Discrete Inequalities of the Gronwall–Bellman Type in \( n \) Independent Variables

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1. Introduction and Notation

The aim of this paper is to study a discrete inequality of the Gronwall–Bellman type in \( n \) independent variables. As far as the author knows, the existing results for \( n > 1 \) \([10, 11]\) related to ours are limited to \( n = 3 \) only. In this paper, we weaken the conditions of the known results for \( n = 3 \) as far as possible and generalize them to \( n \) independent variables in order to get a more compact and elegant form.

Let \( R_r := [0, \infty) \) and let \( \mathbb{N} \) be the set of nonnegative integers. The expression \( u(0) + \sum_{s=0}^{n-1} b(s) \) represents a solution of the linear difference equation \( \Delta u(n) = b(n) \) for all \( n \in \mathbb{N} \), where \( \Delta \) is the operator defined by \( \Delta u(n) = u(n+1) - u(n) \). The expression \( \prod_{s=0}^{n-1} c(s) \) represents a solution of the linear difference equation \( x(n+1) = c(n)x(n) \) for all \( n \in \mathbb{N} \) under the initial condition \( x(0) = 1 \). We assume that \( \sum_{s=0}^{n-1} b(s) = 0 \) and \( \prod_{s=0}^{n-1} c(s) = 1 \).

For \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \), \( \vec{1} = (1, \ldots, 1) \), \( \vec{0} = (0, \ldots, 0) \in \mathbb{N}^n \), we define

\[
\delta u(y) := \sum_{y \leq \vec{1}} u(y) := \sum_{y_1=0}^{1} \cdots \sum_{y_n=0}^{n-1} u(y_1, \ldots, y_n)
\]

and \( x := (x_1, \vec{x}) \), where \( \vec{x} := (x_2, \ldots, x_n) \). The natural partial ordering on \( \mathbb{N}^n \) is defined by

\( x \leq y \) if and only if \( x_i \leq y_i \) for \( i = 1, 2, \ldots, n \).

The difference operators on \( \mathbb{N}^n \) are defined as follows:

\[
\Delta u_{x_1}(x_1, x_2, \ldots, x_n) := u(x_1 + 1, x_2, \ldots, x_n) - u(x_1, x_2, \ldots, x_n),
\]

\[
\Delta u_{x_2}(x_1, x_2, \ldots, x_n) := u(x_1, x_2 + 1, x_3, \ldots, x_n) - u(x_1, x_2, x_3, \ldots, x_n),
\]

\[
\vdots
\]

\[
\Delta u_{x_n}(x_1, x_2, \ldots, x_n) := u(x_1, x_2, \ldots, x_{n-1} + 1) - u(x_1, x_2, \ldots, x_n),
\]

322
and

$$Δ x_{1}x_{1}(x_{1}, x_{2}, \ldots, x_{n}) := Δ u_{x_{1}}(x_{1}, x_{2}, x_{3}, \ldots, x_{n}) - Δ u_{x_{1}}(x_{1}, x_{2}, \ldots, x_{n}),$$

and so on.

2. **Main Results**

We begin with the following theorem.

**Theorem 1.** Let $u(x), f(x)$ and $h(x)$ be real-valued nonnegative functions defined on $N^n$ and let $H(t) \in C[R_+, R_+]$ be a nondecreasing function such that

$$Q(r) := \int_{r_{0}}^{r} \frac{ds}{H(s)}$$

exists for $r \geq 0$ with $r_0 > 0$ fixed, but arbitrary. If the inequality

$$u(x) \leq f(x) + \sum_{i=0}^{x-1} h(t) H(u(t)), \quad x \in N^n$$

holds, then

$$u(x) \leq Q^{-1} \left[ Q(\tilde{f}(x)) + \sum_{i=0}^{x-1} h(t) \right], \quad \text{for } 0 \leq x \leq b$$

where

(i) $Q^{-1}$ is the inverse function of $Q$,  
(ii) $\tilde{f}(x) := \max \{f(y); 0 < y \leq x\}$, 
(iii) $b \in N^n$ is chosen so that

$$Q(\tilde{f}(x)) + \sum_{i=0}^{x-1} h(t) \in \text{Range}(Q), \quad \text{for } 0 \leq x \leq b.$$

**Proof:** Let

$$v(x) := \sum_{i=0}^{x-1} h(t) H(u(t)),$$

then

$$u(x) \leq f(x) + v(x), \quad \text{for } 0 \leq x \leq b$$

$$Δ^n v_{x}(x) = h(x) H(u(x)).$$
Since $H$ is nondecreasing, it follows from (ii), (3) and (4) that

$$\Delta^n u_x(x) \leq h(x) H(f(x) + v(x)) \leq h(x) H(f(X) + v(x))$$

for arbitrary $X \geq \tilde{0}$ and $\tilde{0} \leq x \leq X$. Set $V(x) := f(x) + v(x) + \varepsilon (\varepsilon > 0)$, so $u(x) \leq V(x)$ and

$$\Delta^n V_x(x) = \Delta^n v_x(x) \leq h(x) H(V(x)) \leq h(x) H(V(x_1, \ldots, x_{n-1}, x_n + 1)), \quad (5)$$

for $\tilde{0} \leq x \leq X$, which implies

$$\frac{\Delta^{n-1} V_{x_1, \ldots, x_n-1}(x_1, \ldots, x_{n-1}, x_n + 1) - \Delta^{n-1} V_{x_1, \ldots, x_n-1}(x)}{H(V(x_1, \ldots, x_{n-1}, x_n + 1))} \leq h(x).$$

Since $\Delta^k V_{x_1, \ldots, x_k}(x) = \Delta^k v_{x_1, \ldots, x_k}(x) \geq 0$ always, and $=0$ if $x_i = 0$ for $i = k + 1, \ldots, n$, and since $V(x)$ is nondecreasing in each component, it follows from the above inequality that

$$\frac{\Delta^{n-1} V_{x_1, \ldots, x_{n-1}}(x_1, \ldots, x_{n-1}, x_n + 1) - \Delta^{n-1} V_{x_1, \ldots, x_{n-1}}(x)}{H(V(x_1, \ldots, x_{n-1}, x_n + 1))} \leq h(x).$$

Keeping $x_1, \ldots, x_{n-1}$ fixed in the above inequality, setting $x_n = t_n$ and summing over $t_n = 0, 1, \ldots, x_n - 1$, we have

$$\frac{\Delta^{n-1} V_{x_1, \ldots, x_n-2}(x_1, \ldots, x_{n-2}, x_{n-1} + 1, x_n)}{H(V(x_1, \ldots, x_{n-1}, x_n + 1))} \leq \sum_{t_n = 0}^{x_n - 1} h(x_1, \ldots, x_{n-1}, t_n).$$

Since $V(x) \leq V(x_1, \ldots, x_{n-2}, x_{n-1} + 1, x_n)$, we have

$$\frac{\Delta^{n-2} V_{x_1, \ldots, x_{n-2}}(x_1, \ldots, x_{n-2}, x_{n-1} + 1, x_n)}{H(V(x_1, \ldots, x_{n-2}, x_{n-1} + 1, x_n))} \leq \sum_{t_n = 0}^{x_n - 1} h(x_1, \ldots, x_{n-1}, t_n).$$

Keeping $x_1, \ldots, x_{n-2}, x_n$ fixed in the above inequality, setting $x_{n-1} = t_{n-1}$ and summing over $t_{n-1} = 0, 1, \ldots, x_{n-1} - 1$, we have

$$\frac{\Delta^{n-2} V_{x_1, \ldots, x_{n-2}}(x_1, \ldots, x_{n-2}, x_{n-1} + 1, x_n)}{H(V(x_1, \ldots, x_{n-2}, x_{n-1} + 1, x_n))} \leq \sum_{t_{n-1} = 0}^{x_{n-1} - 1} \sum_{t_n = 0}^{x_n - 1} h(x_1, \ldots, x_{n-2}, t_{n-1}, t_n).$$
Continuing in this way, we have

\[ \frac{\Delta V_{x}(x)}{H(V(x))} \leq \sum_{i=0}^{x-1} h(x_1, i). \quad (6) \]

This implies

\[ Q(V(x)) \leq Q(\bar{f}(X) + \varepsilon) + \sum_{i=0}^{x-1} h(i), \quad \text{for } \hat{0} \leq x \leq X. \]

Thus

\[ u(x) \leq V(x) \leq Q^{-1} \left[ Q(\bar{f}(X) + \varepsilon) + \sum_{i=0}^{x-1} h(i) \right], \quad \text{for } \hat{0} \leq x \leq X. \]

Letting \( \varepsilon \downarrow 0 \), we have

\[ u(x) \leq Q^{-1} \left[ Q(\bar{f}(X)) + \sum_{i=0}^{x-1} h(i) \right], \quad \text{for } \hat{0} \leq x \leq X. \quad (7) \]

In particular, (7) holds for \( x = X \leq b \) provided \( b \) is chosen as defined in (iii). Replacing \( X \) by \( x \) in (7) gives, finally,

\[ u(x) \leq Q^{-1} \left[ Q(\bar{f}(x)) + \sum_{i=0}^{x-1} h(i) \right], \quad \text{for } \hat{0} \leq x \leq b. \]

This completes the proof.

**Corollary 1.** Under the hypotheses of Theorem 1, if \( H = \) the identity mapping, then

\[ u(x) \leq \bar{f}(x) \prod_{t_1=0}^{x_1-1} \left[ 1 + \sum_{i=0}^{x-1} h(t_1, i) \right], \quad \text{for } x \in \mathbb{N}^n. \quad (8) \]

**Proof.** It follows from (6) that

\[ \frac{V(x + 1, \bar{x})}{V(x)} \leq 1 + \sum_{i=0}^{x-1} h(x_1, i). \]

Keeping \( \bar{x} = (x_2, \ldots, x_n) \) fixed in this inequality, setting \( x_1 = t_1 \) and taking the product over \( t_1 = 0, 1, \ldots, x_1 - 1 \), we have

\[ V(x) \leq (\bar{f}(X) + \varepsilon) \prod_{t_1=0}^{x_1-1} \left[ 1 + \sum_{i=0}^{x-1} h(t_1, i) \right]. \]
Letting \( \epsilon \to 0 \) and replacing \( X \) by \( x \) as in the proof of Theorem 1, we obtain the required bound in (8).

**Remark 1.** In case \( f \) is nondecreasing in each \( x_i \), we have \( \tilde{f} \equiv f \).

**Remark 2.** For \( n = 1 \) and \( f(x) \equiv \text{constant}, \) Theorem 1 reduces to the result of Hull and Luxemburg [2] (see also Beesack’s lecture notes [1, p. 98]). The continuous analogue of Theorem 1 is due to LaSalle [4].

**Remark 3.** For \( n = 3 \), Theorem 1 improves Theorem 3 of [10] and Corollary 1 is an improvement of [10, Theorem 1] and [11, Theorem 1]. For \( n = 1 \), Corollary 1 improves the results of Miller [5, Lemma 3.2] and Sugiyama [13, Corollary].

The following theorem is an improvement of Theorem 2 of Pachpatte and Singare [10].

**Theorem 2.** Let \( u(x), f(x), h(x), H(r), Q(r) \) and \( Q^{-1}(r) \) be defined as in Theorem 1 with \( H(r) \) subadditive and submultiplicative and let \( g(x), k(x) \) be real-valued nonnegative functions defined on \( \mathbb{N}^n \). If the inequality

\[
\begin{align*}
&u(x) \leq f(x) + g(x) \sum_{y=0}^{x-1} h(y) H \left( u(y) + g(y) \sum_{z=0}^{y-1} k(z) H(u(z)) \right) \\
&+ \sum_{y=0}^{x-1} (h(y) + k(y)) H(f(y)) + \sum_{y=0}^{x-1} (h(y) + k(y)) H(g(y)) \in \text{Range}(Q),
\end{align*}
\]

holds for \( x \in \mathbb{N}^n \), then

\[
\begin{align*}
u(x) \leq & f(x) + g(x) Q^{-1}\left[ \sum_{y=0}^{x-1} (h(y) + k(y)) H(f(y)) \right] \\
&+ \sum_{y=0}^{x-1} (h(y) + k(y)) H(g(y)),
\end{align*}
\]

for \( 0 \leq x \leq b \), where (i) \( b \in \mathbb{N}^n \) is chosen so that

\[
Q \left[ \sum_{y=0}^{x-1} (h(y) + k(y)) H(f(y)) \right] \\
+ \sum_{y=0}^{x-1} (h(y) + k(y)) H(g(y)) \in \text{Range}(Q),
\]

for \( 0 \leq x \leq b \).

**Proof.** Set

\[ w(x) := u(x) + g(x) \sum_{y=0}^{x-1} k(y) H(u(y)), \]
so \( u(x) \leq w(x) \) and \( H(u(x)) \leq H(w(x)) \). It follows from (9) that

\[
w(x) - g(x) \sum_{y=0}^{x-1} k(y) H(u(y)) = u(x) \leq f(x) + g(x) \sum_{y=0}^{x-1} H(w(y)),
\]
or

\[
w(x) \leq f(x) + g(x) \sum_{y=0}^{x-1} (h(y) + k(y)) H(w(y)).
\]

For brevity, set \( b := h + k \) and \( v(x) := \sum_{y=0}^{x-1} b(y) H(w(y)) \). Then

\[
w(x) \leq f(x) + g(x) v(x),
\]

\[
\Delta^n v(x) = b(x) H(w(x)) \leq b(x) H(f(x) + g(x) v(x)).
\]

Since \( H \) is also subadditive and submultiplicative,

\[
\Delta^n v(x) \leq b(x) H(f(x)) + b(x) H(g(x)) H(v(x)) = B(x) + C(x) H(v(x)),
\]

say.

Now by repeated summation and using \( \Delta^k v_{x_1,\ldots,x_k} (x) = 0 \) if \( x_i = 0 \) for \( i = k + 1,\ldots, n \), we get

\[
v(x) \leq \sum_{y=0}^{x-1} B(y) + \sum_{y=0}^{x-1} C(y) H(v(y)) = B_1(x) + \sum_{y=0}^{x-1} C(y) H(v(y)) \leq B(X) + \sum_{y=0}^{x-1} C(y) H(v(y)), \quad 0 \leq x \leq X.
\]

Set

\[
V(x) := B_1(X) + \sum_{y=0}^{x-1} C(y) H(v(y)).
\]

so \( V(x) = B_1(X) \) if any \( x_i = 0 \). Then

\[
\Delta^n V_x (x) = C(x) H(v(x)) \leq C(x) H(V(x)), \quad 0 \leq x \leq X. \quad (11)
\]

If one now proceeds as in Theorem 1, one gets

\[
V(x) \leq Q^{-1} \left[ Q(B_1(X)) + \sum_{y=0}^{x-1} C(y) \right], \quad 0 \leq x \leq X.
\]
CHEH-CHIH YEH

Setting $x = X$ and then replacing $X$ by $x$ in the above inequality we have

$$u(x) \leqslant w(x) \leqslant f(x) + g(x) v(x) \leqslant f(x) + g(x) V(x)$$

$$\leqslant f(x) + g(x) Q^{-1} \left\{ Q \left[ \sum_{y=0}^{x-1} (h(y) + k(y)) H(f(y)) \right] + \sum_{y=0}^{x-1} (h(y) + k(y)) H(g(y)) \right\}$$

for $0 \leqslant x \leqslant b$, where $b$ is chosen as defined in (i). This completes the proof.

**Corollary 2.** Under the hypotheses of Theorem 2, if $H(s) \equiv s$, then

$$u(x) \leqslant f(x) + g(x) \sum_{y=0}^{x-1} (h(y) + k(y)) f(y)$$

$$\cdot \prod_{t_1=0}^{x-1} \left[ 1 + \sum_{t=0}^{y-1} (h + k) g(t_1, t) \right], \text{ for } x \in \mathbb{N}^n. \quad (12)$$

**Proof.** It follows from (11) that

$$A^n V_x(x) \leqslant C(x) V(x), \quad 0 \leqslant x \leqslant X.$$ 

Hence if we proceed as in the proof of Corollary 1, we obtain the desired bound in (12).

**Remark 4.** Corollary 2 is an improvement of [9, Theorem 1] for $n = 1$.

**Remark 5.** For $k \equiv 0$, the inequalities (10) and (12) reduce to the inequalities

$$u(x) \leqslant f(x) + g(x) Q^{-1} \left\{ Q \left[ \sum_{y=0}^{x-1} h(y) H(f(y)) \right] + \sum_{y=0}^{x-1} h(y) H(g(y)) \right\} \quad (13)$$

and

$$u(x) \leqslant f(x) + g(x) \left( \sum_{y=0}^{x-1} h(y) f(y) \right) \left( \prod_{t_1=0}^{x-1} \left[ 1 + \sum_{t=0}^{y-1} h(t_1, t) g(t_1, t) \right] \right) \quad (14)$$

respectively. Inequality (13) extends a part of Theorem 1 of Pachpatte [8], which says mainly that

$$u(n) \leqslant f(n) + g(n) p \left( \sum_{y=0}^{n-1} h(y) H(u(y)) \right), \quad n \in \mathbb{N}$$
implies
\[ u(n) \leq f(n) + g(n) p \left\{ Q^{-1} \left[ Q \left( \sum_{y=0}^{n-1} h(y) H(f(y)) \right) \right] + \sum_{y=0}^{n-1} h(y) H(g(y)) \right\}, \quad n \in \mathbb{N}. \]

In fact, Theorem 1 of [8] can also be extended to \( n \) independent variables. Inequality (14) extends the results of Jones [3, Lemma 3] and Sugiyama [13, Lemma 1].

**Corollary 3.** Under the hypotheses of Theorem 2, if \( g(x) = 1, k(x) \leq 0 \) and is not required to be submultiplicative, then
\[ u(x) \leq f(x) + Q^{-1} \left\{ Q \left[ \sum_{y=0}^{x-1} h(y) H(f(y)) \right] + \sum_{y=0}^{x-1} h(y) \right\} \]
for \( \tilde{\alpha} \leq x \leq b \), where \( b \in \mathbb{N}^n \) is chosen so that
\[ Q \left[ \sum_{y=0}^{x-1} h(y) H(f(y)) \right] + \sum_{y=0}^{x-1} h(y) \in \text{Range}(Q), \quad \text{for} \quad \tilde{\alpha} \leq x \leq b. \]

**Theorem 3.** Let \( u(x), f(x) \) and \( h(x) \) be defined as in Theorem 1 with \( f(x) \) nondecreasing in each \( x_i \) and let \( Q(s) \in \mathcal{C}[\mathbb{R}_+, \mathbb{R}_+] \) be nondecreasing with
\[ +.Q(MQ(f)) \quad \text{fort} \to 1 \quad \text{and} \quad s \to 0. \]
Let \( H(s) \in \mathcal{C}[\mathbb{R}_+, [1, \infty)) \) be a strictly increasing, subadditive and supermultiplicative function. If the inequality
\[ u(x) \leq f(x) + H^{-1} \left[ Q \left( \sum_{y=0}^{x-1} h(y) H(u(y)) \right) \right], \quad x \in \mathbb{N}^n \]
holds, where \( H^{-1} \) is the inverse function of \( H \), then, for \( \tilde{\alpha} \leq x \leq b \)
\[ u(x) \leq f(x) H^{-1} \left\{ 1 + Q \left[ G^{-1} \left( \sum_{y=0}^{x-1} h(y) \right) \right] \right\}, \]
where
\[ G(r) := \int_{0}^{r} \frac{ds}{1 + Q(s)} \quad \text{for} \quad r \geq 0, \]
$G^{-1}$ is the inverse function of $G$ and $b \in N^n$ is chosen so that
\[ \sum_{y=0}^{x-1} h(y) \in \text{Range}(G) \]
and
\[ 1 + Q \left[ G^{-1} \left( \sum_{y=0}^{x-1} h(y) \right) \right] \in \text{Range}(H) \]
for $0 \leq x \leq b$.

Proof. Since $H$ is subadditive, it follows from (15) that
\[ H(u(x)) \leq H(f(x)) + Q \left[ \sum_{y=0}^{x-1} h(y) H(u(y)) \right]. \]

Since $H(f(x)) \geq 1$ is nondecreasing,
\[
\frac{H(u(x))}{H(f(x))} \leq 1 + Q \left[ \sum_{y=0}^{x-1} h(y) \frac{H(u(y))}{H(f(y))} \right]. \tag{16}
\]

Define
\[ w(x) := \sum_{y=0}^{x-1} h(y) \frac{H(u(y))}{H(f(y))}. \]

Thus
\[ w(x) = 0 \quad \text{on } x_i = 0 \text{ for } i = 1, 2, \ldots, n, \]
and
\[ \Delta^n w_x(x) = h(x) \frac{H(u(x))}{H(f(x))}. \tag{17} \]

It follows from (16) and (17) that
\[ \Delta^n w_x(x) \leq h(x)[1 + Q(w(x))]. \]

Thus
\[ \frac{\Delta^n w_x(x)}{1 + Q(w(x))} \leq h(x). \]
As in the proof of Theorem 1, we have
\[
G(w(x_1 + 1, \bar{x})) - G(w(x)) = \int_{w(x)}^{w(x_1 - 1, \bar{x})} \frac{ds}{1 + Q(s)} \leq \frac{\Delta w_{x_1}(x)}{1 + Q(w(x))} \leq \sum_{i=0}^{x_1 - 1} h(x_1, i).
\]
Keeping \( \bar{x} = (x_2, \ldots, x_n) \) fixed in this inequality, setting \( x_1 = t \), and summing over \( t = 0, 1, \ldots, x_1 - 1 \), we have
\[
G(w(x)) - G(w(0, \bar{x})) \leq \sum_{t=0}^{x_1 - 1} h(t).
\]
This and \( G(0) = 0 \) imply
\[
w(x) \leq G^{-1} \left( \sum_{t=0}^{x_1 - 1} h(t) \right).
\]
This and (16) imply
\[
H(u(x)) \leq H(f(x))(1 + Q(w(x))) \leq H(f(x)) \left( 1 + Q \left[ G^{-1} \left( \sum_{t=0}^{x_1 - 1} h(t) \right) \right] \right).
\]
Since \( H \) is also supermultiplicative and increasing, \( H^{-1} \) is submultiplicative. Thus
\[
u(x) \leq f(x) H^{-1} \left( 1 + Q \left[ G^{-1} \left( \sum_{t=0}^{x_1 - 1} h(t) \right) \right] \right).
\]
This completes the proof.

**Remark 6.** For \( n = 2 \), Theorem 3 is very close to Theorem 5 of Singare and Pachpatte [12].

**Remark 7.** For \( n = 1 \), the continuous analogues of Theorems 2 and 3 are given in Theorem 1 of [6] and Theorem 5 of [7], respectively.

**References**