Universal exponents and tail estimates in the enumeration of planar maps

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Abstract

It has been observed that for most classes of planar maps, the number of maps of size \( n \) grows asymptotically like \( c \cdot n^{-5/2} \gamma^n \), for suitable positive constants \( c \) and \( \gamma \). It has also been observed that, if \( d_k \) is the limit probability that the root vertex in a random map has degree \( k \), then again for most classes of maps the tail of the distribution is asymptotically of the form \( d_k \sim c \cdot k^{1/2} q^k \) as \( k \to \infty \), for positive constants \( c, q \) with \( q < 1 \).

We provide a rationale for this universal behaviour in terms of analytic conditions on the associated generating functions. The fact that generating functions for maps satisfy as a rule a quadratic equation with one catalytic variable, allows us to identify a critical condition implying the shape of the above-mentioned asymptotic estimates. We verify this condition on several well-known families of planar maps.

Keywords: planar maps, quadratic method, asymptotic expansions.
1 Introduction

A planar map is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane. A map is rooted if a vertex $v$ and an edge $e$ incident with $v$ are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of $e$ is called the root-face and is usually taken as the outer face. All maps in this paper are rooted.

The enumeration of rooted maps is a classical subject, initiated by Tutte in the 1960’s. He introduced the technique now called “the quadratic method” in order to compute the number $M_n$ of rooted maps with $n$ edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!} 3^n.$$ 

This was later extended by Tutte and his school to several classes of planar maps: 2-connected, 3-connected, bipartite, Eulerian, triangulations, quadrangulations, etc. Using the previous formula, Stirling’s estimate gives $M_n \sim c \cdot n^{-5/2} 12^n$, where $c > 0$ is a constant. In all cases where a “natural” condition is imposed on maps, the asymptotic estimates turn out to be of this kind:

$$c \cdot n^{-5/2} \gamma^n.$$ 

The constants $c$ and $\gamma$ depend on the class under consideration, but one gets systematically an $n^{-5/2}$ term in the estimate.

This phenomenon is discussed by Banderier et al. [1]: ‘This generic asymptotic form is “universal” in so far as it is valid for all known “natural families of maps”.’ The main goal of this paper is to provide an explanation for this universal phenomenon, based on the analysis of the associated counting generating functions.

We turn now to degree distributions. Given a class of planar maps, let $d_{n,k}$ be the probability that the root-vertex has degree $k$ in a map with $n$ edges, and assume that the following limit exists for all $k \geq 1$: $d_k = \lim_{n \to \infty} d_{n,k}$.

Liskovets observed in [7] that for several natural classes of planar maps, the estimates of $d_k$ for large $k$ are of the form

$$d_k \sim c \cdot k^{1/2} q^k,$$

where again the constants $c > 0$ and $0 < q < 1$ depend on the class, but

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the critical exponent $1/2$ appears to be universal, in the sense that it is the same for all classes of planar maps. However we find no attempt to explain this universal phenomenon in [7]. Our main result provides again such an explanation.

It must be noted that there are exceptions to this behaviour. For instance, the class of rooted maps with a unique face: since they are in bijection with plane trees, the subexponential term is $n^{-3/2}$ in this case. Plane trees are also an exception for the degree of the root, since we have the well-known exact expression $d_k = k(1/2)^{k+1}$. Outerplanar maps, which can be encoded by trees, are another exception. Also, maps having some kind of symmetry usually do not follow the universal pattern.

In the following table we list several classes of maps that conform to the universal exponents $n^{-5/2}$ for the univariate enumeration, and to $k^{1/2}$ for the tail of the limit distribution of the root-vertex degree. We display in each case the constants $\gamma$ and $q$ that appear in (1) and (2).

<table>
<thead>
<tr>
<th>Class of maps</th>
<th>$\gamma$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arbitrary</td>
<td>12</td>
<td>5/6</td>
</tr>
<tr>
<td>Eulerian</td>
<td>8</td>
<td>$\sqrt{3}/2$</td>
</tr>
<tr>
<td>3-connected</td>
<td>4</td>
<td>1/2</td>
</tr>
<tr>
<td>Loopless</td>
<td>$256/27$</td>
<td>$3/4$</td>
</tr>
<tr>
<td>2-connected</td>
<td>$27/4$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>Bipartite</td>
<td>8</td>
<td>$3/4$</td>
</tr>
</tbody>
</table>

The last two families have not been discussed explicitly with respect to the degree of the root-vertex, hence we treat them in some detail in Section 2. To illustrate the applicability of the method, we also discuss the related problem of counting near-triangulations with respect to the degree of the root-face.

The goal of this paper is to provide an explanation for these universal phenomena, based on a detailed analysis of the quadratic method. In order to motivate the statements that follow, let us recall the basic technique for counting planar maps. Let $M_{n,k}$ be the number of maps with $n$ edges and in which the degree of the root-face is equal $k$. Let $M(z,u) = \sum m_{n,k} u^k z^n$ be the associated generating function. As shown by Tutte [9], $M(z,u)$ satisfies the quadratic equation

$$M(z,u) = 1 + zu^2 M(z,u)^2 + u z \frac{u M(z,u) - M(z;1)}{u - 1}.$$
Completing the square and setting \( y(z) = M(z, 1) \) for simplicity, the former equation can be rewritten as

\[
[G_1(z, u, y(z))M(z, u) + G_2(z, u, y(z))]^2 = H(z, u, y(z))
\]

where the \( G_i \) and \( H \) depend on the variables indicated (in this particular case only \( H \) depends on \( y \)). The quadratic method consists on binding variables \( z \) and \( u \), assuming that there exists a function \( u(z) \) such that \( H(z, u(z), y(z)) = 0 \) identically. Because of the square in the left-hand side of (4), the derivative \( H_u(z, u(z), y(z)) \) with respect to \( u \) also vanishes. From the system of equations

\[
H(z, u(z), y(z)) = 0, \quad H_u(z, u(z), y(z)) = 0.
\]

one eliminates \( y(z) \) to find \( u(z) \), and then from \( H(z, u(z), y(z)) = 0 \). Once we know \( y(z) = M(z, 1) \), from Equation (3) we obtain \( M(z, u) \). If we carry out this program in this particular case, we find that

\[
y(z) = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2} = 1 + 2z + 9z^2 + 54z^3 + \cdots,
\]

from which we can deduce the explicit form for the numbers \( M_n \). An explicit expression is obtained also for \( M(z, u) \), which encodes completely the distribution of the degree of the root-face. Since planar maps are closed under duality, this is the same distribution as the degree of the root-vertex.

In order to estimate \( M_n \) and \( M_{n,k} \) we use singularity analysis. The singular expansion of \( y(z) \) at its dominant singularity \( z = 1/12 \) is of the form

\[
y(z) = y_0 + y_2(1 - 12z) + y_3(1 - 12z)^{3/2} + O((1 - 2z)^2).
\]

By transfer theorems \([5]\), we obtain \( M_n = [z^n]y(z) \sim c \cdot n^{-5/2}12^n \).

The key point is that there is no square-root term \((1 - 12z)^{1/2}\) in the singular expansion, hence the subexponential term is \( n^{-5/2} \) instead of the classical \( n^{-3/2} \) term that arises in the enumeration of trees.

It can be checked that, for \( u \) near 1, the singularity of \( M(z, u) \) as a function of \( z \) does not change and we have a singular expansion

\[
M(z, u) = y_0(u) + y_2(u)(1 - 12z) + y_3(u)(1 - 12z)^{3/2} + O((1 - 2z)^2),
\]

where the \( y_i(u) \) are analytic and can be computed explicitly. The probability that the root-face has degree \( k \) in a random map with \( n \) edges is equal to \( M_{n,k}/M_n \). Hence the probability generating function (PGF) of the root-face degree is

\[
p_n(u) = \frac{[z^n]M(z, u)}{[z^n]y(z)}.
\]
It follows that the PGF of the limiting distribution is equal to

\[ p(u) = \lim_{n \to \infty} p_n(u) = \frac{y_3(u)}{y_3} = \frac{u \sqrt{3}}{(2 + u)(6 - 5u)^3}, \]

whose dominant singularity is at \( u = 6/5 \). Finally, again by singularity analysis, we obtain the tail estimate

\[ d_k = [u^k]p(u) \sim c \cdot k^{1/2} \left( \frac{5}{6} \right)^k. \]

Our main result says (informally) that the above situation is typical for most families of planar maps: the univariate expansion of \( y(z) \) has \((1 - z/z_0)^{3/2}\) as dominant term, where \( z_0 \) is the singularity of \( y(z) \). Furthermore, the singularity in \( z \) of \( M(z, u) \) does not change if \( u \) is small, and the singular expansion of the limit PFG \( p(u) \) has \((1 - qu)^{3/2}\) as the dominant term, where \( 1/q \) is the singularity of \( p(u) \). More precisely, we concentrate on situations where

\[ F(z, u) = \sum_{n,k} f_{n,k} u^k z^n, \quad y(z) = \sum_n y_n z^n \]

are the unique solutions of equation (4). Usually \( G_1, G_2, \) and \( H \) are polynomials, but the proof of our main result requires only that these functions are analytic in a proper range. In our applications we also have \( y(z) = F(z, 1) \) or \( y(z) = F(y, 0) \), but this is not be required either.

If \((z_0, y_0, u_0)\) is a critical point of the system (5), the Jacobian

\[
\begin{vmatrix}
H_y & H_u \\
H_{uy} & H_{uu}
\end{vmatrix} = \begin{vmatrix}
H_y & 0 \\
H_{uy} & H_{uu}
\end{vmatrix} = H_y H_{uu}
\]

must vanish, that is, \( H_y H_{uu} = 0 \) at \((z_0, u_0, y_0)\). Interestingly enough, for all the natural classes of planar maps amenable to the quadratic method, it turns out that \( H_{uu} = 0 \) and \( H_y \neq 0 \). The critical condition is then

\[ H_{uu}(z_0, u_0, y_0) = 0. \]

This condition is easy to check, since we always work in the realm of algebraic functions and algebraic numbers. Actually the system \( H = H_u = H_{uu} = 0 \) has (usually) only finitely many solutions. For the running example we are using, we have \((z_0, y_0, u_0) = (1/12, 6/5, 4/3)\) and

\[ (6) \quad H = 4(u - 1)u^3z^2y + u^4z^2 - 4u^4z + 6u^3z - 2u^2z + u^2 - 2u + 1. \]

A simple check gives \( H_{uu}(1/12, 6/5, 4/3) = 0 \). It also gives \( H_{uu}(1/12, 6/5, 4/3) \neq 0 \), which is necessary, together with the other non-vanishing conditions, for the method to work properly.
We are ready to state our main result in which we use the notation
\[ C(z_0, \delta) = \{ z \in \mathbb{C} : |z| < z_0 + \delta \} \setminus [z_0, \infty) \]
for a slit circle.

**Theorem 1.1** Let \( F(z, u) = \sum_{n,k} f_{n,k} u^k z^n \) and \( y(z) = \sum_n y_n z^n \) with \( f_{n,k} \geq 0 \) and \( y_n \geq 0 \) be the unique solutions of the equation
\begin{equation}
(G_1(z, u, y(z)) F(z, u) + G_2(z, u, y(z)))^2 = H(z, u, y(z)),
\end{equation}
with functions \( G_1, G_2 \) and \( H \) that are analytic for \( |z| < z_0 + \eta, |u| < u_0 + \eta, |y| < y_0 + \eta \) for some \( \eta > 0 \), where \( z_0 > 0 \) denotes the radius of convergence of \( y(z) \) that satisfies \( 0 < y_0 = y(z_0) < \infty \) and \( u_0 > 0 \) denotes the radius of convergence of the function \( F(z_0, u) \). Assume also that \( z = z_0 \) and \( u = u_0 \) are the only singularities on the circles of convergence of \( y(z) \) and \( F(y_0, u) \) and that they can be continued analytically to slit circles \( C(z_0, \delta') \) and \( C(u_0, \delta'') \), respectively.

Furthermore assume that
\[ H(z_0, u_0, y_0) = 0, \quad H_u(z_0, u_0, y_0) = 0, \quad H_{uu}(z_0, u_0, y_0) = 0 \]
together with
\[ G_1 \neq 0, \quad H_y \neq 0, \quad H_{uy} \neq 0, \quad H_{uu} \neq 0, \quad H_z H_{uy} \neq H_y H_{zu}, \]
\[ H_{uu} H_{uy}^2 - H_y H_{uy} H_{uuu} + 3 H_y H_{uuy} H_{uu} \neq 0 \]
evaluated at \((z_0, u_0, y_0)\). Then

(i) the following asymptotic estimate holds for some constant \( c > 0 \):
\[ y_n \sim c \cdot z_0^{-n} n^{-5/2}, \]

(ii) and for every integer \( k \geq 0 \) the limit \( d_k = \lim_{n \to \infty} \frac{f_{n,k}}{y_n} \) exists and we have, uniformly for \( k \leq C \log n \),
\[ \frac{f_{n,k}}{d_n} \sim \bar{c} \cdot q^k k^{1/2} \]
for some \( \bar{c} > 0 \), \( q = 1/u_0 \) and any constant \( C > 0 \), in particular
\[ d_k \sim \bar{c} \cdot q^k k^{1/2} \quad (k \to \infty). \]

For completeness, we check all the condition in the statement for \( H(z, u, y) \) as in (6), evaluated at the critical point \((z_0, u_0, y_0) = (1/12, 6/5, 4/3)\). In addition to \( H \), we need \( G_1 = 2(1-u)u^2z \). Then
\begin{align*}
G_1 &= -\frac{6}{125}, \quad H_y = \frac{6}{625}, \quad H_{uy} = \frac{9}{125}, \quad H_{uu} = -\frac{50}{9}, \\
H_z H_{uy} - H_y H_{zu} &= \frac{288}{15625}, \quad H_{uu} H_{uy}^2 - H_y H_{uuy} H_{uu} + 3 H_y H_{uuy} H_{uu} = \frac{43}{625}.
\end{align*}
The proof of the main result cannot be presented here. In Section 2 we present further examples of natural families of maps to which Theorem 1.1 applies. The paper concludes with some remarks on the degree distribution in maps and graphs.

## 2 Further Examples

Our first example, arbitrary planar maps, has been discussed in the Introduction. We continue with 2-connected maps (also called non-separable in the literature). In what follows, the size of a map is its number of edges.

### 2-connected maps

A map is 2-connected if it has no cut vertex. Here the resulting equation for the corresponding generating function $B(z, u)$ is

$$B(z, u)^2 - (u^2 z + uB(z) + 1)B(z, u) + (u^3 + u)B(z, 1)z + u^2 z - uz = 0.$$  

Completing the square, the corresponding $H$ function in (4) is equal to

$$H(z, u, y) = u^2 y^2 - (2u^3 z + 2u)y + u^4 z^2 - 2u^2 z + 4uz + 1.$$  

The dominant singularity of $B(z, 1)$ is at $z_0 = 4/27$. Then $y_0 = y(z_0) = 4/3$, and $u_0 = u(z_0) = 3/2$. We check that $H_{uu}(4/27, 3/2, 4/3) = 0$ and can apply Theorem 1.1; all non-vanishing conditions are satisfied.

### Bipartite and Eulerian maps

Let $B(z, u)$ denote the generating function of bipartited maps, where $u$ marks the degree of the root-face. Then we have

$$B(z, u) = 1 + u^2 zB(z; u)^2 + u^2 z \frac{B(z, u) - B(z, 1)}{u^2 - 1}.$$  

The fact that the equation is in terms of $u^2$ is because in a bipartite map the root face has even degree. We then obtain

$$H(z, u, y) = (4u^6 z^2 - 4u^4 z^2)y + u^4 - 2u^2 + 6u^4 z + 1 - 2u^2 z + u^4 z^2 - 4u^6 z.$$  

The usual computations give the critical values $z_0 = 1/8$, $u_0 = 2/\sqrt{3}$, $y_0 = 5/4$. Again we check the critical condition $H_{uu}(z_0, u_0, y_0) = 0$, together with the additional non-vanishing conditions so that we can apply Theorem 1.1. By duality, we obtain the distribution of the degree of the root-vertex degree in Eulerian maps.

In order to obtain the distribution of the degree of the root-vertex in bipartite maps, we can use a bijection between 2-colored maps and 3-colored triangulations ([6]) which implies that the number of bipartite maps of size $n$
whose root-vertex has degree $k$ is the same as the number of bipartite maps of size $n$ whose root-face has degree $2k$. Hence both distributions are essentially the same, and the critical value $u_0$ for the degree of the root-vertex is the square of the corresponding value for the root-face, that is, $u_0 = (2/\sqrt{3})^2 = 4/3$.

**Triangulations**

This example is a bit different and illustrates the flexibility of the method. In this section all maps are simple, that is, they have no multiple edges. A triangulation is a map in which every face is a triangle. A near-triangulation is a map in which all faces are triangles except possibly the outer face, and in addition there is no chord. Vertices in the outer face of a near-triangulation are called external, and internal otherwise. To enumerate triangulations, Tutte [8] proceeded as follows. Let $T_{n,k}$ be the number of near-triangulations with $n$ internal vertices and $k + 3$ external vertices, and let $T(z, u) = \sum T_{n,k} z^n u^k$. Notice that $T(z, 0)$ enumerates triangulations according to the number of internal vertices.

By removing the triangle containing the root-edge in a near triangulation one obtains a sequence of near-triangulations. This decomposition gives rise to the equation

$$u^2 T(z, u)^2 + (z + zuy(z) - u - u^2)T(z, u) + u - zy(z) = 0,$$

where $y(z) = T(z, 0)$. The basic function to be analyzed is once more the discriminant of the quadratic equation, and is equal to

$$H(z, u, y) = z^2 u^2 y^2 + 2(z^2 u + zu^2 - zu^3)y + (z - u - u^2)^2 - 4u^3.$$

The critical values are $z_0 = 27/256$, $u_0 = 3/16$, $y_0 = 32/27$. Notice that in this case $u_0 < 1$; this is because the univariate generating function is now obtained by setting $u = 0$. But this is only a detail and the analysis is the same. Indeed, we check the critical condition $H_{uu}(z_0, u_0, y_0) = 0$, together with the non-vanishing conditions.

If we denote by $S(z, u)$ the generating function of near-triangulations, where now $z$ marks the total number of vertices minus 3, then $S(z, u) = T(z, zu)$. The critical values $(z_0, u_0)$ for $S(z, u)$ are determined by $z_0 = z_0$ and $z_0 u_0 = u_0$, so that $u_0 = u_0 = z_0 = 16/9$, and we can apply Theorem 1.1.

**References**


