A Note on the Lattice Definability of Bernstein Algebras*

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ABSTRACT

We prove that for exclusive and normal Bernstein algebras, the type is determined by the lattice of subalgebras. It is also seen that the lattice of subalgebras does not determine an arbitrary Bernstein algebra up to isomorphism.

0. INTRODUCTION

The origin of Bernstein algebras lies in genetics and in the study of the stationary evolution operators; see Lyubich [10]. Holgate [7] was the first to give a formulation of Bernstein's problem in the language of nonassociative algebras. For a summary of known results see Wörz-Busekros [11, Chapter 9].

On the other hand, the study of the relationship between the lattice of subalgebras and other structural properties of the algebra can be found in Barnes [3, 2] for associative and Lie algebras and in J. A. Laliena [8] for alternative algebras. For Jordan algebras similar studies have been done by J. A. Laliena [9] and completed by J. A. Anquela in [1].

Concerning Bernstein algebras, a first approach to the problem of determining the algebra by knowing its lattice of subalgebras can be found in [4] and in [6].

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1. PRELIMINARIES

A finite-dimensional commutative algebra $A$ over a field $K$ is called baric if there exists a nontrivial homomorphism $\omega : A \to K$, called the weight homomorphism. A baric algebra is called a Bernstein algebra if

$$(x^2)^2 = \omega(x)^2 x^2$$

for all $x$ in $A$.

In the following, let $K$ be a commutative infinite field of characteristic different from 2. Let us list several results on Bernstein algebras which can be found in [11].

For every Bernstein algebra, the nontrivial homomorphism $\omega : A \to K$ is uniquely determined.

Every Bernstein algebra $A$ possesses at least one nonzero idempotent.

Every Bernstein algebra $A$ with nonzero idempotent $e$ can be decomposed into the internal direct sum of subspaces (we will denote such a sum by $\oplus$):

$$A = Ke \oplus \text{Ker } \omega,$$

with

$$\text{Ker } \omega = U_e \oplus V_e,$$

where

$$U_e = \{ex | x \in \text{Ker } \omega \} = \{x \in A | ex = \frac{1}{2}x\},$$

$$V_e = \{x \in A | ex = 0\}.$$

The subspaces $U_e$ and $V_e$ of $A$ satisfy

$$U_eV_e \subseteq U_e, \quad V_e^2 \subseteq U_e, \quad U_e^2 \subseteq V_e, \quad U_e V_e^2 = \langle 0 \rangle.$$

Although the decomposition of a Bernstein algebra depends on the choice of the idempotent $e$, the dimension of the subspace $U_e$ of $A$ is an invariant of $A$. If $\dim_K A = n + 1$, then one can associate to $A = Ke \oplus U_e \oplus V_e$ a pair of integers $(r + 1, s)$, called the type of $A$, where $r = \dim_K U_e$, $s = \dim_K V_e$, so that $r + s = n$. In the same way, Wörz-Busekros shows in [11] that $\dim_K U_e^2$ and $\dim_K (U_e V_e + V_e^2)$ are also invariants of the algebra $A$. 
A Bernstein algebra is said to be **trivial** if \((\ker \omega)^2 = 0\), **exclusive** if \(U_r^2 = 0\), and **normal** if \(U_r V_r + V_r^2 = 0\). Note that these definitions do not depend on the choice of the idempotent.

Let \(A\) be an algebra over a commutative field \(K\). We denote by \(\mathcal{L}(A)\) the lattice of all subalgebras of \(A\). By an \(\mathcal{L}\)-isomorphism (lattice isomorphism) of the algebra \(A\) onto an algebra \(B\) over the same field, we mean an isomorphism

\[
\mathcal{L}(A) \rightarrow \mathcal{L}(B)
\]

of \(\mathcal{L}(A)\) onto \(\mathcal{L}(B)\).

We put \(l(A)\), the **length** of \(A\), for the supremum of the lengths of all the chains in \(\mathcal{L}(A)\) (by the length of a chain we mean its cardinality minus one). Clearly we have \(\dim_K A \geq l(A)\), and if the algebra \(A\) is finite-dimensional then \(l(A)\) is the maximum, not only the supremum.

In [4], the following results are proved:

**Theorem 1.1.** For all subalgebras of a Bernstein algebra, length and dimension coincide. Hence the dimension of subalgebras is invariant under lattice homomorphism.

**Theorem 1.2.** Let \(A, B\) be Bernstein algebras such that there is an \(\mathcal{L}\)-isomorphism between them. Then they must be isomorphic if any of the following conditions holds:

1. The dimension of \(A\) is less than or equal to three.
2. The algebra \(A\) is a trivial Bernstein algebra of any dimension and type.
3. The algebra \(A\) is a normal Bernstein algebra of type \((2, n - 1)\) \((\dim_K A = n + 1)\).

We will also need the following table, which can be found in [4]. In this table we have a list (up to isomorphism) of all subalgebras of length two that can be found inside a Bernstein algebra. Since they have length two, the number of proper subalgebras distinguishes them up to \(\mathcal{L}\)-isomorphism:

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Multiplication table</th>
<th>Subalgebras</th>
<th>No. of Subalgebras</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Ke + Ku)</td>
<td>(e^2 = e, eu = \frac{1}{2} u, u^2 = 0)</td>
<td>(Ku, K(e + au))</td>
<td>(</td>
</tr>
<tr>
<td>(Ke + Kv)</td>
<td>(e^2 = e, ev = 0, v^2 = 0)</td>
<td>(Ke, Kv)</td>
<td>2</td>
</tr>
<tr>
<td>(B_{(1)} = Kz + Kw)</td>
<td>(z^2 = 0, zw = 0, w^2 = 0)</td>
<td>(Kw, K(z + aw))</td>
<td>(</td>
</tr>
<tr>
<td>(B_{(2)} = Kz + Kw)</td>
<td>(z^2 = 0, zw = 0, w^2 = z)</td>
<td>(Kz)</td>
<td>1</td>
</tr>
<tr>
<td>(B_{(3)} = Kz + Kw)</td>
<td>(z^2 = 0, zw = z, w^2 = 0)</td>
<td>(Kz, Kw)</td>
<td>2</td>
</tr>
</tbody>
</table>
Here \( \alpha \) is in \( K \), and we write \(|K|\) for the cardinality of the field \( K \), which is infinite in this case.

2. NORMAL AND EXCLUSIVE BERNSTEIN ALGEBRAS

We will begin with an example which shows that (1.2) cannot be extended to arbitrary Bernstein algebras.

**Example 2.1.** Take the 4-dimensional Bernstein algebra listed as (5) in [5], that is to say,

\[
A(\alpha) = Ke + Ku + Kv + Kw,
\]

where

\[
e^2 = e, \quad eu = \frac{1}{2}u, \quad ev = ew = 0,
\]

\[
u^2 = 0, \quad uv = 0, \quad uw = 0,
\]

\[
v^2 = u, \quad w^2 = au, \quad vw = 0, \quad \alpha \notin -K^2.
\]

It is straightforward to see that the lattices of the \( A(\alpha) \)'s are isomorphic. In [5] it is shown that these algebras are not in general isomorphic.

Nevertheless, although the algebra cannot be completely determined, some structural characteristic can be known from the study of the lattice of subalgebras, as we will see in the following.

**Theorem 2.2.** If two exclusive Bernstein algebras are \( \mathcal{L} \)-isomorphic, then they have the same type.

**Proof.** Let \((A, \omega), (\tilde{A}, \tilde{\omega})\) be exclusive Bernstein algebras, and let \( \phi : \mathcal{L}(A) \rightarrow \mathcal{L}(\tilde{A}) \) be an \( \mathcal{L} \)-isomorphism. We will see that if \( A \) is of type \((r + 1, s)\), where \( r + s = n \), then \( \tilde{A} \) is of type \((r + 1, s)\), by induction on \( n \).

The cases \( n = 1, 2 \) follow from (1.2). Let us suppose the result is true for all \( 1 \leq k \leq n - 1, \, n \geq 3 \).

If one of the algebras considered is trivial, then they must be isomorphic from (1.2). Hence we can restrict ourselves to nontrivial algebras.

Let us see that if \((\tilde{r} + 1, \tilde{s})\) is the type of \( \tilde{A} \), we have \( \tilde{r} \geq r \). By a symmetric argument, the reverse inequality will be also true and \( \tilde{r} = r \) and \( \tilde{s} = s \), since \( A \) and \( \tilde{A} \) have the same dimension from (1.1).
\( \phi(\text{Ker} \omega) \) is a subalgebra of \( \tilde{A} \) of dimension \( n \). We have two possibilities:

**Case 1:** \( \phi(\text{Ker} \omega) = \text{Ker} \tilde{\omega} \). Take \( e \), a nonzero idempotent of \( A \). The subspace \( Ke + U_e \) is a subalgebra of \( A \), since \( A \) is exclusive, and it is different from the whole algebra, since \( A \) is not trivial. We have

\[
\phi(Ke + U_e) \leq \text{Ker} \tilde{\omega} = \phi(\text{Ker} \omega).
\]

Hence, \( \phi(Ke + U_e) \) is a subalgebra of \( \tilde{A} \) which is itself a Bernstein algebra. The algebra \( Ke + U_e \) is trivial of type \((r + 1, 0)\). From (1.2), the algebra \( \phi(Ke + U_e) \) must have the same type, which implies \( \tilde{r} \geq r \).

**Case 2:** \( \phi(\text{Ker} \omega) \neq \text{Ker} \tilde{\omega} \). Now, \( \phi^{-1}(\text{Ker} \tilde{\omega}) \) is a subalgebra (of dimension \( n \)) of \( A \) that is not contained in \( \text{Ker} \omega \). Thus, it is a Bernstein algebra and we can write one of its Peirce decompositions:

\[
\phi^{-1}(\text{Ker} \tilde{\omega}) = Ke + W \bot H,
\]

where \( W \subseteq U_e \), \( H \subseteq V_e \), and \( e \) is a nonzero idempotent contained in \( \phi^{-1}(\text{Ker} \tilde{\omega}) \). Hence, \( W = U_e \) or \( H = V_e \).

(1) If \( W = U_e \), then \( \dim_K H = n - r - 1 = s - 1 \). Suppose \( s = 1 \), i.e., \( H = 0 \). Considering a basis \( \{u_1, \ldots, u_s\} \) of \( U_e \), we can construct the idempotents:

\[
e_i = e + u_i, \quad 0 \leq i \leq r
\]

(where \( u_0 = 0 \), that is to say, \( e_0 = e \)). These idempotents are linearly independent, and for \( i \neq j \),

\[
e_i e_j = \frac{1}{2}(e_i + e_j).
\]

Moreover, for any \( \alpha, \beta \) in \( K \),

\[
(\alpha e_i + \beta e_j)^2 = (\alpha + \beta)(\alpha e_i + \beta e_j).
\]

Thus, \( Ke_i \vee Ke_j = Ke_i + Ke_j \) and it is an algebra with \(|K| + 1\) subalgebras. Put \( Kx_i = \phi(Ke_i) \), a subalgebra of \( \phi(Ke + U_e) = \text{Ker} \tilde{\omega} \), for all \( 0 \leq i \leq r \). Now, for \( i \neq j \), \( Kx_i \vee Kx_j \) is a subalgebra of dimension 2 of \( \tilde{A} \), contained in \( \text{Ker} \tilde{\omega} \), with \(|K| + 1\) subalgebras. It must be isomorphic to \( B_{(1)} \) (see the table.
in Section 1). In particular, \( x_i x_j = 0 \) for all \( 0 \leq i, j \leq r \). Hence
\[
Kx_0 + \cdots + Kx_r = Kx_0 \vee \cdots \vee Kx_r = \phi(Ke_0 \vee \cdots \vee Ke_r)
\]
\[
= \phi(Ke_0 + \cdots + Ke_r) = \phi(Ke + U_e) = \text{Ker} \tilde{\omega}.
\]

Then \((\text{Ker} \tilde{\omega})^2 = 0\) and \(\tilde{A}\) is a trivial algebra, contradicting our assumption. Thus \( s > 2 \). Let \( \{v_1, \ldots, v_{r+1}\} \) be a basis of \( H \) such that \( \{v_1, \ldots, v_{r+1}, u\} \) is a basis of \( V_e \). Let us consider the subalgebra
\[
A_1 = Ke + U_e + \langle v_2, \ldots, v_r \rangle.
\]
It is clear that \( A_1 \neq Ke + U_e + H \) and has dimension \( n \), so that \( \phi(A_1) \neq \text{Ker} \tilde{\omega} \) is a subalgebra of \( \tilde{A} \) which is itself an exclusive Bernstein algebra. By the induction assumption, \( \phi(A_1) \) is a Bernstein algebra of type \( (r + 1, s - 1) \). This implies \( \tilde{r} > r \).

(2) If \( W \neq U_e \), the subalgebra \( Ke + U_e \) is not contained in \( Ke + W + H \). Thus, \( \phi(Ke + U_e) \not\subseteq \text{Ker} \tilde{\omega} \), and \( \tilde{r} > r \) follows as in case 1.

**Theorem 2.3.** If two normal Bernstein algebras are \( \mathcal{L}\)-isomorphic, then they have the same type.

**Proof.** Let \( (A, \omega), (\tilde{A}, \tilde{\omega}) \) be normal Bernstein algebras of types \( (r + 1, s), (\tilde{r} + 1, \tilde{s}) \), respectively. Let \( \phi : \mathcal{L}(A) \to \mathcal{L}(\tilde{A}) \) be an \( \mathcal{L}\)-isomorphism.

We can assume neither of them is trivial, and it is enough to prove \( \tilde{s} \geq s \), similarly to (2.2).

**Case 1:** \( \phi(\text{Ker} \omega) = \text{Ker} \tilde{\omega} \). Take \( e \) a nonzero idempotent in \( A \). The subalgebra \( Ke + V_e \) is mapped by \( \phi \) onto a subalgebra of \( \tilde{A} \) which is itself a Bernstein algebra. Using (1.2), since \( Ke + V_e \) is trivial, its image is trivial of the same type, yielding \( \tilde{s} \geq s \).

**Case 2:** \( \phi(\text{Ker} \omega) \neq \text{Ker} \tilde{\omega} \). As in (2.2), we can write
\[
\phi^{-1}(\text{Ker} \tilde{\omega}) = Ke + W + H, \text{ where } e^2 = e \neq 0, \ W = U_e, \text{ or } H = V_e.
\]
(1) If \( H \neq V_e \), then \( Ke + V_e \) is not contained in \( Ke + W + H \), and thus
\[
\phi(Ke + V_e) \not\subseteq \text{Ker} \tilde{\omega}.
\]

Using (1.2), as above, \( \tilde{s} \geq s \).

(2) If \( H = V_e \), then \( \dim_k W = r - 1 \). We claim that for all \( u \in U_e - W \), \( u^2 = 0 \). Otherwise \( Ku + Ku^2 \), which is a subalgebra (since \( u^3 = 0 \) in any
Bernstein algebra; see [11]), has exactly one proper subalgebra, which implies \( \phi(Ku + Ku^2) \subseteq \text{Ker } \omega \) (see the table in Section 1). Hence

\[ A = Ke + U_e + V_e = Ke + Ku + W + H = (Ke + W + H) \lor (Ku + Ku^2), \]

and \( \phi(A) \subseteq \text{Ker } \omega \), which is impossible. Take \( u_o \) in \( U_e - W \). For any \( w \) in \( W \), \( u_o + w \in U_e - W \). Hence \( (u_o + w)^2 = 0 \). Thus

\[ w^2 = -2u_0 w \quad \text{for all } w \text{ in } W. \]

Writing the previous equality for the element \( 2w \), we get

\[ 4w^2 = -2u_0(2w) = -4u_0 w = 2w^2. \]

Hence, \( w^2 = 0 \).

We have proved \( W^2 = 0 \), \( u_0 W = 0 \), and \( U_e^2 = 0 \), since \( U_e = Ku_0 + W \). This means that \( A \) is trivial, since it was normal, which contradicts our assumption.

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REFERENCES


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