f_i, 0, \ldots, 0\right). \]

Here each \( f_i \) is a nonzero function. In particular, since we assume that \( \lambda_0 \) is a maximal fiber point for \( M_1 \) and by the proof of Lemma 7,

\[ f_i(\lambda_0) \neq 0. \]  \( (5) \)
For simplifying symbols, let \( A_i \) \( (1 \leq i \leq d_1) \) be a \( d_1 \times d_2 \) matrix with only one nonzero row \( (\frac{h_{i1}}{f_i}, \ldots, \frac{h_{i,d_2}}{f_i}) \), which is the \( i \)-th row. Similarly, let \( B_i \) \( (d_1 + 1 \leq i \leq d_1 + d') \) be a \( d' \times d_2 \) matrix with only one nonzero row \( (\frac{k_{i1}}{f_i}, \ldots, \frac{k_{i,d_2}}{f_i}) \), which is the \( i \)-th row. Then \( F_i \) can be rewritten as: for \( i = 1, \ldots, d_1, \)

\[
F_i = \left( \begin{array}{ccc}
0, & \ldots, & 0, f_i, 0, \ldots, 0, (0, \ldots, 0, f_i, 0, \ldots, 0)A_i, 0, \ldots, 0 \\
\end{array} \right).
\]

and for \( i = d_1 + 1, \ldots, d_1 + d' \),

\[
F_i = \left( \begin{array}{ccc}
0, & \ldots, & 0, (0, \ldots, 0, f_i, 0, \ldots, 0)B_i, 0, \ldots, 0, f_i, 0, \ldots, 0 \\
\end{array} \right).
\]

Similarly, we can find the following \( d_2 + d' \) elements in \( M_2 \)

\[
G_i = (G_i, T_1 G_i, \ldots, T_{N-d} G_i), \quad 1 \leq i \leq d_2 + d',
\]

where for \( i = 1, \ldots, d_2, \)

\[
G_i = \left( \begin{array}{ccc}
(0, \ldots, 0, g_i, 0, \ldots, 0)C_i, 0, \ldots, 0, g_i, 0, \ldots, 0, 0, \ldots, 0 \\
\end{array} \right).
\]

and for \( i = d_2 + 1, \ldots, d_2 + d', \)

\[
G_i = \left( \begin{array}{ccc}
(0, \ldots, 0, g_i, 0, \ldots, 0)D_i, 0, \ldots, 0, 0, \ldots, 0, g_i, 0, \ldots, 0 \\
\end{array} \right).
\]

with \( g_i \neq 0 \), and each \( C_i \) or \( D_i \) is a \( d_2 \times d_1 \) or \( d' \times d_1 \) matrix respectively. Similarly, we have

\[
g_i(\lambda_0) \neq 0.
\]

In particular, it follows from (5) and (9) that

\[
\dim(\text{span}\{F_1(\lambda_0), \ldots, F_{d_1}(\lambda_0), G_1(\lambda_0), \ldots, G_{d_2}(\lambda_0)\}) = d_1 + d_2.
\]

**Step II:** In this step we are mainly concerned with solving Eq. (12) by analyzing its coefficient matrix. Our previous choices of vectors \( F_i \) and \( G_i \) in Step I make explicit analysis of the coefficient matrix of (12) possible.

In order to consider the fiber dimension of \( M_1 \cap M_2 \), we consider those \( (d + d') \)-tuples \((\tilde{r}_1, \ldots, \tilde{r}_{d+d'})\), with multiplier entries and not all being zeros, such that

\[
\tilde{r}_1 F_1 + \cdots + \tilde{r}_{d_1+d'} F_{d_1+d'} = \tilde{r}_{d_1+d'+1} G_1 + \cdots + \tilde{r}_{d+d'} G_{d_2+d'} \quad (\neq 0).
\]

Next we need the reduction of Eq. (11) from \( F_i \) and \( G_i \) to \( F_i \) and \( G_i \). To do this, observe that for a vector \( f \) in \( M_1 \cap M_2 \), if the first \( d \) entries of \( f \) are all zero, then \( f \) must be zero since \( M = M_1 \cap M_2 \) is a \( d \)-graph subspace. Now because the linear transformations \( T_j \) are
commuting with multiplications induced by multipliers, we only need to consider the tuples \((\tilde{r}_1, \ldots, \tilde{r}_{d+d'})\) such that

\[
\tilde{r}_1 F_1 + \cdots + \tilde{r}_{d_1+d'} F_{d_1+d'} = \tilde{r}_{d_1+d'}+G_{d_1} + \cdots + \tilde{r}_{d+d'} G_{d_2+d'} \quad (\neq 0).
\]

Now we rearrange the above equation as

\[
\begin{align*}
& (r_1 F_1 + \cdots + r_{d_1} F_{d_1}) + (r_{d_1+1} G_1 + \cdots + r_{d_1+d_2} G_{d_2}) \\
& \quad + (r_{d_1+d_2+1} F_{d_1+1} + \cdots + r_{d_1+d_2+d'} F_{d_1+d'}) \\
& \quad + (r_{d_1+d_2+d'+1} G_{d_2+1} + \cdots + r_{d_1+d_2+2d'} G_{d_2+2d'}) = 0.
\end{align*}
\]

Therefore, we need to consider the following \(d \times (d+d')\) coefficient matrix of Eq. (12)

\[
\Delta = \begin{pmatrix}
W_1 & CV_1 & 0 & DV_2 \\
AW_1 & V_1 & BW_2 & 0 \\
0 & 0 & W_2 & V_2
\end{pmatrix},
\]

where the columns are \(F_i^T\) and \(G_i^T\) with \(T\) denoting transposition; namely, \(W_1\) is the \(d_1 \times d_1\) diagonal matrix

\[
W_1 = \text{diag}(f_1, \ldots, f_{d_1})
\]

and \(AW_1\) is a \(d_2 \times d_1\) matrix with columns

\[
\left((0, \ldots, 0, f_i, 0, \ldots, 0)A_i\right)^T, \quad 1 \leq i \leq d_1.
\]

Similarly,

\[
W_2 = \text{diag}(f_{d_1+1}, \ldots, f_{d_1+d'}),
\]

\[
V_1 = \text{diag}(g_1, \ldots, g_{d_2}),
\]

\[
V_2 = \text{diag}(g_{d_2+1}, \ldots, g_{d_2+d'}).
\]

Moreover, \(BW_2, CV_1\) and \(DV_2\) are understood in the same way as \(AW_1\).

Let \(\Theta\) be the first \(3 \times 3\) block matrix of \(\Delta\), that is,

\[
\Theta = \begin{pmatrix}
W_1 & CV_1 & 0 \\
AW_1 & V_1 & BW_2 \\
0 & 0 & W_2
\end{pmatrix}.
\]

**Claim.** The determinant of \(\Theta\) is a nonzero analytic function.
Proof. It is sufficient to show that
\[
\det \begin{pmatrix} W_1 & CV_1 \\ AW_1 & V_1 \end{pmatrix} \neq 0
\] (13)
since \(W_2\) is a diagonal matrix with nonzero diagonal entries. \(\square\)

Note that the column vectors in (13) are the \((E_1 + E_2)\)-components of
\[
F_1, \ldots, F_{d_1}, G_1, \ldots, G_{d_2}.
\] (14)

Let us recall three facts here:

1. \(E'\)-components of all vectors in (14) are zero.
2. For any vector in \(M\), if its \((E_1 + E_2 + E')\)-component is zero, then the vector is itself zero, because \(M\) is a \(d\)-graph subspace.
3. Vectors in (14), if evaluated at \(\lambda_0\), are independent. See (10).

Now (13) follows from the above three facts, because otherwise it implies that vectors in (14) are always dependent when evaluated at any point \(\lambda\).

**Step III:** In this step, we (explicitly) solve Eq. (12) and observe that the degree of freedom in the solution is \(d'\), hence completing the proof of Theorem 2.

First, recall that the inverse matrix of \(\Theta\) is given by
\[
\Theta^{-1} = \frac{1}{\det \Theta} (A_{ij})_{i,j=1}^d,
\]
where \(A_{ij}\) are the \((d-1) \times (d-1)\) minors of \(\Theta\), and they are all multipliers of \(D\).

If we write \(\Delta\) as \(\Delta = (\Theta, \Theta_1)\), then
\[
(A_{ij})_{i,j=1}^d \cdot \Delta = (\det(\Theta) \cdot I_d, \det(\Theta) \cdot \Theta^{-1} \cdot \Theta_1)
\]
at the level of matrix multiplication, where \(I_d\) is the identity matrix with size \(d\).

Note that \(\det(\Theta) \cdot \Theta^{-1} \cdot \Theta_1\) is a \(d \times d'\) matrix and we write it as
\[
\det(\Theta) \cdot \Theta^{-1} \cdot \Theta_1 = \Theta' = (h_{ij})_{i=1,\ldots,d, j=1,\ldots,d'}.
\]

Then the following equation (15) is obtained by multiplying Eq. (12) with \((A_{ij})_{i,j=1}^d\).
\[
(\det(\Theta) \cdot I_d, \Theta') \begin{pmatrix} r_1 \\ \vdots \\ r_{d+d'} \end{pmatrix} = 0.
\] (15)

Hence any solution of (15) is also a solution of (12).

Now it is not hard to see that the solutions of (12) have \(d'\) many free variables. To be more precise, we write down explicitly the following \(d'\) tuples of \(R = (r_1, \ldots, r_{d+d'})\) which are the solutions of Eq. (15), hence of Eq. (12),
\[ R_i = (h_{i1}, \ldots, h_{id}, 0, \ldots, 0, -\det(\Theta), 0, \ldots, 0), \quad 1 \leq i \leq d'. \]  

(16)

So we have the following \( d' \) vectors in \( \mathcal{M}_1 \cap \mathcal{M}_2 \):

\[ h_{i, d_i+1} \mathcal{G}_1 + \cdots + h_{i, d_i+d_2} \mathcal{G}_{d_2} - \det(\Theta) \mathcal{G}_{d_2+i} \in \mathcal{M}_1 \cap \mathcal{M}_2, \quad i = 1, \ldots, d'. \]  

(17)

Moreover, by the particular forms of \( G_i \) in (7), (8), we know that vectors in (17) show that \( \mathcal{M}_1 \cap \mathcal{M}_2 \) occupies at least \( H(k) \otimes E' \). This completes the proof of the theorem since

\[ fd(\mathcal{M}_1) + fd(\mathcal{M}_2) - fd(\mathcal{M}) = (d_1 + d') + (d_2 + d') - (d_1 + d_2 + d') = d'. \]  

\[ \Box \]

4. An application: subadditivity of Samuel multiplicity

In commutative algebra, the additivity of Samuel multiplicity ([5, p. 273, p. 279], [9, p. 52]) is of fundamental importance for applications in algebraic geometry and a parallel version in operator theory is proved, say, for the Hardy space \( H^2(\mathbb{D}) \) and the Dirichlet space \( \mathcal{D} \) over the unit disk [7].

The purpose of this section is to show that Theorem 2 can lead to a subadditivity result (18) for Samuel multiplicities. Although the proof is short, this type of results seems to be new in operator theory literature, so we record it here.

For an operator \( T \in B(H) \) acting on a Hilbert space \( H \) such that \( \dim(H/T H) < \infty \), the Samuel multiplicity is defined by [6]

\[ e(T, H) = \lim_{k \to \infty} \frac{\dim(H/T^k H)}{k}, \]

which is well defined and is indeed a finite integer.

For an invariant subspace \( \mathcal{M} \subset \mathcal{D} \otimes \mathbb{C}^N \), we define \( e(\mathcal{M}^\perp) \) to be \( e(\mathcal{M}^\perp, S_2) \). Here \( S_2 \) is the compression of \( M_2 \) to \( \mathcal{M}^\perp \), the orthogonal complement of \( \mathcal{M} \) in \( \mathcal{D} \otimes \mathbb{C}^N \). This section concerns subadditivity of \( e(\mathcal{M}^\perp) \); namely, the relationship between \( e(\mathcal{M}_1^\perp \vee \mathcal{M}_2^\perp) \) and \( e(\mathcal{M}_1^\perp) + e(\mathcal{M}_2^\perp) \).

In this paper \( \vee \) denotes the closed span of two subspaces.

**Theorem 8.** For any two invariant subspaces \( \mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{D} \otimes \mathbb{C}^N, N \in \mathbb{N} \), we have

\[ e(\mathcal{M}_1^\perp \vee \mathcal{M}_2^\perp) \leq e(\mathcal{M}_1^\perp) + e(\mathcal{M}_2^\perp). \]  

(18)

Indeed, we have

\[ e(\mathcal{M}_1^\perp \vee \mathcal{M}_2^\perp) + e(\mathcal{M}_1^\perp \cap \mathcal{M}_2^\perp) \leq e(\mathcal{M}_1^\perp) + e(\mathcal{M}_2^\perp). \]  

(19)

**Proof.** In [7], the second author obtained

\[ fd(\mathcal{M}) + e(\mathcal{M}^\perp) = N \]  

(20)

for any invariant subspace \( \mathcal{M} \subset \mathcal{D} \otimes \mathbb{C}^N \). Meanwhile, by Theorem 2
Combining (20) and (21), one gets (19). □

**Remark.** One can also extend the above result to the complete NP case, but this will require one to extend the corresponding result from [7], which will cause unnecessary complexity for this paper.

5. Complete cellular indecomposable property

Next we introduce a stronger version of (CIP), which is clearly motivated by Theorems 1 and 2. In this section, $H$ denotes a Hilbert space of analytic functions over a domain $\Omega \subset \mathbb{C}$. Moreover, $M_z$, the multiplication by the coordinate function, is assumed to be bounded and all invariant subspaces are with respect to $M_z$.

**Definition 9.** $H$ has the complete cellular indecomposable property (CCIP) if for any invariant subspace $\mathcal{M} \subset H \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, two invariant subspaces $M_1, M_2 \subset \mathcal{M}$ such that

\[ fd(M_1) + fd(M_2) > fd(\mathcal{M}) \]

have a nontrivial intersection

$M_1 \cap M_2 \neq \{0\}$.

It is also natural to consider the following weaker definition, replacing $\mathcal{M}$ by the whole space $H \otimes \mathbb{C}^N$.

**Definition 10.** $H$ has (CCIP') if any two invariant subspaces $\mathcal{M}_1, \mathcal{M}_2 \subset H \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, such that

\[ fd(\mathcal{M}_1) + fd(\mathcal{M}_2) > N \]

have a nontrivial intersection

$\mathcal{M}_1 \cap \mathcal{M}_2 \neq \{0\}$.

The purpose of this section is to show that under a natural complementary condition (C) the weaker property (CCIP') implies the stronger version (CCIP).

**Definition 11.** $H$ is said to satisfy the complementary condition (C) if for any invariant subspace $\mathcal{M} \subset H \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, there is another invariant subspace $\mathcal{M}'$ such that
1. \( fd(M') + fd(M) = N; \)
2. \( fd(M \vee M') = N; \)
3. \( M \) and \( M' \) have a positive angle.

The condition (C) appears to be a fairly general property, and we conjecture that it holds for the Hardy space, the Dirichlet space, and even Bergman space. Yet even the Hardy space case is not previously known, and we intend to pursue these problems in a forthcoming work.

Here the angle between \( M_1, M_2 \), denoted by \( \text{angle}(M_1, M_2) \), is defined to be \( \theta \in [0, \frac{\pi}{2}] \) such that

\[
\cos(\theta) = \sup \{ |\langle f, g \rangle| : \|f\| = \|g\| = 1, \ f \in M_1, \ g \in M_2 \}. \tag{22}
\]

The invariant subspace \( M' \) satisfying the above conditions is called a complementary space of \( M \).

**Theorem 12.** If \( H \) satisfies the complementary condition (C), then the weaker (CCIP', Definition 10) implies the stronger (CCIP, Definition 9).

**Proof.** We argue by contradiction. Assume that \( H \) satisfies (CCIP') and there is an invariant subspace \( M \subset H \otimes \mathbb{C}^N \) such that it has two invariant subspaces \( M_1, M_2 \subset M \) satisfying

\[
fd(M_1) + fd(M_2) > fd(M) \quad \text{and} \quad M_1 \cap M_2 = \{0\}.
\]

Consider \( M' \), a complementary space of \( M \), and \( M_2 + M' \), which is automatically closed since \( M \) and \( M' \), hence \( M_2 \) and \( M' \), have a positive angle, by condition 3.

First, we show

\[
fd(M_2 + M') = fd(M_2) + fd(M'). \tag{23}
\]

Choose a \( \lambda \in \Omega \) such that it is a maximal fiber point for \( M, M', \) and \( M \vee M' \). In particular,

\[
\dim M(\lambda) = fd(M) \quad \text{and} \quad \dim M'(\lambda) = fd(M').
\]

By condition 2,

\[
\dim (M \vee M')(\lambda) = \dim [M(\lambda) + M'(\lambda)] = N,
\]

which implies

\[
M(\lambda) + M'(\lambda) = \mathbb{C}^N.
\]

On the other hand,

\[
M(\lambda) \cap M'(\lambda) = \{0\} \tag{24}
\]

since \( \dim M(\lambda) + \dim M'(\lambda) = N \) by condition 1.
It follows from (24) that \( \mathcal{M}_2(\lambda) \cap \mathcal{M}'(\lambda) = \{0\} \).

If we further assume that \( \lambda \) is a maximal fiber point for \( \mathcal{M}_2 \) and \( \mathcal{M}_2 + \mathcal{M}' \), then

\[
fd(\mathcal{M}_2 + \mathcal{M}') = \dim(\mathcal{M}_2 + \mathcal{M}')(\lambda)
\]

which is equal to

\[
\dim \mathcal{M}_2(\lambda) + \dim \mathcal{M}'(\lambda)
\]

since the latter two spaces have a trivial intersection. So Eq. (23) is proved.

Now by the assumption of (CCIP'),

\[
fd(\mathcal{M}_1) + fd(\mathcal{M}_2 + \mathcal{M}') = fd(\mathcal{M}_1) + fd(\mathcal{M}_2) + fd(\mathcal{M}')
\]

\[
> fd(\mathcal{M}) + fd(\mathcal{M}')
\]

\[
= N,
\]

(25)

hence there exists a nonzero intersection element

\[
m_1 = m_2 + m' \in \mathcal{M}_1 \cap (\mathcal{M}_2 + \mathcal{M}'),
\]

where \( m' \in \mathcal{M}' \) and \( m_i \in \mathcal{M}_i, i = 1, 2 \). So

\[
m_1 - m_2 = m' \in \mathcal{M} \cap \mathcal{M}' = \{0\}.
\]

So \( m' = 0 \) and

\[
m_1 = m_2 \in \mathcal{M}_1 \cap \mathcal{M}_2.
\]

Contradiction. \( \square \)

6. A generalization of Bourdon’s result on codimension-one property

Recall that Bourdon’s result says that the cellular indecomposable property (CIP) implies the well-known codimension-one property. For a subspace \( \mathcal{M} \) in a Hilbert space \( H \) of analytic functions over a domain \( \Omega \subset \mathbb{C} \), we say that \( \mathcal{M} \) has the division property at \( \lambda \in \Omega \) if

\[
(z - \lambda)g \in \mathcal{M}
\]

for some \( g \in H \) implies \( g \in \mathcal{M} \).

**Theorem.** (See Bourdon [2].) Suppose that \( H \) is a Hilbert space of analytic functions over the unit disk \( \mathbb{D} \) satisfying that
1. the polynomials are dense in $H$;
2. $H$ has the division property at zero;
3. the linear functional of evaluation at each point of $\mathbb{D}$ is continuous.

If $H$ has the cellular indecomposable property (CIP), then for any nonzero invariant subspace $M \subset H$,

$$\dim (M \ominus (z - \lambda)M) = 1, \quad |\lambda| < r_1(Mz),$$

where $r_1(Mz)$ is a positive constant.

Note that when considering vector-valued spaces, it is by now customary in operator theory to replace codimension-one by codimension-$N$. In this section we show that one can indeed obtain a codimension-$N$ version of Bourdon’s result using (CCIP), see Theorem 13. Moreover, we will prove a stronger result (Theorem 14) which is the main result of this section.

**Assumptions.** In this and the next sections, $\Omega$ denotes a domain in $\mathbb{C}$ containing the origin and $H$ is a Hilbert space of analytic functions over $\Omega$ satisfying that

1. $Mz$, the multiplication by the coordinate function $z$, is bounded on $H$;
2. the linear functional of evaluation at each point of $\Omega$ is continuous;
3. $H$ has the division property at each point $\lambda \in \Omega$.

It is known that the above condition (3) implies that for $\lambda \in \Omega$, $Mz - \lambda$ has a closed range, which is just the kernel of the evaluation at $\lambda$. So $Mz - \lambda$ is bounded below. For more details, see [12]. This in turn implies that $Mz - \lambda$ is semi-Fredholm. Since $Mz - \lambda$ has a trivial kernel, for any invariant $M$,

$$\dim (M \ominus zM) = \dim (M \ominus (z - \lambda)M)$$

(26)

by general Fredholm theory. Note that this co-dimension can be infinite.

**Theorem 13.** If $H$ has the complete cellular indecomposable property (CCIP), then any invariant subspace $M \subset H \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, satisfies

$$\dim (M \ominus zM) \leq N.$$  

(27)

Recall that (27) is proved for the Dirichlet space by Richter in [13]. Then it is improved to be an equality by the second author in [7],

$$\dim (M \ominus zM) = fd(M).$$

(28)

Note that $fd(M)$ is, by definition, at most $N$.

Next we show that this equality (28) indeed holds for any space with (CCIP). So Theorem 13 will follow from Theorem 14.
Theorem 14. If $H$ has the complete cellular indecomposable property (CCIP), then any invariant subspace $M \subset H \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, satisfies

$$\dim(M \oplus zM) = fd(M). \quad (29)$$

This result extends the one variable case of Corollary 4.6 in [3]. Before proving Theorem 14, the following two lemmas are needed.

Lemma 15. If an invariant subspace $M \subset H \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, satisfies (29), then $M$ has the division property at any maximal fiber point.

Proof. Let $fd(M) = d$. Choose a $\lambda \in \Omega$ to be a maximal fiber point for $M$ and take $f_1, \ldots, f_d \in M$ such that

$$f_1(\lambda), \ldots, f_d(\lambda)$$

form a basis for $M(\lambda)$.

Let $P_\lambda$ be the projection onto $M \oplus (z - \lambda)M$ and

$$h_i = P_\lambda(f_i), \quad 1 \leq i \leq d.$$.

Obviously,

$$f_i(\lambda) = h_i(\lambda),$$

hence $h_1(\lambda), \ldots, h_d(\lambda)$ are linearly independent. So are

$$h_1, \ldots, h_d,$$

which implies that $h_1, \ldots, h_d$ form a basis for $M \oplus (z - \lambda)M$ since

$$\dim(M \oplus (z - \lambda)M) = \dim(M \oplus zM) = fd(M) = d.$$.

If $(z - \lambda)g \in M$, write

$$(z - \lambda)g = c_1 h_1 + \cdots + c_d h_d + (z - \lambda)g'$$

for some $g' \in M$. Let $z = \lambda$ in the above identity, one has that

$$c_1 = \cdots = c_d = 0$$

since $h_1(\lambda), \ldots, h_d(\lambda)$ are linearly independent. Hence $g = g' \in M$, as desired. \qed

Lemma 16. If $g \in H \otimes \mathbb{C}^N$, $N \in \mathbb{N}$, and $g(\lambda) \neq 0$, then $[g]$ has the division property at $\lambda$. 
Proof. We first observe that both $fd([g])$ and $\dim([g] \ominus z[g])$ are one. Then note that $\lambda$ is a maximal fiber point for $[g]$. So the proof follows from Lemma 15. □

Now we are ready to prove Theorem 14.

**Proof of Theorem 14.** Given (CCIP), we first show the following claim.

**Claim.** If $\lambda \in \Omega$ is a maximal fiber point, namely, $\dim M(\lambda) = fd(M)$, then $M$ has the division property at $\lambda$.

**Proof.** Assume that $(z - \lambda)g \in M$, and we need to show $g \in M$. We first deal with the case $g(\lambda) \neq 0$.

Let $fd(M) = d$ and pick $f_1, \ldots, f_d \in M$ such that

$$\dim(\text{span}\{f_1(\lambda), \ldots, f_d(\lambda)\}) = d. \quad (30)$$

Now consider

$$M' = [g, f_1, \ldots, f_d],$$

the invariant subspace generated by $g, f_1, \ldots, f_d$. Observe that

$$fd([g, f_1, \ldots, f_d]) = fd([(z - \lambda)g, f_1, \ldots, f_d])$$

and since $(z - \lambda)g \in M$, we have

$$fd(M') = d.$$ 

So,

$$fd([g]) + fd([f_1, \ldots, f_d]) > fd(M')$$

and by (CCIP), we have

$$[g] \cap [f_1, \ldots, f_d] \neq \{0\}.$$

**Subclaim.** $\dim([f_1, \ldots, f_d] \ominus (z - \lambda)[f_1, \ldots, f_d]) = d.$

**Proof.** Denote $[f_1, \ldots, f_d]$ by $M_1$ and it is easy to see

$$\dim(M_1 \ominus (z - \lambda)M_1) \leq d \quad (31)$$

since $M_1$ is generated by $d$ elements. Next decompose

$$f_i = f_i^1 + f_i^2, \quad 1 \leq i \leq d,$$
with $f^1_i \in \mathcal{M}_1 \ominus (z - \lambda)\mathcal{M}_1$ and $f^2_i \in (z - \lambda)\mathcal{M}_1$. To show the equality in (31) it is sufficient to show that

$$\{{f^1_1, \ldots, f^1_d}\}$$

are linearly independent in $\mathcal{M}_1$. To show (32) it is sufficient to show that

$$\{{f^1_1(\lambda), \ldots, f^1_d(\lambda)}\}$$

are linearly independent in $\mathbb{C}^N$. Now (33) follows from

$$f_i(\lambda) = f^1_i(\lambda)$$

and the fact that $\{f_i(\lambda)\}$ are linearly independent (30). So the subclaim is proved.

Let us continue with the proof of the claim. By Lemma 15, Lemma 16, and the subclaim, both invariant subspaces $[g]$ and $[f_1, \ldots, f_d]$ have the division property at $\lambda$.

It is easy to see that if two invariant subspaces have the division property at $\lambda$, then so does their intersection, if nontrivial. Moreover, if an invariant subspace has the division property at $\lambda$, then it contains a function which is nonvanishing at $\lambda$. So we can pick

$$h \in [g] \cap [f_1, \ldots, f_d]$$

such that

$$h(\lambda) \neq 0.$$  

Meanwhile, there are polynomials $p_n, q^1_n, \ldots, q^d_n$ such that

$$p_n g \to h \quad \text{and} \quad \sum_{i=1}^d q^i_n f_i \to h, \quad \text{as} \quad n \to \infty. \quad (34)$$

Because the evaluation at $\lambda$ is continuous,

$$p_n(\lambda) g(\lambda) \to h(\lambda), \quad \text{as} \quad n \to \infty.$$  

So

$$p_n(\lambda) \to c = \frac{h(\lambda)}{g(\lambda)}, \quad \text{as} \quad n \to \infty. \quad (35)$$

Note that $c$ is a nonzero constant. Hence, (34) and (35) can be rewritten such that

$$\left\| \frac{p_n(z) - p_n(\lambda)}{z - \lambda} (z - \lambda) g - \sum_{i=1}^d q^i_n f_i + c g \right\|_{H \otimes \mathbb{C}^N} \to 0, \quad \text{as} \quad n \to \infty. \quad (36)$$

Since $(z - \lambda) g \in \mathcal{M}$ and $f_i \in \mathcal{M}$, we have $g \in \mathcal{M}$. The claim is proved when $g(\lambda) \neq 0$. 
For general \( g \) such that \((z - \lambda)g \in \mathcal{M}\), we still need to show \( g \in \mathcal{M} \). Write

\[
g = (z - \lambda)^c g_1
\]

for some positive integer \( c \) and \( g_1(\lambda) \neq 0 \). Note that \( g_1 \in H \) since we assume that \( H \) has the division property at \( \lambda \). Then, similar to the above arguments, by picking an \( h \),

\[
h \in [g_1] \cap [f_1, \ldots, f_d]
\]

such that \( h(\lambda) \neq 0 \), one has a similar statement as (36) which will show that

\[
g_1 \in [(z - \lambda)g_1, f_1, \ldots, f_d].
\]

So

\[
g = (z - \lambda)^c g_1 \in [(z - \lambda)^{c+1} g_1, (z - \lambda)^c f_1, \ldots, (z - \lambda)^c f_d] \subset \mathcal{M}.
\]

The claim is proved. \( \square \)

To continue with the proof of Theorem 14, it is an easy general fact that

\[
\dim(\mathcal{M} \ominus z\mathcal{M}) = \dim(\mathcal{M} \ominus (z - \lambda)\mathcal{M}) \geq fd(\mathcal{M}). \tag{37}
\]

If the above inequality (37) is strictly greater, then we can find \( d + 1 \) linearly independent functions

\[
g_1, \ldots, g_d, g_{d+1} \in \mathcal{M} \ominus (z - \lambda)\mathcal{M}.
\]

On the other hand,

\[
g_1(\lambda), \ldots, g_d(\lambda), g_{d+1}(\lambda)
\]

must be linearly dependent in \( \mathbb{C}^N \) since

\[
fd(\mathcal{M}) = d,
\]

so there are not all zero constants \( c_1, \ldots, c_{d+1} \) such that

\[
\sum_{i=1}^{d+1} c_i g_i(\lambda) = \left( \sum_{i=1}^{d+1} c_i g_i \right)(\lambda) = 0,
\]

which implies that, by the division property of \( \mathcal{M} \) at \( \lambda \), or by the claim,

\[
\sum_{i=1}^{d+1} c_i g_i \in (z - \lambda)\mathcal{M}.
\]

Contradiction. This completes the proof of Theorem 14. \( \square \)
7. A partial converse for CIP

Recall that Bourdon’s result states that (CIP) implies the codimension-one property. Moreover, Bourdon proves a partial converse: If \( M_z \) on \( H \) is such that each nontrivial invariant subspace has codimension one,

\[
\text{cod}(\mathcal{M}) \triangleq \dim(\mathcal{M} \ominus z\mathcal{M}) = 1,
\]

then any two nontrivial invariant subspaces \( \mathcal{M}_1, \mathcal{M}_2 \) have a zero angle [2].

Note that the converse of Bourdon’s result is not true as illustrated by the following example. Let 

\[
d\mu = dA + \chi_S|dz|, \text{ where } A \text{ is the (normalized) area measure on the disk, } |dz| \text{ the (normalized) Lebesgue measure on the unit circle and } \chi_S \text{ the characteristic function of the upper semicircle } S \subset \mathbb{T}.
\]

Furthermore, let \( H = P^2(\mu) \), the closure of polynomials in \( L^2(d\mu) \). Then it is well known to experts that, just like the Bergman space, one can find two zero sequences of \( H \) such that their union is not a zero sequence. This can also be shown directly by imitating the proof of Horowitz’s theorem, see Theorem 3 of Chapter 4 in [4]. Then for these two zero sequences, their corresponding invariant subspaces have a trivial intersection. On the other hand, by [1], any nontrivial invariant subspace of \( H \) has codimension one.

In Section 6 we showed that (CCIP) implies the codimension-\( N \) property; indeed, we proved a stronger result (Theorem 14); namely, given (CCIP), one has

\[
\text{cod}(\mathcal{M}) = \dim(\mathcal{M} \ominus z\mathcal{M}) = fd(\mathcal{M})
\]

for any invariant subspace \( \mathcal{M} \subset H \otimes \mathbb{C}^N, N \in \mathbb{N} \).

**Definition 17.** We say that an invariant subspace \( \mathcal{M} \subset H \otimes \mathbb{C}^N, N \in \mathbb{N} \), satisfies the **cod-fd condition** if its codimension is equal to the fiber dimension; namely, \( \text{cod}(\mathcal{M}) = fd(\mathcal{M}) \).

The purpose of this section is to give a partial converse for Theorem 14. Recall that \( H \) in this section satisfies the assumptions in Section 6.

**Theorem 18.** Suppose that each invariant subspace of \( H \otimes \mathbb{C}^N, N \in \mathbb{N} \), satisfies the cod-fd condition. If two invariant subspaces \( \mathcal{M}_1, \mathcal{M}_2 \subset H \otimes \mathbb{C}^N \) satisfy

\[
fd(\mathcal{M}_1) + fd(\mathcal{M}_2) > N,
\]

then \( \angle(\mathcal{M}_1, \mathcal{M}_2) = 0 \).

Observe that if \( \mathcal{M}_1 \cap \mathcal{M}_2 \neq \{0\} \), then \( \angle(\mathcal{M}_1, \mathcal{M}_2) = 0 \). Also observe that it is not enough to just assume that \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have the cod-fd condition.

**Proof of Theorem 18.** Let \( fd(\mathcal{M}_2) = t \) and take \( g_1, \ldots, g_t \in \mathcal{M}_2 \) such that for some point, hence for almost every point, \( \lambda \in \Omega \),

\[
\dim(\text{span}\{g_1(\lambda), \ldots, g_t(\lambda)\}) = t.
\]

Define
\[N_0 = M_1\]

and

\[N_i = [M_1, g_1, \ldots, g_i], \quad i = 1, \ldots, t.\]

Then the following finite, increasing sequence

\[\{fd(N_i)\}_{i=0}^t\]

has to stabilize at some stage, due to the fact that \(fd(N_i) \leq N\) and the assumption (38). That is, there exists an \(r < t\) such that

\[fd(N_r) = fd(N_{r+1}) = d\]

for the first time.

Choose \(\lambda_0 \in \Omega\) such that

\[\dim(N_r(\lambda_0)) = \dim(N_{r+1}(\lambda_0)) = d\]

and

\[g_1(\lambda_0), \ldots, g_{r+1}(\lambda_0)\]

are linearly independent. Observe that for any \(\lambda \in \Omega\),

\[fd(N_{r+1}) = fd([M_1, g_1, \ldots, g_r, (z - \lambda)g_{r+1}]).\]

Then \(\lambda_0\) is also a maximal fiber point for

\[M' = [M_1, g_1, \ldots, g_r, (z - \lambda_0)g_{r+1}]\]

since

\[\dim(M'(\lambda_0)) = \dim(N_r(\lambda_0)) = \dim(N_{r+1}(\lambda_0)) = fd(N_{r+1}) = fd(M').\]

By the assumption of the theorem, \(M'\) has the cod-fd condition. Hence by Lemma 15, \(M'\) has the division property at \(\lambda_0\). It follows that

\[g_{r+1} \in [M_1, g_1, \ldots, g_r, (z - \lambda_0)g_{r+1}].\]

So there exist functions \(m_n \in M_1\) and polynomials \(p_n^1, \ldots, p_n^{r+1}\) such that

\[\left\|m_n + p_n^1 g_1 + \cdots + p_n^r g_r + (z - \lambda_0)p_n^{r+1} g_{r+1} - g_{r+1}\right\|_{H \otimes C^N} \to 0, \quad n \to \infty. \quad (40)\]
Let
\[ R_n = p_1^ng_1 + \cdots + p_r^ng_r + (z - \lambda_0)p_{r+1}^ng_{r+1}, \]
and we claim
\[ \inf_n \| R_n \|_{H \otimes \mathbb{C}^N} > 0. \quad (41) \]
Otherwise, there is a subsequence
\[ \| R_{n_j} \|_{H \otimes \mathbb{C}^N} \to 0, \quad \text{as } j \to \infty. \]
Meanwhile, the evaluation at \( \lambda_0 \) is continuous, so
\[ R_{n_j}(\lambda_0) \to 0, \quad \text{as } j \to \infty, \]
which implies that
\[ g_{r+1}(\lambda_0) \in \text{span}\{g_1(\lambda_0), \ldots, g_r(\lambda_0)\}. \]
This contradicts the fact that \( g_1(\lambda_0), \ldots, g_{r+1}(\lambda_0) \) are linearly independent, or (39).

By (40) and (41), we also have the fact that
\[ \inf_n \| m_n \|_{H \otimes \mathbb{C}^N} > 0. \]
Hence (40) leads to
\[ \left\| \frac{m_n}{\| m_n \|} + \frac{R_n}{\| R_n \|} \right\|_{H \otimes \mathbb{C}^N} \to 0, \quad \text{as } n \to \infty. \]
This implies that \( \text{angle}(\mathcal{M}_1, \mathcal{M}_2) = 0. \quad \square \)

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