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The full automorphism group of a cyclic p-gonal surface

A. Wootton

University of Portland, 5000 North Willamette Blvd., Portland, OR 97203, USA

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Abstract

If p is prime, a compact Riemann surface X of genus $g \geqslant 2$ is called cyclic p-gonal if it admits a cyclic group of automorphisms C_p of order p such that the quotient space X/C_p has genus 0. If in addition C_p is not normal in the full automorphism A, then we call X a non-normal p-gonal surface. In the following we classify all non-normal p-gonal surfaces.

Keywords: Compact Riemann surface; Automorphism group; Hyperelliptic surface

1. Introduction

A compact Riemann surface X of genus $g \geqslant 2$ which admits a cyclic group of automorphisms C_p of prime order p such that the quotient space X/C_p has genus 0 is called a *cyclic p-gonal surface* or a p-gonal surface for brevity. The group C_p is called a p-gonal group for X. If in addition C_p is normal in the full automorphism group of X, then we call X a *normal p-gonal surface*. Else we call X a *non-normal p-gonal surface* and any group G with $C_p \leqslant G \leqslant \mathrm{Aut}(X)$ in which C_p is not normal a *non-normal p-gonal overgroup*. The primary result of this paper is a classification of all non-normal p-gonal surfaces. By classification, we mean that for each such surface X, we find the full automorphism group and the signature for the normalizer of a surface group for X. These results coupled with the results of [22] can then be used to find a defining affine model for X.

E-mail address: wootton@up.edu.

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The study of such surfaces and other related surfaces is already extensive in the literature, see for example [2–4,7–10,12–15,19–21,23]. The first family of cyclic p-gonal surfaces to be classified in this sense were the hyperelliptic surfaces (when $p = 2$). The groups and signatures for this family were first presented in [4] and the full automorphism groups were found in [7]. It is natural to try to generalize the results of the classification of hyperelliptic surfaces. However, the main contributing factor to this classification was the fact that a hyperelliptic surface is always normal 2-gonal and for $p > 2$ this is no longer true. Though a complete classification of cyclic p-gonal surfaces is not possible using the methods developed for the hyperelliptic classification, the methods can be generalized to find all *normal p-gonal overgroups* (groups with a normal cyclic p-gonal subgroup) and corresponding group signatures for a general p, see for example [16] or [22]. Then, as with the hyperelliptic case, the results of [6] can be used to determine which of these groups are full automorphism groups of a normal p-gonal surface (see Section 4 for details). With these results in consideration, the remaining problem is to find all non-normal p-gonal overgroups which act as full automorphism groups on p-gonal surfaces.

In [1], Accola showed that if X is a genus g p-gonal surface and $g > (p-1)^2$, then X is normal p-gonal, so in order to classify non-normal p-gonal surfaces, we only need to consider surfaces of genus $g \leqslant (p-1)^2$. This result motivates a study of p-gonal surfaces of genus $g \leqslant (p-1)^2$ for small values of p to provide insight into the general problem. Results regarding trigonal surfaces ($p = 3$) can be found in [2,3,10,23] and results for a closely related family of surfaces in [8]. Similar methods could be used to study p-gonal surfaces for other small values of p. Another way to gain insight into this problem is to study families of non-normal p-gonal surfaces with additional restrictions, see for example [14,15,21]. In these collective works, a number of different non-normal p-gonal surfaces were found including certain unique surfaces (like Klein's genus 3 surface) and infinite families of surfaces (like the pth Fermat curve for each prime $p > 3$). With the insight provided by these results, we shall finish this classification by showing that there are no additional non-normal p-gonal surfaces to those which already appear in the literature.

Our approach to the problem is first to classify certain families of p-gonal surfaces and then show that each non-normal p-gonal surface must lie in one of these families. We start in Section 2 by developing a number of preliminary results regarding automorphism groups of compact Riemann surfaces, uniformization and Fuchsian groups—discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. Following this, in Section 3, we shall prove the main result needed for this classification. In Sections 4, 5 and 6, we shall present classification results for three different special classes of cyclic p-gonal surfaces—normal p-gonal surfaces, p-gonal surfaces for primes 2, 3, 5 and 7 (primes we shall henceforth refer to as *small primes*) and p-gonal surfaces which admit a group of automorphisms divisible by p^2. Using the results for these special families and the main result of Section 3, we shall complete the classification in Section 7. We finish in Section 8 with a brief summary of our results.

2. Preliminary results

A compact Riemann surface X of genus $g \geqslant 2$ can be realized as a quotient of the upper half plane $\mathbb{H}/\Lambda$ where Λ is a torsion free Fuchsian group called a *surface group* for X. Under such a realization, a group G acts as a group of automorphisms on X if and only if $G = \Gamma/\Lambda$ for some Fuchsian group Γ containing Λ as a normal subgroup of index $|G|$. We call Γ the Fuchsian group corresponding to G usually denoting it by Γ_G, and if Λ has been fixed, we call G the automorphism group corresponding to Γ. If G is a group of automorphisms of X with surface

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\pi_\Lambda"'] \arrow[rr, bend left, "\pi_\Gamma"] & \mathbb{H}/\Lambda = X \arrow[r, "\pi_G"'] & \mathbb{H}/\Gamma = X/G \end{tikzcd}$$

Fig. 1. Holomorphic quotient maps and surface identifications.

group Λ and Γ is the Fuchsian group corresponding to G, we identify the orbit spaces $\mathbb{H}/\Gamma$ and X/G and the quotient map $\pi_G : X \to X/G$ is branched over the same points as $\pi_\Gamma : \mathbb{H} \to \mathbb{H}/\Gamma$ with the same ramification indices as illustrated in Fig. 1.

We define the signature of a Fuchsian group Γ to be the tuple $(g_\Gamma; m_1, m_2, \ldots, m_r)$ where the quotient space $\mathbb{H}/\Gamma$ has genus g_Γ and the quotient map π_Γ branches over r points with ramification indices m_i for $1 \leqslant i \leqslant r$. We call g_Γ the *orbit genus* of Γ and the numbers $m_1, \ldots, m_r$ the *periods* of Γ. The signature of Γ provides information regarding a presentation for Γ (see for example [5, Theorem 3.2]):

Theorem 2.1. *If Γ is a Fuchsian group with signature $(g_\Gamma; m_1, \ldots, m_r)$ then there exist group elements $\alpha_1, \beta_1, \ldots, \alpha_{g_\Gamma}, \beta_{g_\Gamma}, \zeta_1, \ldots, \zeta_r \in \mathrm{PSL}(2, \mathbb{R})$ such that*

(1) $\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_\Gamma}, \beta_{g_\Gamma}, \zeta_1, \ldots, \zeta_r \rangle$,
(2) *defining relations for Γ are*

$$\zeta_1^{m_1}, \zeta_2^{m_2}, \ldots, \zeta_r^{m_r}, \prod_{i=1}^{g_\Gamma} [\alpha_i, \beta_i] \prod_{j=1}^{r} \zeta_j,$$

(3) *each elliptic element* (*the elements of finite order*) *lies in a unique conjugate of $\langle \zeta_i \rangle$ for suitable i.*

We call a set of elements of Γ satisfying Theorem 2.1 *canonical generators* for Γ. Notice that if Γ is a surface group for a surface of genus g_Γ, since it is torsion free, it must have signature $(g_\Gamma; -)$.

Theorem 2.1 implies that if $\Gamma_1 \leqslant \Gamma$, then any elliptic element of Γ_1 must be conjugate to an elliptic element of Γ. This motivates the following definition.

Definition 2.2. Suppose $\Gamma_1 \leqslant \Gamma$ are Fuchsian groups, $\eta \in \Gamma_1$ is an elliptic element and η is Γ-conjugate to a power of ζ, some elliptic element of Γ. Then we say η is induced by ζ.

In fact, by Theorem 2.1, any elliptic generator of Γ_1 in a set of canonical generators for Γ_1 must be conjugate to a power of a unique elliptic generator of Γ in a set of canonical generators of Γ (though other generators of Γ_1 could be conjugate to that same generator of Γ, see Theorem 2.4 and Remark 2.5).

We fix some notation. Henceforth, let X denote a cyclic p-gonal surface of genus g, Λ a surface group for X, C_p a p-gonal group for X and Γ_p the Fuchsian group corresponding to C_p. Also, let A denote the full automorphism group of X, Γ_A its corresponding Fuchsian group, N the normalizer of C_p in A, Γ_N its corresponding Fuchsian group and $K = N/C_p = \Gamma_N/\Gamma_p$. After appropriate identifications, we have the tower of groups and epimorphisms illustrated in

$$\begin{tikzcd}
\Gamma_A \arrow[r, "\rho_{\Gamma_A}"'] \arrow[d, "\vee"'] \arrow[dd, bend right, "\vee"'] \arrow[ddd, bend right=60, "\triangledown"'] & A \arrow[d, "\vee"'] & \\
\Gamma_N \arrow[r, "\rho_{\Gamma_A}"'] \arrow[d, "\triangledown"'] & N \arrow[r, "\rho_K"'] \arrow[d, "\triangledown"'] & K \\
\Gamma_p \arrow[r, "\rho_{\Gamma_A}"'] \arrow[d, "\triangledown"'] & C_p \arrow[r, "\rho_K"'] \arrow[d, "\triangledown"'] & 1 \\
\Lambda \arrow[r, "\rho_{\Gamma_A}"'] & 1 &
\end{tikzcd}$$

Fig. 2. Groups and quotients.

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "\pi_\Lambda"'] \arrow[rr, bend left, "\pi_{\Gamma_p}"] \arrow[rrr, bend left=50, "\pi_{\Gamma_N}"] \arrow[rrrr, bend left=60, "\pi_{\Gamma_A}"] & \mathbb{H}/\Lambda \arrow[r, "\pi_{C_p}"'] \arrow[rr, bend right, "\pi_N"'] \arrow[rrr, bend right=50, "\pi_A"'] & \mathbb{H}/\Gamma_p \arrow[r, "\pi_K"'] & \mathbb{H}/\Gamma_N \arrow[r] & \mathbb{H}/\Gamma_A
\end{tikzcd}$$

Fig. 3. Holomorphic maps and quotient spaces.

Fig. 2 and corresponding to this, the tower of surfaces and holomorphic maps between them illustrated in Fig. 3.

Remark 2.3. We observe that the kernel Λ of the quotient map $\rho_{\Gamma_A} : \Gamma_A \to A$ is torsion free and we usually refer to such a map as a *surface kernel epimorphism*. As suggested above, if a group G acts on a surface X, then there exists a surface kernel epimorphism $\rho : \Gamma_G \to G$. Conversely, if there exists a surface kernel epimorphism from a Fuchsian group Γ onto G, then there exists a surface X on which G acts—namely the surface $X = \mathbb{H}/\operatorname{Ker}(\rho)$. Note that a necessary and sufficient condition for an epimorphism $\rho : \Gamma \to G$ to be a surface kernel epimorphism is that the elliptic generators of Γ preserve their orders under ρ.

Our observations imply that the signatures of the groups Γ_p, Γ_N and Γ_A must be closely related and exactly how can be determined through examination of the ramification data of the maps between the quotient surfaces. For example, since the map $\pi_{C_p} : X \to X/C_p$ is a Galois cover of the Riemann sphere of degree p, at any point it will either be totally ramified of order p, or unramified. It follows that Γ_p has signature $(0; \underbrace{p, \dots, p}_{R \text{ times}})$ for some integer $R > 2$ and consequently both Γ_A and Γ_N must have elliptic elements divisible by p. We postpone a major discussion of the geometric method of signature determination until the later relevant sections (see Proposition 4.1). However, since the signatures of Γ_p, Γ_N and Γ_A provide a presentation for these groups, information about the signature of each of these groups and how they relate can be derived through purely group theoretic methods. We summarize below (for details, see [17]).

Theorem 2.4. *Let Γ have signature $(g_\Gamma; m_1, \dots, m_r)$. Then Γ contains a subgroup Γ_1 of finite index with signature $(g_{\Gamma_1}; n_{1,1}, n_{1,2}, \dots, n_{1,\theta_1}, \dots, n_{r,\theta_r})$ if and only if*

(1) *there exists a finite permutation group G transitive on $[\Gamma : \Gamma_1]$ points and an epimorphism $\Phi : \Gamma \to G$ such that the permutation $\Phi(\zeta_j)$ has precisely θ_j cycles of length less then m_j, the lengths of these cycles being*

$$m_j/n_{j,1}, \dots, m_j/n_{j,\theta_j}$$

where ζ_j is the jth canonical elliptic generator of Γ,

(2) $$[\Gamma : \Gamma_1] = \left(2g_{\Gamma_1} - 2 + \sum_{j=1}^{r}\sum_{i=1}^{\theta_j}\left(1 - \frac{1}{n_{j,i}}\right)\right) \Big/ \left(2g_\Gamma - 2 + \sum_{j=1}^{r}\left(1 - \frac{1}{m_i}\right)\right).$$

Remark 2.5. In Singerman's original proof of this result, the map Φ is induced by the action of Γ on the left cosets of Γ_1. He explicitly showed that if ζ is a canonical generator of Γ of order m, then the number of canonical generators of Γ_1 induced by ζ equals the number of cycles of $\Phi(\zeta)$ of lengths less than m. Moreover, the orders of these elliptic generators are given by n_i where the m/n_i run over the lengths of each of the cycles of $\Phi(\zeta)$ less than m.

Of particular importance is the role of the normalizer Γ_N of Γ_p in Γ_A. Therefore, we observe that in the special case when Γ_1 is normal in Γ, we have the following result (see for example Lemma 3.6 of [5]).

Corollary 2.6. *Let Γ be a Fuchsian group with signature $(g_\Gamma; m_1, \dots, m_r)$ and $\Gamma_1 \leqslant \Gamma$ a normal subgroup of finite index such that $\zeta_j \Gamma_1$ has order u_j in the quotient group Γ/Γ_1 where ζ_j is the jth canonical elliptic generator of Γ. Then the orbit genus g_{Γ_1} of Γ_1 is given by*

$$g_{\Gamma_1} - 1 = [\Gamma : \Gamma_1](g_\Gamma - 1) + \frac{[\Gamma : \Gamma_1]}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{u_j}\right),$$

and the periods of Γ_1 are $n_{j,i} = \frac{m_i}{u_i}$, $1 \leqslant i \leqslant \frac{[\Gamma:\Gamma_1]}{u_j}$, $1 \leqslant j \leqslant r$, where $n_{j,i} = 1$ are deleted.

3. Fuchsian groups and signatures

One of the crucial steps in the classification of non-normal p-gonal surfaces is determination of the possible signatures for Γ_A. In order to do this, we shall first show how the elliptic generators of Γ_A and Γ_N are related. For this, we need the following useful result.

Lemma 3.1. *Suppose $\zeta \in \Gamma_A$ induces an elliptic element of Γ_p. Then it induces an elliptic element of Γ_N of the same order.*

Proof. If ζ induces $\mu \in \Gamma_p$, then, $\mu = \gamma\zeta^s\gamma^{-1}$ for some $\gamma \in \Gamma_A$ and $s \in \mathbb{N}$. Under the epimorphism $\rho : \Gamma_A \to A$, we have $\rho(\mu) = \rho(\gamma)\rho(\zeta)^s\rho(\gamma)^{-1}$, so it follows that $\langle\rho(\gamma)\rho(\zeta) \times \rho(\gamma^{-1})\rangle \leqslant N$ (since C_p is generated by $\rho(\mu)$). Therefore, we must have $\rho(\gamma\zeta\gamma^{-1}) \in N$, so $\gamma\zeta\gamma^{-1} \in \Gamma_N$. In particular, $\gamma\zeta\gamma^{-1}$ is induced by ζ and is clearly of the same order. $\square$

Observe that Proposition 3.1 implies that if $\zeta \in \Gamma_A$ is a canonical elliptic generator of Γ_A, then it must induce a canonical elliptic generator of Γ_N of the same order. With this in mind, we are now ready to prove the key result in this classification. The idea dates back to Singerman's original proof of Theorem 2.4 though restricts to the special case when X is a cyclic p-gonal surface.

Theorem 3.2. *Suppose $\zeta \in \Gamma_A$ is a canonical generator of Γ_A which induces a canonical generator of Γ_p and $\eta \in \Gamma_N$ is a canonical generator of Γ_N induced by ζ of the same order. Then the number of canonical generators of Γ_p induced by ζ is equal to the number of canonical generators of Γ_p induced by η.*

Proof. If $\Gamma_A = \Gamma_N$ (so X is normal p-gonal), then the result holds trivially, so assume X is not normal p-gonal. In order to determine the number of canonical generators of Γ_p induced by ζ, we need to consider the map $\Phi : \Gamma_A \to S_{[\Gamma_A:\Gamma_p]}$ induced by the action of Γ_A on the cosets of Γ_p (see Remark 2.5). The kernel of this map will be the intersection of the conjugates of Γ_p, so will coincide with Λ. In particular, the image of Φ will be isomorphic to A and the action of Γ_A on the cosets of Γ_p by left multiplication will be the same as that of A on the cosets of C_p. Therefore, by identifying A with the image of Γ_A under Φ, we consider how $\Phi(\zeta) \in A$ permutes the left cosets of C_p.

Since ζ induces an element of Γ_p, it must have order kp for some integer k and hence so will the element $\Phi(\zeta)$. Using Theorem 2.4 and Remark 2.5, it follows that the number of canonical generators of Γ_p induced by ζ will be equal to the number of cycles of length k in the image $\Phi(\zeta)$. This means we need to determine the cycle structure of $\Phi(\zeta)$, and we shall do this by explicitly examining the action of $\Phi(\zeta)$ on the cosets of C_p.

First we shall show that without loss of generality, we may assume $\Phi(\zeta)^k \in C_p$. By assumption, ζ induces a canonical generator of Γ_p, so there exist $\gamma \in \Gamma_A$ and an integer l relatively prime to p such that $\gamma\zeta^{lk}\gamma^{-1}$ is a canonical generator for Γ_p. It follows that $\Phi(\gamma\zeta^{lk}\gamma^{-1}) \in C_p$, and in fact $\Phi(\gamma\zeta^k\gamma^{-1}) \in C_p$ because l is relatively prime to p. Since A is a permutation group, cycle structure is preserved under conjugation, so the cycle structure of $\Phi(\zeta)$ will be the same as that of $\Phi(\gamma)\Phi(\zeta)\Phi(\gamma^{-1})$. As we are only interested in the cycle structure of $\Phi(\zeta)$, it suffices to determine the cycle structure of $\Phi(\gamma)\Phi(\zeta)\Phi(\gamma^{-1})$, and by the above, we know $\Phi(\gamma\zeta\gamma^{-1})^k \in C_p$, thus we may assume $\Phi(\zeta)^k \in C_p$.

In order to determine the cycle structure of $\Phi(\zeta)$, we first consider the action on the coset C_p. The action of $\Phi(\zeta)$ on C_p gives a cycle of the form

$$C_p \to \Phi(\zeta)C_p \to \Phi(\zeta)^2C_p \to \cdots.$$

To determine the length of this cycle, we need to find the smallest integer t such that $\Phi(\zeta)^tC_p = C_p$. However, $\Phi(\zeta)^tC_p = C_p$ only when $\Phi(\zeta)^t \in C_p$, so it follows that t is a multiple of k. Thus, the action of $\Phi(\zeta)$ on C_p produces the following cycle which has length k.

$$\begin{tikzcd} C_p \arrow[r, "\Phi(\zeta)"] & \Phi(\zeta)C_p \arrow[r, "\Phi(\zeta)"] & \cdots \arrow[r, "\Phi(\zeta)"] & \Phi(\zeta)^{k-1}C_p \arrow[lll, bend left, "\Phi(\zeta)"] \end{tikzcd}$$

By a similar argument, if $n \in N$, then the cycle

$$nC_p \to \Phi(\zeta)nC_p \to \Phi(\zeta)^2 nC_p \to \cdots$$

also has length k. To see this, observe that $\Phi(\zeta)^t nC_p = nC_p$ if and only if $n^{-1}\Phi(\zeta)^t n \in C_p$. Since $n \in N$, this occurs if and only if $\Phi(\zeta)^t \in C_p$, so t must be a multiple of k. Therefore, the action of $\Phi(\zeta)$ on nC_p for $n \in N$ produces the following cycle which has length k.

$$\begin{tikzcd} nC_p \arrow[r, "\Phi(\zeta)"] & \Phi(\zeta)nC_p \arrow[r, "\Phi(\zeta)"] & \cdots \arrow[r, "\Phi(\zeta)"] & \Phi(\zeta)^{k-1}nC_p \arrow[lll, bend left, "\Phi(\zeta)"] \end{tikzcd}$$

Since there are $|N|/p$ cosets of C_p with coset representatives in N, and each cycle is of length k, the image of $\Phi(\zeta)$ will contain at least $|N|/kp$ cycles of length k. Consequently, there will be at least $|N|/kp$ canonical generators of Γ_p induced by ζ.

Now consider a coset hC_p where $h \notin N$. Considering the action of $\Phi(\zeta)$ on hC_p, we get

$$hC_p \to \Phi(\zeta)hC_p \to \Phi(\zeta)^2 hC_p \to \cdots.$$

We need to determine the length of this cycle or equivalently, the smallest value of t such that $\Phi(\zeta)^t \in hC_p h^{-1}$. Since $\Phi(\zeta)^k \in C_p$ and $C_p \cap hC_p h^{-1} = \{e\}$ (we are assuming $h \notin N$), $\Phi(\zeta)^t \in h^{-1}C_p h$ implies $\Phi(\zeta)^t = e$. The smallest value of t for which this happens is kp. Thus, the action of $\Phi(\zeta)$ on hC_p produces a cycle which has length kp. In particular, each coset hC_p for $h \notin N$ lies in a cycle of length kp and consequently no additional canonical generators of Γ_p are induced by ζ. Thus there are a total of $|N|/kp$ canonical generators of Γ_p induced by $\zeta \in \Gamma_A$.

If ζ induces $\eta \in \Gamma_N$, to determine the number of canonical generators of Γ_p induced by η, we apply an identical argument and get $|N|/kp$. Hence ζ and η induce the same number of canonical generators of Γ_p. $\square$

The following useful result is immediate.

Corollary 3.3. *The number of canonical generators of Γ_N inducing canonical generators of Γ_p is equal to the number of canonical generators of Γ_A inducing canonical generators of Γ_p. Moreover, they share the same orders.*

To summarize, we have shown that if we know the signature of Γ_N, then the signature of Γ_A is very closely related. This means to classify p-gonal surfaces, if we can determine all possible signatures for Γ_N, we can use this fact with other results regarding Fuchsian groups to determine the possible signatures for Γ_A and consequently the restrictions imposed on possible quotient groups. An interesting consequence to Theorem 3.2 is the following.

Corollary 3.4. *If $A \neq C_p$, then $N > C_p$.*

4. The classification of normal p-gonal surfaces

For the next three sections, we combine prior results with new techniques to explain how to classify three different families of cyclic p-gonal surfaces. In this section, we consider the family of normal cyclic p-gonal surfaces (so in this case $A = N$). In order to do this, first we shall explain how to determine all normal p-gonal overgroups (recall that a subgroup $H \leqslant A$ is a normal p-gonal overgroup if it contains a p-gonal subgroup which is normal). Though all such groups are known, see for example [16] or [22], we summarize how they were found as the methods are key to the classification of non-normal p-gonal overgroups. Following this, we explain how to determine whether or not a given normal p-gonal overgroup is indeed the full automorphism group of a normal p-gonal surface and hence how to classify normal p-gonal surfaces.

Suppose that $H \leqslant A$ is a normal p-gonal overgroup. Then H will contain a cyclic p subgroup C_p which is normal and such that the quotient space X/C_p has genus 0. Since the group $\tilde{K} = H/C_p \leqslant K = N/C_p$ acts on the quotient space X/C_p, it follows that $\tilde{K}$ is a finite group of automorphisms of the Riemann sphere. All such groups are well known and we tabulate them in Table 1. The branching data is a vector whose length is the number of critical values of the map $\pi_{\tilde{K}}: \Sigma \to \Sigma/\tilde{K}$ and whose entries are the number of fibers over each critical value where Σ denotes the Riemann sphere. It follows that any normal p-gonal overgroup must satisfy the short exact sequence in Fig. 4 where $\tilde{K}$ is a finite group of automorphisms of the Riemann sphere.

Complete solutions to this short exact sequence for $p \geqslant 3$ can be found in [16], or [22, Appendix B], and for $p = 2$, in [4]. If Γ_H denotes the Fuchsian group corresponding to H, we can use the following more general result to determine the possible signatures for Γ_H.

Proposition 4.1. *Suppose Γ_1 is a Fuchsian group with signature $(0; p, \ldots, p)$ where p is a prime and Γ is a Fuchsian group containing Γ_1 as a normal subgroup and let $\tilde{K}$ denote the quotient group. Then $\tilde{K}$ is an automorphism group of the Riemann sphere and the signatures for Γ_1 and Γ are related as follows.*

(1) *If $\tilde{K} \neq C_n$ and (d_1, d_2, d_3) is the branching data of the quotient map $\pi_{\tilde{K}}: \mathbb{H}/\Gamma_1 \to \mathbb{H}/\Gamma$, the signature of Γ is $(0; a_1d_1, a_2d_2, a_3d_3, \underbrace{p, \ldots, p}_{l\ \text{times}})$ where a_1, a_2, and a_3 are either 1 or p*

Table 1
Groups of automorphisms of the Riemann sphere and branching data

Group	Branching data
C_n	(n, n)
D_n	$(2, 2, n)$
A_4	$(2, 3, 3)$
S_4	$(2, 3, 4)$
A_5	$(2, 3, 5)$

$$1 \to C_p \to H \to \tilde{K} \to 1$$

Fig. 4. Short exact sequence for normal p-gonal overgroups.

depending upon whether any branch points of π_{Γ_1} *coincide with ramification points of* $\pi_{\tilde{K}}$. *For such a group* Γ, *the signature of* Γ_1 *is* $(0; \underbrace{p, \ldots, p}_{r \text{ times}})$ *where*

$$r = l|\tilde{K}| + \frac{(a_1 - 1)|\tilde{K}|}{(p-1)d_1} + \frac{(a_2 - 1)|\tilde{K}|}{(p-1)d_2} + \frac{(a_3 - 1)|\tilde{K}|}{(p-1)d_3}.$$

In particular, if $\zeta_1, \ldots, \zeta_{l+3}$ *are a set of canonical generators for* Γ, *then* ζ_i *induces* $(a_i - 1)|\tilde{K}|/((p-1)d_i)$ *generators of* Γ_1 *for* $1 \leqslant i \leqslant 3$ *and* $|\tilde{K}|$ *generators of* Γ_1 *for all other* ζ_i.

(2) *If* $\tilde{K} = C_n$, *the signature of* Γ *is* $(0; a_1 n, a_2 n, \underbrace{p, \ldots, p}_{l \text{ times}})$ *where* a_1 *and* a_2 *are either* 1 *or* p *depending upon whether any branch points of* π_{Γ_1} *coincide with ramification points of* $\pi_{\tilde{K}}$. *For such a group* Γ, *the signature of* Γ_1 *is* $(0; \underbrace{p, \ldots, p}_{r \text{ times}})$ *where*

$$r = ln + \frac{(a_1 - 1)}{(p-1)} + \frac{(a_2 - 1)}{(p-1)}.$$

In particular, if $\zeta_1, \ldots, \zeta_{l+2}$ *are a set of canonical generators for* Γ, *then* ζ_i *induces* $(a_i - 1)/(p-1)$ *generators of* Γ_1 *for* $1 \leqslant i \leqslant 2$ *and* n *generators of* Γ_1 *for all other* ζ_i.

Proof. This is a slightly generalized version of Proposition 3 in [21]. □

In order to determine all normal p-gonal overgroups, we proceed as follows. Start with a possible group H which satisfies the short exact sequence in Fig. 4 (so it is a candidate for a normal p-gonal overgroup). Next we use Proposition 4.1 to find all possible signatures for Fuchsian groups for which there *may* exist a surface kernel epimorphism $\rho : \Gamma_H \to H$ with $\operatorname{Ker}(\rho) \leqslant \Gamma_p \leqslant \Gamma_H$. Then we either show that such a map ρ exists by explicit construction or show that no such map can exist (see Remark 2.3). Since for a fixed p the number of groups and signatures is completely determined, this is fairly straightforward (see [22] for an overview). We can now use these results to classify normal p-gonal surfaces.

If H is a normal p-gonal overgroup for some p-gonal surface given by the surface kernel epimorphism $\rho : \Gamma_H \to H$, we associate the triple (H, Γ_H, ρ) called a generating triple for H. In order to classify normal p-gonal surfaces, for each generating triple (H, Γ_H, ρ) we need to determine whether there exists a p-gonal surface $X = \mathbb{H}/\operatorname{Ker}(\rho)$ for some fixed Fuchsian group Γ_H with full automorphism group H, or whether H is always contained in some larger automorphism group G. To this end, the following definition will be useful.

Definition 4.2. A Fuchsian group Γ_1 is finitely maximal if there does not exist a Fuchsian group Γ with $\Gamma_1 \leqslant \Gamma$ and $[\Gamma : \Gamma_1] < \infty$.

The signatures of all Fuchsian groups which are not finitely maximal were determined by Singerman in [18]. Since they are the only ones relevant to this classification, we tabulate those with orbit genus 0 in Table 2. For our purposes, if (H, Γ_H, ρ) is a generating triple for a normal p-gonal overgroup, unless the signature of Γ_H appears in Singerman's list, then there will always exist a finitely maximal Fuchsian group with that signature, and so $\operatorname{Ker}(\rho)$ cannot be normal and

Table 2
Singerman's list for genus 0 groups, [18]

Case	Signature Γ_1	Signature Γ	$[\Gamma : \Gamma_1]$
$N4$	$(0; t, t, t, t)$, $t \geqslant 3$	$(0; 2, 2, 2, t)$	4
$N5$	$(0; s, s, t, t)$, $s + t \geqslant 5$	$(0; 2, 2, s, t)$	2
$N6$	$(0; t, t, t)$, $t \geqslant 4$	$(0; 3, 3, t)$	3
$N7$	$(0; t, t, t)$, $t \geqslant 4$	$(0; 2, 3, 2t)$	6
$N8$	$(0; t, t, s)$, $t \geqslant 3$, $t + s \geqslant 7$	$(0; 2, t, 2s)$	2
$T1$	$(0; 7, 7, 7)$	$(0; 2, 3, 7)$	24
$T2$	$(0; 2, 7, 7)$	$(0; 2, 3, 7)$	9
$T3$	$(0; 3, 3, 7)$	$(0; 2, 3, 7)$	8
$T4$	$(0; 4, 8, 8)$	$(0; 2, 3, 8)$	12
$T5$	$(0; 3, 8, 8)$	$(0; 2, 3, 8)$	10
$T6$	$(0; 9, 9, 9)$	$(0; 2, 3, 9)$	12
$T7$	$(0; 4, 4, 5)$	$(0; 2, 4, 5)$	6
$T8$	$(0; t, 4t, 4t)$, $t \geqslant 2$	$(0; 2, 3, 4t)$	6
$T9$	$(0; t, 2t, 2t)$, $t \geqslant 3$	$(0; 2, 4, 2t)$	4
$T10$	$(0; 3, t, 3t)$, $t \geqslant 3$	$(0; 2, 3, 3t)$	4
$T11$	$(0; 2, t, 2t)$, $t \geqslant 4$	$(0; 2, 3, 2t)$	3

of finite index in any overgroup of Γ_H. In particular, unless the signature of Γ_H appears in Singerman's list, then there will always exist a normal p-gonal surface with full automorphism group H. If the signature of Γ_H does appear in Singerman's list, then for each possible overgroup Γ of Γ_H and for each epimorphism $\rho : \Gamma_H \to H$, we need to determine whether or not $\mathrm{Ker}(\rho)$ is normal in Γ. If there exists an epimorphism ρ such that $\mathrm{Ker}(\rho)$ is not normal in Γ for each possible Γ, then there exists a normal p-gonal surface with full automorphism group H and corresponding Fuchsian group Γ. Else, no such surface exists.

The general problem of whether or not the kernel of a surface kernel epimorphism $\rho_{\Gamma_1} : \Gamma_1 \to G$ onto a finite group G is normal in Γ where the signatures of Γ and Γ_1 appear in Singerman's list was considered in [6]. The results from [6] can be used explicitly to determine the existence or non-existence of normal p-gonal surfaces with given ramification behavior and full automorphism group (for the hyperelliptic case, see [8]). Due to the lengthy calculations required and because it is not really relevant to our main result, we omit details and complete results and instead illustrate the method of determination of maximal automorphism groups of normal p-gonal surfaces with a very explicit example.

Example 4.3. Suppose that Γ_H has signature $(0; t, t, t, t)$ for some $t \geqslant 3$ and $\iota_1, \ldots, \iota_4$ are canonical generators for Γ_H. Using Proposition 4.1, we must have $t = p$. Also, since $p \neq 2$, the only possible quotient groups are C_p (in which case $\Gamma_H = \Gamma_p$) and $C_p \times C_p$ (in which case, using Proposition 4.1 just two elliptic generators of Γ_H induce elliptic generators of Γ_p). We consider the case when the quotient group is C_p, so $\Gamma_H = \Gamma_p$. Let Γ_L denote the Fuchsian overgroup of Γ_p with signature $(0; 2, 2, 2, p)$ and Γ_J an overgroup with signature $(0; 2, 2, p, p)$.

If $H = \langle x \rangle$ and $p = 3$, then any epimorphism $\rho : \Gamma_p \to C_p$ is of the form $\rho(\iota_1) = x$, $\rho(\iota_2) = x$, $\rho(\iota_3) = x^2$, $\rho(\iota_4) = x^2$ up to a permutation of the generators $\iota_1, \ldots, \iota_4$ and an automorphism of H. In particular, using Theorem 5.1 of [6], $\mathrm{Ker}(\rho)$ will also be normal in both Γ_L and Γ_J. Observe however that Γ_p is also normal in Γ_L and Γ_J, so $X = \mathbb{H}/\mathrm{Ker}(\rho)$ is normal 3-gonal. Invoking Theorem 5.1 of [6], the full automorphism group of X is $\Gamma_L/\mathrm{Ker}(\rho) = D_6$. In particular, there is no normal 3-gonal surface X with full automorphism group C_3 whose corresponding Fuchsian group has signature $(0; 3, 3, 3, 3)$.

If $H = C_p = \langle x\rangle$ and $p \neq 3$, then define an epimorphism $\rho : \Gamma_p \to C_p$ as $\rho(\iota_1) = x$, $\rho(\iota_2) = x$, $\rho(\iota_3) = x$, $\rho(\iota_4) = x^{-3}$. Clearly there is no automorphism of C_p satisfying Theorem 5.1 of [6] for this epimorphism and hence $\mathrm{Ker}(\rho)$ is not normal in any of the possible overgroups of Γ_p with the two signatures given in Singerman's list. In particular, there is a normal p-gonal surface X with full automorphism group C_p and whose corresponding Fuchsian group has signature $(0; p, p, p, p)$ for $p \neq 3$.

Since in principle we know how to classify normal p-gonal surfaces, we shall henceforth assume that X is a non-normal p-gonal surface.

5. The classification for small primes

The next case we consider is when p is a small prime (that is $p \in \{2, 3, 5, 7\}$). The key step to the classification of such surfaces is due the following which is a consequence of the main result in [1].

Theorem 5.1. *If X is a cyclic p-gonal surface and $g > (p-1)^2$, then X is normal p-gonal.*

Since for each small prime p we have $(p-1)^2 \leqslant 36$, all possible cases for non-normal p-gonal surfaces will appear in Breuer's lists, see [5], which contain *all* automorphism groups that act on surfaces of genus up to 48 and the corresponding group signature data. Therefore, to classify all such surfaces, we just need to proceed through Breuer's lists and pick out groups and signatures satisfying the necessary conditions. We summarize.

Theorem 5.2. *Suppose p is a small prime. Then the full automorphism group and the signature of its corresponding Fuchsian group is one of those tabulated in Table* 3 (*where CD denotes the central diagonal subgroup of* $C_4 \times \mathrm{SL}(2,3)$ *of order* 2).

Proof. By Theorem 5.1, a 2-gonal surface will always be normal 2-gonal. In particular, no non-normal 2-gonal surface exists. For $p \in \{3, 5, 7\}$, we know that if $g > (p-1)^2$, then X is normal p-gonal. For the cases when $g \leqslant (p-1)^2$, we can proceed through Breuer's lists to determine

Table 3
Non-normal p-gonal overgroups for small primes

Prime	Signature of Γ_A	Genus	Automorphism group
3	$(0; 2, 3, 8)$	2	$GL(2, 3)$
3	$(0; 2, 3, 12)$	3	$(C_4 \times \mathrm{SL}(2, 3))/(CD)$
3	$(0, 2, 2, 2, 3)$	4	$(C_3 \times C_3) \rtimes V_4$
3	$(0; 2, 4, 6)$	4	$(C_3 \times C_3) \rtimes D_4$
5	$(0; 2, 4, 5)$	4	S_5
5	$(0; 2, 3, 10)$	6	$(C_5 \times C_5) \rtimes S_3$
5	$(0; 2, 2, 2, 5)$	16	$(C_5 \times C_5) \rtimes V_4$
5	$(0; 2, 4, 10)$	16	$(C_5 \times C_5) \rtimes D_4$
7	$(0; 2, 3, 7)$	3	$\mathrm{PSL}(2, 7)$
7	$(0; 2, 3, 14)$	15	$(C_7 \times C_7) \rtimes S_3$
7	$(0; 2, 2, 2, 7)$	36	$(C_7 \times C_7) \rtimes V_4$
7	$(0; 2, 4, 14)$	36	$(C_7 \times C_7) \rtimes D_4$

the non-normal p-gonal surfaces. However, rather than considering the complete lists (there are 1495 possible groups and signature types for $g = 36$ alone), we can refine these lists to eliminate groups and signature types not satisfying the necessary criteria.

First, we observe that the orbit genus of Γ_A must be 0 and there must be periods divisible by p. Second, since we are searching for automorphism groups of non-normal p-gonal surfaces, $|N| > p$ by Corollary 3.4. This means if $p^2 \mid |A|$, then $|A| \geqslant 2p^2$ (since C_p will be normal in a group of order p^2 but not normal in A since we are assuming X is a non-normal p-gonal surface). If $p^2 \nmid |A|$, then C_p is a Sylow p-subgroup of A. Since we are assuming A is a non-normal p-gonal overgroup, C_p cannot be unique, so using the Sylow Theorems, there must be at least $p+1$ subgroups of order p. Therefore, because $N > C_p$, it follows that $|A| \geqslant 2(p+1)p$, so in both cases we have $|A| \geqslant 2p^2$. Finally, the groups Γ_p and Γ_A must satisfy (2) of Theorem 2.4. Using these conditions, we reduce the list of possibilities to a much smaller list, and the remaining groups and signatures can be considered on a case-to-case basis. □

With the results we have so far proved, we shall henceforth assume that X is a non-normal p-gonal surface and $p \geqslant 11$.

6. The classification of non-normal p-gonal overgroups of order divisible by p^2

The last preliminary case we shall examine is when A is a non-normal p-gonal overgroup and $p^2 \mid |A|$ for $p \geqslant 11$ (though most results hold for all primes). Our method will be to show that most of these surfaces are non-normal Belyĭ p-gonal surfaces (defined below) and such surfaces were classified in [21]. The remaining cases can then be considered on a case-to-case basis.

Definition 6.1. A non-normal Belyĭ p-gonal surface X is a non-normal p-gonal surface admitting a normal p-gonal overgroup H such that the map $\pi_H : X \to X/H$ is branched over exactly three points.

We shall make explicit use of the following which is one of the main results of [13]. In particular, this result implies that X is normal p-gonal if and only if there exists a unique p-gonal group of X in A.

Theorem 6.2. *If X is a cyclic p-gonal surface and A is the full automorphism group of X, then all p-gonal groups are conjugate in A.*

The main result of this section is the classification given in Theorem 6.8. Since the proof is fairly technical and heavily computational, we break it down into a series of Lemmas.

Lemma 6.3. *Suppose G is an automorphism group of a compact Riemann surface X of genus g and $|G| \geqslant 13(g-1)$. Then the signature of the corresponding Fuchsian group Γ_G is one of those tabulated in Table* 4.

Proof. See for example Lemma 3.2 of [20]. □

We shall first describe the Sylow subgroups of A and their corresponding Fuchsian groups. Let S denote a Sylow subgroup of A which contains a cyclic p-gonal subgroup and let Γ_S denote its corresponding Fuchsian group.

Table 4
Signatures for large automorphism groups

Signature	Additional conditions
$(0; 3, 3, n)$	$4 \leqslant n \leqslant 5$
$(0; 2, 6, 6)$	
$(0; 2, 5, 5)$	
$(0; 2, 4, n)$	$5 \leqslant n \leqslant 10$
$(0; 2, 3, n)$	$7 \leqslant n \leqslant 78$

Lemma 6.4. *S has order p^2. Moreover, if S is cyclic, then Γ_S has signature $(0; p, p^2, p^2)$, and if S is elementary abelian, then S has signature $(0; p, p, p)$ or $(0; p, p, p, p)$.*

Proof. Suppose S has order p^n where $n \geqslant 3$. Since we are assuming X is a non-normal p-gonal surface, we have $g \leqslant (p-1)^2$, so for $p \geqslant 13$,

$$|S| = p^n \geqslant p^3 > p(g-1) \geqslant 13(g-1)$$

and for $p = 11$, we have $g \leqslant 100$ giving

$$|S| = 11^n \geqslant 11^3 > 13(100-1) \geqslant 13(g-1).$$

In particular, the signature for the Fuchsian group Γ_S corresponding to S must appear in Table 4. However, because $|S| = p^n$, each period of Γ_S must be divisible by p and for $p \geqslant 11$, no such signature exists in Table 4. Hence the Sylow-subgroups of A have order p^2.

To determine the signature for Γ_S we can use Proposition 4.1. Since S has order p^2, S is either cyclic or elementary abelian. In either case, the quotient group $S/C_p = \Gamma_S/\Gamma_p$ is cyclic of order p, so Proposition 4.1 implies Γ_S has signature $(0; a_1 p, a_2 p, \underbrace{p, \dots, p}_{l \text{ times}})$ for some l where $a_1 = 1$ or p and similarly with a_2. We consider the two different possibilities for S.

If S is cyclic, in order for there to exist a surface kernel epimorphism from Γ_S to S, there must be at least two elliptic elements of order p^2, so Γ_S must have signature $(0; p^2, p^2, \underbrace{p, \dots, p}_{l \text{ times}})$. Using Corollary 2.6, the genus of X is $g = lp(p-1)/2$. Since X is non-normal p-gonal, we must have $g \leqslant (p-1)^2$, so the only possibility is $l = 1$. Thus Γ_S must have signature $(0; p, p^2, p^2)$.

If S is elementary abelian, all non-identity elements will have order p, so Γ_S must have signature $(0; p, p, \underbrace{p, \dots, p}_{l \text{ times}})$. Again using Corollary 2.6, the genus of X is $g = (lp/2 - 1)(p-1)$, so if X is non-normal p-gonal, the only possibilities are $l = 1$ and $l = 2$. Thus Γ_S must have signature $(0; p, p, p)$ or $(0; p, p, p, p)$. □

For two of the three cases in Lemma 6.4, the group Γ_S is a triangle group and so any corresponding surface X will be a non-normal Belyĭ p-gonal surface. Therefore, the only case we need to consider in detail is the case where S is elementary abelian and Γ_S has signature $(0; p, p, p, p)$, so for the rest of this section we shall assume this is the case. Under these assumptions, we have the following.

Lemma 6.5. *S is the unique Sylow subgroup of A.*

Proof. By the Sylow theorems and because p^2 divides $|A|$, if S is not normal in A, then $|A| \geqslant p^2(p+1) > 13(g-1)$ for $p \geqslant 11$. In particular, the signature of Γ_A must appear in Table 4. Since we are assuming $p \geqslant 11$ and because Γ_A must have periods divisible by p, Γ_A must have signature $(0; 2, 3, k)$ with $k \leqslant 78$. This immediately implies $p < 78$. For the remaining possibilities for p, we know Γ_A must contain a subgroup with signature $(0; p, p, p, p)$, so we can use Theorem 2.4 to loop through all the possibilities for the signatures of Γ_A which contain such a group. The result is five different possible signatures for Γ_A: $(0; 2, 3, 33)$ with $|A| = 1452$, $(0; 2, 3, 39)$ with $|A| = 2028$, $(0; 2, 3, 51)$ with $|A| = 3468$, $(0; 2, 3, 57)$ with $|A| = 4332$ and $(0; 2, 3, 69)$ with $|A| = 6348$. However, in all cases, using GAP [11], for all possibilities for A, the Sylow p-subgroup is normal. $\square$

Let $\rho : \Gamma_S \to S$ denote the restriction of ρ_{Γ_A} to Γ_S. In general, if C is any cyclic subgroup of S of order p, then we can calculate the orbit genus of the Fuchsian group Γ_C corresponding to C by considering the map $\rho_C \circ \rho : \Gamma_S \to S/C$ and using Corollary 2.6 (where ρ_C is the quotient map from S to S/C). Specifically, after simplification, the orbit genus g_C of Γ_C is given by the formula

$$g_C = \left(\frac{j}{2} - 1\right)(p-1)$$

where j is the number of canonical generators of Γ_S with non-trivial image in the quotient group S/C under the map $\rho_C \circ \rho$. If C is cyclic p-gonal, then the Fuchsian group Γ_C must have orbit genus 0 and so $j = 2$. This means that just two of the elliptic generators have non-trivial image under $\rho_C \circ \rho$. Equivalently, and more importantly, this means two of the elliptic generators will have trivial image under $\rho_C \circ \rho$, and so it follows that exactly two of the elliptic generators lie in C under the epimorphism ρ. We can use these observations to describe the group A and the epimorphism ρ in much more detail.

Lemma 6.6. *A has just two cyclic p-gonal subgroups and $[\Gamma_A : \Gamma_N] = 2$. Moreover, if $C_p = \langle x \rangle$ and $C'_p = \langle y \rangle$ denote the two cyclic p-gonal subgroups of S and $\iota_1, \iota_2, \iota_3$ and ι_4 denote canonical generators for Γ_S then up to an automorphism in* Aut (S) *and a permutation of the ι_i, we have $\rho(\iota_1) = x$, $\rho(\iota_2) = x^{-1}$, $\rho(\iota_3) = y$ and $\rho(\iota_4) = y^{-1}$.*

Proof. Since we are assuming X is non-normal p-gonal, and since S is the unique Sylow subgroup, Theorem 6.2 implies that S must contain more than one cyclic p-gonal subgroup. As usual, let C_p denote a cyclic p-gonal subgroup and let C'_p denote some other cyclic p-gonal subgroup of S. Our previous observations imply the image of exactly two of the canonical generators will lie in C_p and the image of the other two in C'_p. This means that if C is any other cyclic subgroup of S of order p, then all four generators $\iota_1, \ldots, \iota_4$ will have non-trivial image under $\rho_C \circ \rho$ and so the orbit genus of Γ_C will be $g_C = (p-1) \neq 0$. In particular, C is not a cyclic p-gonal subgroup. The fact that $[\Gamma_A : \Gamma_N] = 2$ follows directly from this result together with Theorem 6.2 and the fact that any A-conjugate of a p-gonal group is also p-gonal (since the corresponding Fuchsian groups will be conjugate in PSL $(2, \mathbb{R})$ and signature is invariant under conjugation in PSL $(2, \mathbb{R})$).

To finish, we need to show that ρ has the indicated form. Up to a permutation of ι_1, ι_2, ι_3 and ι_4, since two lie in C_p and two lie in C'_p, we may assume $\rho(\iota_1), \rho(\iota_2) \in C_p$ and $\rho(\iota_3), \rho(\iota_4) \in C'_p$. If follows that $\rho(\iota_1) = \rho(\iota_2)^{-1}$ and $\rho(\iota_3) = \rho(\iota_4)^{-1}$ since the product $\rho(\iota_1)\rho(\iota_2)\rho(\iota_3)\rho(\iota_4)$ must be the identity. Then after composition with the automorphism $\phi \in \mathrm{Aut}(S)$ defined by $\phi(\rho(\iota_1)) = x$ and $\phi(\rho(\iota_3)) = y$, we get the desired epimorphism. □

With such an explicit description for the epimorphism ρ, we can apply results from [6] to determine larger groups in which Λ is normal and consequently larger groups of automorphisms of X.

Lemma 6.7. *The group Λ is normal in Γ_L with signature* $(0; 2, 2, 2, p)$ *and the quotient group $\Gamma_L/\Lambda \cong V_4 \ltimes S = L$ is a non-normal p-gonal overgroup for both p-gonal subgroups of S. Moreover, if L is not the full automorphism group of X, then X is a non-normal Belyĭ p-gonal surface.*

Proof. Using Lemma 6.6, the epimorphism ρ satisfies the conditions of N4 in [6], so the kernel $\mathrm{Ker}(\rho) = \Lambda$ is normal in Γ_L with signature $(0; 2, 2, 2, p)$ and with quotient group $L = V_4 \ltimes S$. We need to show that neither of the cyclic p-gonal subgroups are normal.

Suppose C is a normal subgroup of L of order p and let Γ_C be the corresponding Fuchsian group. Then the quotient Γ_L/Γ_C has order $4p$ and if C is a cyclic p-gonal group, then Γ_L/Γ_C must be isomorphic to a group of automorphisms of the Riemann sphere. In particular, since $p \geqslant 11$ and L contains no elements of order 4 (so neither does the quotient group), the only possibility is D_{2p}. Let $\rho_{\Gamma_C} : \Gamma_L \to \Gamma_L/\Gamma_C \cong D_{2p}$ denote the quotient map and $\psi_1, \ldots, \psi_4$ canonical generators where ψ_4 has order p. Observe that ψ_4 must have non-trivial image under ρ_{Γ_C} since D_{2p} cannot be generated by three or less elements of order 2 whose product is the identity. Using Corollary 2.6, it follows that Γ_C has no periods equal to p and orbit genus strictly greater than 0, so C cannot be cyclic p-gonal since Γ_C would have to have signature $(0; p, \ldots, p)$.

Next we need to show that if $L \neq A$ then X is a non-normal Belyĭ p-gonal surface which is equivalent to showing that the group Γ_N is a triangle group. First we observe that since Γ_p is normal in Γ_S, we must have $\Gamma_S \leqslant \Gamma_N$. Referring to Singerman's list, if Γ_N is not a triangle group, it is either a Fuchsian group with signature $(0; 2, 2, 2, p)$ or a Fuchsian group with signature $(0; 2, 2, p, p)$, so it suffices to show that Γ_N cannot have either of these signatures.

To show Γ_N cannot have signature $(0; 2, 2, 2, p)$, we imitate the proof above. To show that Γ_N cannot have signature $(0; 2, 2, p, p)$, we observe if $A \neq L$, then $[\Gamma_A : \Gamma_S] > 4$, so since $[\Gamma_A : \Gamma_N] = 2$ we must have $[\Gamma_N : \Gamma_S] > 2$. However, if Γ_J has signature $(0; 2, 2, p, p)$, then $[\Gamma_J : \Gamma_S] = 2$, so it follows that Γ_N cannot have signature $(0; 2, 2, p, p)$. □

Since Γ_L is a finitely maximal Fuchsian group, it follows that there exists a surface (in fact an infinite family of surfaces) which is non-normal p-gonal with full automorphism group L and corresponding Fuchsian group Γ_L. Any surface with $A \neq L$ is a non-normal Belyĭ p-gonal surface, so was classified in [21]. Therefore, combining all our results, we have the following.

Theorem 6.8. *Suppose X is cyclic p-gonal for $p \geqslant 11$, $p^2 \mid |A|$ but X is not normal p-gonal. Then the genus of X, the group A and the signature of the Fuchsian group corresponding to A is one of those given in Table* 5.

Table 5
Automorphism groups and signatures

Case	Signature	Automorphism group	Genus
1	$(0; 2, 3, 2p)$	$(C_p \times C_p) \rtimes S_3$	$\frac{(p-1)(p-2)}{2}$
2	$(0; 2, 2, 2, p)$	$(C_p \times C_p) \rtimes V_4$	$(p-1)^2$
3	$(0; 2, 4, 2p)$	$(C_p \times C_p) \rtimes D_4$	$(p-1)^2$

7. Computations for general p

We are now ready to finish the classification of non-normal p-gonal surfaces. Due to our previous results, we may henceforth assume that X is a non-normal p-gonal surface, $p \geqslant 11$ and p^2 does not divide $|A|$. Before we proceed with the classification, we make some observations regarding the signatures of Γ_A and Γ_N and the order of A. Recall that the group K was defined to be the group N/C_p and its branching data is given in Table 1.

Lemma 7.1. *There exist integers $n_1, \ldots, n_{\nu_1}, o_1/p, \ldots, o_\tau/p$ relatively prime to p such that $(n_1, \ldots, n_{\nu_1}, o_1/p, \ldots, o_\tau/p)$ is the branching data of K where $\nu_1 + \tau \leqslant 3$. Moreover, there exist an integer l and integers $m_1, \ldots, m_{\nu_2}$ with $\nu_2 \leqslant \nu_1$ and the property that each integer $n_1, \ldots, n_{\nu_1}$ must divide at least one of the integers $m_1, \ldots, m_{\nu_2}$ such that*

(1) *the signature of Γ_N is $(0; n_1, \ldots, n_{\nu_1}, o_1, \ldots, o_\tau, \underbrace{p, \ldots, p}_{l\ times})$,*
(2) *the signature of Γ_A is $(0; m_1, \ldots, m_{\nu_2}, o_1, \ldots, o_\tau, \underbrace{p, \ldots, p}_{l\ times})$.*

Proof. First, since $p^2 \nmid |A|$, it follows that $p \nmid |K|$. In particular, all entries of the branching data of K will be relatively prime to p and any elliptic elements of Γ_N of order divisible by p must induce elliptic generators of Γ_p. Using Proposition 4.1, it follows that Γ_N will have signature $(0; n_1, \ldots, n_{\nu_1}, o_1, \ldots, o_\tau, \underbrace{p, \ldots, p}_{l \text{ times}})$ where $(n_1, \ldots, n_{\nu_1}, o_1/p, \ldots, o_\tau/p)$ is the branching data of the map π_K (note that $\nu_1 + \tau = 2$ if $K = C_n$ and $\nu_1 + \tau = 3$ else), and the integers $n_1, \ldots, n_{\nu_1}, o_1/p, \ldots, o_\tau/p$ are all relatively prime to p.

To determine the signature of Γ_A, we first observe that the number of periods of Γ_A is at most equal to the number of periods of Γ_N, so applying Corollary 3.3 it follows that Γ_A has signature $(0; m_1, \ldots, m_{\nu_2}, o_1, \ldots, o_\tau, \underbrace{p, \ldots, p}_{l \text{ times}})$ where $\nu_2 \leqslant \nu_1$ and each period $n_1, \ldots, n_{\nu_1}$ has to divide at least one of the periods $m_1, \ldots, m_{\nu_2}$. $\square$

Next observe that since $p^2 \nmid |A|$ and X is non-normal p-gonal, using the Sylow theorems it follows that $|A| = bp(ap + 1)$ where $bp = |N|$, $a \geqslant 1$ and $(b, p) = 1$. In fact we can use our observations to impose further restrictions on a and b.

Lemma 7.2. *If $|A| = bp(ap + 1)$, then $ab \leqslant 13$.*

Proof. Since X is non-normal p-gonal, $g \leqslant (p-1)^2$, so

$$|A| = bp(ap + 1) > ba(g - 1).$$

If $ab \geqslant 13$, then $|A| > 13(g - 1)$ and Γ_A must have one of the signatures in Table 4. Since $p \geqslant 11$, we only need consider signatures of type $(0; 2, 3, k)$ where $11 \leqslant k \leqslant 78$ and k is divisible by some prime $p > 7$. Note that if Γ_A has signature $(0; 2, 3, k)$, then only one of its elliptic generators will induce an elliptic generator of Γ_p, so using the notation of Lemma 7.1, we have $\nu_2 = 2$. It follows that in the signature for Γ_N, we have $\nu_1 = 2$ or $\nu_1 = 3$. If $\nu_1 = 2$, then Γ_N is a triangle group, so the pair (Γ_A, Γ_N) must appear in Singerman's list. However, through observation for $p \geqslant 11$, no such pair appears. Therefore, we only need consider the case when $\nu_1 = 3$.

Again applying Lemma 7.1, if $\nu_1 = 3$, then we must have $k = p$ and Γ_N has signature $(0; n_1, n_2, n_3, p)$ where (n_1, n_2, n_3) is the branching data for K and each of the n_i must divide 2 or 3. The only signatures satisfying these criteria are $(0; 2, 3, 3, p)$ where $K = A_4$ for all primes $11 \leqslant p \leqslant 78$, and $(0; 2, 2, 2, p)$ where $K = V_4$, for all primes $11 \leqslant p \leqslant 78$. However, the signatures of Γ_A and Γ_N must also satisfy (2) of Theorem 2.4 and through explicit calculation it can be shown that none of these signatures do for each possible p. Hence there are no possible pairs for Γ_A and Γ_N if $ab > 13$, so $ab \leqslant 13$. □

Using Lemma 7.1, the number of periods of Γ_A is at most equal to the number of periods of Γ_N. However, if they have the same number of periods, then the signatures of Γ_A and Γ_N must appear in Singerman's list. Referring to the list, this can only happen if both are triangle groups or both have orbit genus zero and 4 periods. If they are both triangle groups, the corresponding surface X is a non-normal Belyĭ p-gonal surface and all such surfaces were classified in [21]. For the non-triangle case in Singerman's list, we have the following.

Lemma 7.3. *There are no possible choices of Γ_A and Γ_N where both have four periods.*

Proof. We only need to consider the two different signature pairs which appear in Singerman's list. For the first case, when Γ_N has signature $(0; t, t, t, t)$, Proposition 4.1 implies $t = p$ and N is either C_p or $C_p \times C_p$. However, we cannot have $N = C_p$ by Corollary 3.3, and $N = C_p \times C_p$ is not possible since we are assuming p^2 does not divide $|A|$.

For the second case, when Γ_N has signature $(0; s, s, t, t)$, with $s \neq t$, Proposition 4.1 implies either $t = p$ or $s = p$, so we assume $t = p$. Using the list of possible groups in [22], either $N = C_{sp}$ or $N = C_p \rtimes C_s$. If $A = \Gamma_A/\Lambda$, where Γ_A has signature $(0; 2, 2, s, p)$, then $[A : N] = 2$ and in particular, $N \lhd A$. However, for both choices of N, C_p is unique, so will also be normal in A. This contradicts that A is a non-normal p-gonal overgroup. □

For all other possible pairs of signatures for Γ_A and Γ_N, the number of periods of Γ_N is strictly greater than the number of those of Γ_A. Since $ab \leqslant 13$, we must have $b = |K| \leqslant 13$, so the only possibilities for K are C_n with $n \leqslant 13$, D_n for $n \leqslant 6$ and A_4. Using Proposition 4.1, for each of these groups we can find the possible signatures of Γ_N and then using Lemma 7.1, we can determine the possible signatures for Γ_A. We list all possibilities for the signatures in Table 6 where cases 1–2 are with $K = C_n$, cases 3–6 are with $K = D_n$ and cases 7–10 are with $K = A_4$.

Next observe that if the signatures of Γ_N and Γ_A are as given in Lemma 7.1, since $|A| = bp(ap+1)$, it follows that $[\Gamma_A : \Gamma_N] = ap + 1$, so (2) of Theorem 2.4 implies

$$(ap+1)\left(-2 + l\left(1 - \frac{1}{p}\right) + \sum_{i=1}^{\nu_1}\left(1 - \frac{1}{n_i}\right) + \sum_{i=1}^{\tau}\left(1 - \frac{1}{o_i}\right)\right)$$
$$= -2 + l\left(1 - \frac{1}{p}\right) + \sum_{i=1}^{\nu_2}\left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{\tau}\left(1 - \frac{1}{o_i}\right).$$

Table 6
Signature pairs not in Singerman's list

Case	Signature of Γ_N	Possible signatures for Γ_A	p
1	$(0; n, n, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; m, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{-2m+nalm+nm+n}{na(-m-1+lm)}$
2	$(0; n, np, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; np, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{-1+a+aln+n}{an(-1+l)}$
$3A$	$(0; 2, 2, n, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; m_1, m_2, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{-m_2m_1+nalm_1m_2+nm_1+nm_2}{an(-m_1-m_2+lm_1m_2)}$
$3B$		$(0; m, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{-m+nalm+nm+n}{na(-m-1+lm)}$
$4A$	$(0; 2, 2, np, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; m, np, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{am+almn+n}{an(-1+lm)}$
$4B$		$(0; np, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{a+aln+n}{an(-1+l)}$
5	$(0; 2p, 2, n, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; m, 2p, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{nm-2m+nam+2nalm+2n}{2na(-1+lm)}$
6	$(0; 2p, 2, np, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; 2p, np, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{n+2a+an+2aln}{2aln}$
$7A$	$(0; 2, 3, 3, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; m_1, m_2, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{-m_1m_2+6lm_1m_2+6m_1+6m_2}{6(-m_1-m_2+lm_1m_2)}$
$7B$		$(0; m, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{5m+6lm+6}{6(-m-1+ln)}$
8	$(0; 2, 3, 3p, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; m, 3p, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{m+2ln+2}{2(-1+ln)}$
9	$(0; 2p, 3, 3, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; m, 2p, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{5m+6lm+6}{6(-1+lm)}$
10	$(0; 2p, 3p, 3, \underbrace{p, \ldots, p}_{l \text{ times}})$	$(0; 2p, 3p, \underbrace{p, \ldots, p}_{l \text{ times}})$	$\frac{3+2l}{2l}$

Solving this equation for p, we get

$$p = \frac{al + \sum_{i=1}^{\nu_2}(1 - \frac{1}{m_i}) - \sum_{i=1}^{\nu_1}(1 - \frac{1}{n_i})}{-2 + l + \sum_{i=1}^{\nu_1}(1 - \frac{1}{n_i}) + \sum_{i=1}^{\tau}(1 - \frac{1}{o_i})}.$$

In particular, we can consider p as a function of $a, n_1, \ldots, n_{\nu_1}, m_1, \ldots, m_{\nu_2}, o_1, \ldots, o_\tau$ and for each of the possible cases in Table 6, we tabulate the formula for p.

With these results, we are now ready to finish the classification of non-normal p-gonal surfaces.

Theorem 7.4. *There are no additional p-gonal surfaces to those already found.*

Proof. Assume that $p \geqslant 11$, X is a non-normal p-gonal surface, p^2 does not divide $|A|$, and the number of elliptic generators of Γ_A is strictly less than those of Γ_N. Then the signatures for Γ_A

and Γ_N must appear in Table 6. Thus to finish the problem, we just need to analyze the different signature pairs given in Table 6. All possible cases follow a similar argument, so we describe one case in detail and omit calculations for the remaining cases.

We consider case 1 where $K = C_n$, so $b = n$. Since $ab \leqslant 13$, we only have a small number of possibilities for b and a. Specifically, for a fixed a with $1 \leqslant a \leqslant 13$, we have $2 \leqslant b \leqslant 13/a$. Since the arguments for all choices of a and b are similar, we illustrate with the case when $a = 3$ and $b = n = 4$. For this case, we get

$$p(l,m) = \frac{(m + 6lm + 2)}{6(-m - 1 + lm)}.$$

Differentiating with respect to l, we get

$$\frac{\partial p}{\partial l} = -\frac{m(7m + 8)}{6(-m - 1 + lm)^2} < 0,$$

so p is a decreasing function with respect to l. Likewise,

$$\frac{\partial p}{\partial m} = -\frac{(-1 + 8l)}{6(-m - 1 + lm)^2} < 0,$$

so p is a decreasing function with respect to m. Therefore, the largest value of p will occur when l and m are least, so when $l = 2$ and $m = 4$ (since $n \mid m$). Evaluating, we get $p \leqslant 3$, so this reduces to the small prime case considered in Theorem 5.2.

In general, for all other cases in Table 6, we first impose any necessary conditions given on the variables in the expression for p. Once all conditions have been imposed, we evaluate each case individually regarding p as a function of the remaining variables. In all cases, it can be shown that p is a decreasing function in a sufficient number of variables to obtain an upper bound for p. Specifically, for all cases we get $p \leqslant 9$, so all cases reduce to the small prime case already considered in Theorem 5.2. Thus no non-normal p-gonal overgroups exist in addition to those we have already determined. $\square$

8. Summary of results

Since the classification of non-normal p-gonal surfaces arose from the close examination of a number of different families, we summarize our results.

Table 7
Non-normal signatures and groups

Prime	Signature of Γ_A	Signature of Γ_N	g	A
3	$(0; 2, 3, 8)$	$(0; 2, 2, 2, 3)$	2	$GL(2, 3)$
3	$(0; 2, 3, 12)$	$(0; 3, 4, 12)$	3	$\mathrm{SL}(2, 3)/CD$
5	$(0; 2, 4, 5)$	$(0; 4, 4, 5)$	4	S_5
7	$(0; 2, 3, 7)$	$(0; 3, 3, 7)$	3	$\mathrm{PSL}(2, 7)$
$p \geqslant 5$	$(0; 2, 3, 2p)$	$(0; 2, p, 2p)$	$\frac{(p-1)(p-2)}{2}$	$(C_p \times C_p) \rtimes S_3$
$p \geqslant 3$	$(0; 2, 2, 2, p)$	$(0; 2, 2, p, p)$	$(p-1)^2$	$(C_p \times C_p) \rtimes V_4$
$p \geqslant 3$	$(0; 2, 4, 2p)$	$(0; 2, 2p, 2p)$	$(p-1)^2$	$(C_p \times C_p) \rtimes D_4$

Theorem 8.1. *Suppose X is a non-normal p-gonal surface. Then the signatures of Γ_A and Γ_N, the full automorphism group of X, the genus of X and where appropriate the different possibilities for p is one of those given in Table 7.*

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