PERGAMON

Applied Mathematics Letters 13 (2000) 53-57

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Existence of Three Solutions for a Two Point Boundary Value Problem

G. BONANNAO
Dipartimento di Informatica, Matematica, Elettronica e Trasporti
Facolta' di Ingegneria, Universita' di Reggio Calabria
via Graziella (Feo di Vito), 89100 Reggio Calabria, Italy
bonanno@ing.unirc.it

(Received and accepted July 1999)

Communicated by R. P. Agarwal

Abstract—The existence of an open interval of parameters so that an ordinary Dirichlet problem has at least three solutions is established. A completely novel assumption is emphasized. The approach is based on variational methods and critical points. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Three solutions, Multiplicity results, Two point boundary value problem, Eigenvalue problem.

INTRODUCTION

In this note, we consider the following autonomous ordinary Dirichlet problem:

$$\begin{aligned} u'' + \lambda f(u) &= 0, \\ u(0) = u(1) &= 0, \end{aligned} \tag{1}$$

where λ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Under a completely novel assumption on the function $\xi \to \int_0^\xi f(t)\,dt$ (see Remark 1), we prove the existence of an open interval $\Lambda \subseteq]0, +\infty[$ such that for every $\lambda \in \Lambda$, problem (1) has at least three classical solutions.

The same conclusion has been obtained in [1] for elliptic equations (however, the result also holds for ordinary equations) under different assumptions on f. In the field of ordinary equations, other authors have studied multiplicity results for special cases of (1) (see, for instance, [2,3]) and, in any case, they established the existence of two solutions (see also [4]).

Our approach is based on a three critical points theorem proved in [1], recalled below for the reader's convenience (Theorem 1), and on a technical lemma (Proposition 1) that allow us to apply it. Our main result is Theorem 2. Moreover, a sample application is presented (Example 1).

RESULTS

First, we recall the three critical points theorem of [1].

0893-9659/00/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(00)00033-1

THEOREM 1. *(See [1].) Let X be a separable and reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteax derivative is compact. Assume that*

$$\lim_{\|u\|\to+\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty,$$

for all $\lambda \in [0, +\infty[$, and that there exists a continuous concave function $h : [0, +\infty[\longrightarrow \mathbb{R}$ such that

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Then, there exist an open interval $\Lambda \subseteq]0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

Here, and in the sequel, X is the Sobolev space $W_0^{1,2}([0,1])$ endowed with the norm $\|u\| = (\int_0^1 |u'(t)|^2\, dt)^{1/2}$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and g is the function defined putting

$$g(\xi) = \int_0^\xi f(t)\, dt,$$

for every $\xi \in \mathbb{R}$.

Our main result is the following theorem.

THEOREM 2. *Assume that there exist four positive constants c, d, a, s, with $c < \sqrt{2}d$ and $s < 2$, such that*

(i)
$$f(t) \geq 0,$$
for every $t \in [-c, \max\{c, d\}]$,

(ii)
$$\frac{g(c)}{c^2} < \frac{1}{4}\frac{g(d)}{d^2},$$

(iii)
$$g(\xi) \leq a\,(1 + |\xi|^s),$$
for all $\xi \in \mathbb{R}$.

Then, there exist an open interval $\Lambda \subseteq]0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions belonging to $C^2([0,1])$ whose norms in $W_0^{1,2}([0,1])$ are less than q.

The proof of Theorem 2 is based on the following technical lemma.

PROPOSITION 1. *Under Assumptions (i) and (ii) of Theorem 2, there exist $r > 0$ and $u \in X$ such that*

$$2r < \|u\|^2$$

and

$$\max_{|\xi|\leq\sqrt{r/2}} g(\xi) < 2r\frac{\int_0^1 g(u(x))\, dx}{\|u\|^2}.$$

PROOF. We put

$$u(x) = \begin{cases} 4\,dx, & \text{if } x \in \left[0, \dfrac{1}{4}\right[, \\[2ex] d, & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right], \\[2ex] 4\,d(1-x), & \text{if } x \in \left]\dfrac{3}{4}, 1\right], \end{cases}$$

and

$$r = 2c^2.$$

Clearly $u \in X$ and $\|u\|^2 = 8d^2$. Hence, taking into account that $c < \sqrt{2}d$, one has

$$2r < \|u\|^2.$$

Moreover, owing to Assumption (i), one has

$$\int_0^1 g(u(x))\,dx \geq \int_{1/4}^{3/4} g(u(x))\,dx = \frac{1}{2} g(d).$$

Hence, one has

$$\frac{\int_0^1 g(u(x))\,dx}{\|u\|^2} \geq \frac{1}{16}\frac{g(d)}{d^2}.$$

Finally, taking into account that

$$\frac{\max_{|\xi| \leq \sqrt{r/2}} g(\xi)}{2r} = \frac{g(c)}{4c^2},$$

from Assumption (ii), the conclusion is obtained. ∎

PROOF OF THEOREM 2. For each $u \in X$, we put

$$\Phi(u) = \frac{1}{2}\|u\|^2, \qquad \Psi(u) = -\int_0^1 \left(\int_0^{u(x)} f(t)\,dt\right) dx,$$
$$J(u) = \Phi(u) + \lambda\Psi(u).$$

It is well known that the critical points in X of the functional J are precisely the weak solutions of problem (1). So, our end is to apply Theorem 1 to Φ and Ψ. Clearly, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuosly Gâteaux differentiable functional whose Gâteax derivative is compact.

Furthermore, thanks to (iii) and to Poincaré inequality, one has

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty,$$

for all $\lambda \in [0, +\infty[$.

Now, taking into account that for every $u \in X$, one has

$$\max_{0 \leq x \leq 1} |u(x)| \leq \frac{1}{2}\|u\|,$$

it follows that

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} (-\Psi(u)) \leq \max_{|\xi| \leq \sqrt{r/2}} g(\xi),$$

for each $r > 0$.

So, owing to Proposition 1, there exist $r > 0$ and $u \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{(-\Psi(u))}{\Phi(u)}.$$

Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < \rho < r\frac{(-\Psi(u))}{\Phi(u)}$$

and define $h(\lambda) = \lambda\rho$ for every $\lambda \geq 0$, from Proposition 3.1 of [5], we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Therefore, we can apply Theorem 1. It follows that there exist an open interval $\Lambda \subseteq]0, +\infty[$ and a positive real number q such that, for every $\lambda \in \Lambda$, the functional $J = \Phi + \lambda\Psi$ has three critical points that are three weak solutions of problem (1) whose norms in $W_0^{1,2}([0,1])$, are less than q. By using classical methods, it is easy to verify that the weak solutions belong to $C^2([0,1])$ and that they are classical solutions; hence, the conclusion is obtained. ∎

REMARK 1. The assumption of Theorem 2,

(ii′) there exist two positive constants c and d, with $c < kd$ $(k \geq 1)$ such that

$$\frac{g(c)}{c^2} < A\frac{g(d)}{d^2},$$

is, to the best of our knowledge, a completely novel condition to obtain multiplicity results in differential equations.

We explicitly observe that the constant A cannot be greater than $1/k$ and that the condition $c < kd$ cannot be dropped, as example $f(u) = 1$ shows.

At the moment, we do not know if the best constant actually is $A = 1/4$, but only that $A \not> \sqrt{2}/2$.

REMARK 2. Of course, in Theorem 2 we can assume

(j)

$$\int_0^d g(\xi)\,d\xi \geq 0;$$

(jj)

$$\frac{g(\xi)}{c^2} < \frac{1}{4}\frac{g(d)}{d^2},$$

for every $\xi \in [-c, c]$,

instead of (i) and (ii).

REMARK 3. Assumption (iii) cannot be dropped, as example $f(u) = e^u$ shows. In fact, problem (1) in this case, for every $\lambda \geq 0$, has at most two solutions (see [2]) and Assumptions (i) and (ii) are satisfied (choosing, for instance, $c = 1$ and $d = 6$).

Finally, we give an application of Theorem 2.

EXAMPLE 1. Let d be such that $(e^d - 1)/d^2 > (1/4)(e - 1)$ (for instance, $d = 6$). Put

$$f(t) = \begin{cases} e^t, & t \leq d, \\ \sqrt{t} + e^d - \sqrt{d}, & t > d, \end{cases}$$

one has

$$g(\xi) = \begin{cases} e^\xi - 1, & \xi \leq d, \\ \frac{2}{3}\sqrt{\xi}^3 + \left(e^d - \sqrt{d}\right)\xi + e^d(1-d) + \frac{1}{3}\sqrt{d}^3 - 1, & \xi > d. \end{cases}$$

It is easy to verify that with $c = 1$, $a = e^d$, and $s = 3/2$ assumptions of Theorem 2 are satisfied, so there exist an open interval $\Lambda \subseteq]0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1), in this case, admits at least three solutions belonging to $C^2([0,1])$ whose norms in $W_0^{1,2}([0,1])$ are less than q.

REFERENCES

1. B. Ricceri, On a three critical points theorem, (preprint).
2. I.M. Gelfand, Some problems in the theory of quasilinear equations, *Amer. Math. Soc. Translations* **29**, 295–381, (1963).
3. P. Korman and T. Ouyang, Exact multiplicity results for a class of boundary-value problems with cubic nonlinearities, *J. Math. Anal. Appl.* **194**, 328–341, (1995).
4. P. Korman and T. Ouyang, Exact multiplicity results for two classes of boundary value problem, *Diff. Integral Eqns.* **6**, 1507–1517, (1993).
5. B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, *Mathl. Comput. Modelling,* Special Issue on "Advanced topics in nonlinear operator theory" (Edited by R.P. Agarwal and D. O'Regan) (to appear).