On Unitary Collineation Groups

ALAN R. HOFFER

Department of Mathematics, University of Oregon, Eugene, Oregon 97403

Communicated by Richard Brauer

Received May 22, 1970

INTRODUCTION

If \( \Pi \) is a finite projective plane of order \( q^2 \) over the field \( K = \text{GF}(q^2) \), then \( \Pi \) admits a polarity \( \delta \) which is induced on \( \Pi \) by a nondegenerate hermitian form on the underlying vector space. The subgroup of the little projective group \( \text{PSL}_2(q^2) \) which centralizes \( \delta \) is the projective special unitary group \( \text{PSU}_2(q) \). The purpose of this paper is to show that the reverse of this situation also holds, namely, if \( \Pi \) is a finite projective plane of order \( q^2 \) which admits a collineation group \( G \) isomorphic to \( \text{PSU}_2(q^2) \), then \( \Pi \) is a desarguesian plane and \( G \) acts on \( \Pi \) in the usual manner, that is, as \( \text{PSU}_2(q) \) acts on \( \text{PG}_2(q^2) \).

A discussion of \( \text{PSU}_2(q) \) may be found in the works of L. Dickson [3] and B. Huppert [5]. For a thorough discussion of projective planes the reader may refer to the book by P. Dembowski [2].

1. PRELIMINARIES

If \( \Pi \) denotes a projective plane, a polarity of \( \Pi \) is a 1-1 involutory correspondence \( \delta \) between the set of points and the set of lines of \( \Pi \) which preserves the incidence relation \( \mid \), i.e., the point \( x \) is incident with the line \( L \), \( x \mid L \), if and only if \( L^\delta \mid x^\delta \). A point \( x \) is an absolute point of \( \Pi \) with respect to the polarity \( \delta \) if \( x \mid x^\delta \), and dually. If \( \Pi \) is a finite projective plane of order \( q^2 \), then \( \delta \) is a unitary polarity of \( \Pi \) if \( \delta \) is a polarity which admits \( q^3 + 1 \) absolute points. When \( \delta \) is a unitary polarity, the configuration \( \Phi \) formed by the absolute points and nonabsolute lines is a unital [2] and satisfies the properties that \( \Phi \) contains \( q^2(q^2 - q + 1) \) lines, \( q^3 + 1 \) points; each point of \( \Phi \) is on \( q^2 \) lines of \( \Phi \), and each line of \( \Phi \) contains \( q + 1 \) points of \( \Phi \). Each point of \( \Pi \) is on some absolute line, and dually.

If \( \Pi \) is the desarguesian plane \( \text{PG}_2(q^2) \) over the field \( \text{GF}(q^2) \), let \( \delta \) denote the unitary polarity of \( \Pi \) which arises from a nondegenerate hermitian form on the underlying vector space, and let \( \Phi \) denote the unital associated with \( \delta \).
If \( G = \text{PSU}_3(q) \) is the subgroup of \( \text{PSL}_3(q^3) \) which centralizes \( \delta \), then the order of \( G \) is \((q^3 + 1)q^3(q^3 - 1)/d\), where \( d = (q + 1, 3) \), and \( G \) is 2-transitive on the points of \( \Phi \), i.e., on the \( q^3 + 1 \) absolute points of \( \Pi \). For each point \( x \) in \( \Phi \), the stabilizer \( G_x \) is the semidirect product of a group \( Q \) of order \( q^3 \) by a cyclic group \( K \) of order \( (q^2 - 1)/d \) which is the stabilizer of two distinct absolute points of \( \Pi \). \( Q \) is regular (and transitive) on the remaining \( q^3 \) absolute points of \( \Pi \) and the center \( T(x) \) of \( Q \) is an elementary abelian \( p \)-group of order \( q \), where \( q = p^k \). \( T(x) \) is the group of \( (x, x^a) \)-elations in \( G \). \( G \) is transitive on the lines of \( \Phi \), and for each such line \( L \), the stabilizer \( G_L \) has order \((q + 1)^2(q^2 - q)/d\) and contains a normal subgroup \( D(L) \) which is a cyclic group of order \((q + 1)/d\). \( D(L) \) is the group of \((L, L)\)-homologies in \( G \). All the homology groups \( D(L) \), for \( L \) in \( \Phi \), are conjugate in \( G \) as are all the elation groups \( T(x) \), for \( x \) in \( \Phi \).

2. The Action of \( \text{PSU}_3(q) \) on a Plane of Order \( q^2 \)

For most of what follows we assume that \( \Pi \) is an arbitrary finite projective plane of order \( q^2 \) and that \( \Pi \) admits a collineation group \( G \) which is isomorphic to \( \text{PSU}_3(q) \). Since \( \Pi \) is desarguesian if \( q = 2 \), we may assume that \( q \) is greater than 2.

**Lemma 1.** If \( \Pi \) is a projective plane of order \( q^2 \) which admits a collineation group \( G \) isomorphic to \( \text{PSU}_3(q) \), then \( G \) fixes no point or line of \( \Pi \) and \( G \) has exactly one point orbit and a line orbit of length \( q^3 + 1 \) on which \( G \) is 2-transitive.

**Proof.** Let \( q = p^k \), where \( p \) is an odd prime and let \( \Delta_i \) denote the point orbits of \( \Pi \) of lengths \( n_i \), respectively. For \( q > 2 \), \( G \) is a simple group \([3]\) and is faithfully represented as a transitive permutation group on each \( \Delta_i \) for which \( n_i \neq 1 \). The smallest number of elements on which \( \text{PSU}_3(q) \) may be represented is \( q^3 + 1 \), except for \( q = 5 \), in which the smallest number is \( 50 \) \([6]\). Since there are \( q^2 + 1 \) lines on each point of \( \Pi \) and \( q^2 + 1 \) points on each line of \( \Pi \), and for \( q \neq 5 \), \( q^2 + 1 < q^3 + 1 \) and for \( q = 5 \), \( q^2 + 1 < 50 \), it must be that \( G \) fixes no point or line of \( \Pi \).

We refer to Mitchell's list of maximal subgroups of \( G \) and search this list for possible indices \( n_i \) in the interval \( q^3 + 1 \leq n_i \leq q^4 + q^2 + 1 \) when \( q \neq 5 \). For \( q = 5 \) we require that \( 50 \leq n_i \leq q^4 + q^3 + 1 = 651 \).

The following is the list of maximal subgroups of \( G \) \([6]\):

1. Groups of order \( q^3(q^2 - 1)/d \).
2. Groups of order \( q(q^2 - 1)(q + 1)/d \).
3. Groups of order \( 6(q + 1)/d \).
4. Groups of order \( 3(q^2 - q + 1)/d \).
5. Groups of order $q(q^2 - 1)$.
6. $\text{PSU}_d(p^m)$, $m \mid k$, $k/m$ is odd; order $p^{3m}(p^{3m} + 1)(p^{3m} - 1) d^{-1}$.
7. Groups containing $\text{PSU}_d(p^m)$ as a normal subgroup of index 3 if 3 divides $q + 1$ and $m$ divides $k$ where $k/m$ is odd.
8. Groups of order 216 if 9 divides $q + 1$; groups of orders 72 and 36 if 3 divides $q + 1$.
9. Groups of order 168 if $\sqrt{-7} \notin \text{GF}(p^k)$; or if $p = 7$.
10. Groups of order 360 if $\sqrt{3} \in \text{GF}(p^k)$ and a cube root of unity does not exist in $\text{GF}(p^k)$; or if $k$ is even and $p = 3$.
11. Groups of order 720 for $p = 5$ and $k$ is odd.
12. Groups of order 2520 for $p = 5$ and $k$ is odd.

For $q \not= 5$, the subgroups of type 1 admit the possible indices $(q^3 + 1)a$, where $a < (q^2 + q + 1)(q + 1)^{-1}$; the groups of type 2, for all $q$, and type 3, for $q = 3$, admit the indices $q^2(q^2 - q + 1)$; and the groups of type 4 through 12 have indices $n_i$ greater than $q^4 + q^3 + 1$. Since neither $(q^3 + 1)a$ nor $q^2(q^2 - q + 1)$ divide $N = q^4 + q^2 + 1$, and $N = (q^3 + 1) + q^2(q^2 - q + 1)$, there are exactly two point (and line) orbits of $G$ on $\Pi$ of lengths $q^3 + 1$ and $q^2(q^2 - q + 1)$. For $q = 5$, we again have possible indices $(q^3 + 1)a$, and $q^2(q^2 - q + 1)$ which arise from the groups of type 1 and type 2, respectively. The groups of types 4 through 10 have indices greater than 651. Subgroups of the groups of type 11 have order 720/b and index 175b. For $175b \leq 651$, we require $b = 1, 2,$ or 3, so the possible orbit lengths arising from the groups of type 11 are 175, 350 or 525. Subgroups of the groups of type 12 have order 2520/c and index 50c. For $50 \leq 651$, we require $1 \leq c \leq 12$. So for $q = 5$, the possible orbit lengths $n_i$ are $126a$, for $0 \leq a \leq 5$, 175, 350, 525, and 50c, for $0 \leq c \leq 12$. Since we must satisfy the Diophantine equation $\sum m_i \cdot n_i = 651$, for $m_i > 0$, only two possibilities survive, namely, $n_1 = 126$, $n_2 = 525$; or $n_1 = 126$, $n_i = 175$, $i = 2, 3, 4$. Hence, $G$ has exactly one orbit of length $q^3 + 1$.

For $q$ even, say $q = 2^k > 2$, the following is the list of maximal subgroups of $G$, where $d = 1$ or 3 according as $k$ is even or odd [4]:

1. Groups of order $2^{2k}(2^{2k} - 1) d^{-1}$.
2. Groups of order $2^k(2^{2k} - 1)(2^k + 1) d^{-1}$.
3. Groups of order $6(2^k + 1)^2 d^{-1}$.
4. Groups of order $3(2^{2k} - 2^k + 1) d^{-1}$.
5. $\text{PSU}_d(2^m)$ of order $2^{3m}(2^{3m} + 1)(2^{3m} - 1)$, where $k/m$ is an odd prime.
6. Groups containing $\text{PSU}_d(2^m)$ as a normal subgroup of index 3 when $m$ is odd and $k = 3m$.
7. A group of order 36 when $k = 1$. 

For $q > 2$, the only indices less than $q^4 + q^2 + 1$ arise from the groups of type 1 or type 2. In the former case, the possible indices are $(q^3 + 1)a$, where $a < (q^2 + q + 1)(q + 1)^{-1}$, and in the latter only the index $q^2(q^2 - q + 1)$ occurs. Hence, as in the case when $q$ is odd, $G$ fixes no point or line of $\Pi$, and for $q$ even $G$ admits exactly two point and two line orbits of lengths $q^3 + 1$ and $q^2(q^2 - q + 1)$.

The transitive representations of $\text{PSU}_3(q)$ of degree $q^3 + 1$ correspond to the action of $\text{PSU}_3(q)$ on the subgroups which fix the center and axis of a group of elations and all such type 1 subgroups are conjugate. So $G$ is 2-transitive on the orbit of length $q^3 + 1$.

**Lemma 2.** If $\Pi$ is a projective plane of order $q^3$ which admits a collineation group $G$ isomorphic to $\text{PSU}_3(q)$, then $\Pi$ contains a unital which is fixed by $G$.

**Proof.** Let $\Delta$ denote the point orbit of $\Pi$ of length $q^3 + 1$ and $\Phi$ the following collections of points and lines of $\Pi$: the points of $\Phi$ are those of $\Delta$ and the lines of $\Phi$ are the lines of $\Pi$ which contain at least two points of $\Delta$. Since $G$ is 2-transitive on $\Delta$, each line in $\Phi$ contains the same number of points of $\Delta$; denote this number as $k$. So $\Phi$ is a $(b, v, r, k, \lambda)$-design with $v = q^3 + 1$, and $\lambda = 1$. Using the relations $r(k - 1) = \lambda(v - 1)$, and $bk = vr$, we have $r = q^3(k - 1)$ and $b = (q^3 + 1)q^3/k(k - 1)$. Here $r$ is the number of lines of $\Phi$ on each point of $\Phi$ and since $r \leq q^2$, we have $k - 1 \geq q$. Also $k$ divides $q^3 + 1$ and so $k = q + 1$. This implies that $b = q^2(q^2 - q + 1)$, and $r = q^3$. So $\Phi$ is a unital and is clearly fixed by $G$.

**Lemma 3.** (Seib). Let $\Pi$ be a finite projective plane of order $q^3$, $\Phi$ a unital embedded in $\Pi$, and $\alpha$ a planar involution of $\Pi$ which fixes $\Phi$. If $\Pi_0$ is the subplane fixed point-wise by $\alpha$, then exactly $q + 1$ points and $q^2$ lines of $\Phi$ are in $\Pi_0$; also

(i) If $q$ is even, the points of $\Phi$ in $\Pi_0$ lie on a line.

(ii) If $q$ is odd, the points of $\Phi$ in $\Pi_0$ form an oval.

For a proof of Lemma 3, see Seib's paper [8].

**Lemma 4.** If $\Pi$ is a projective plane of order $q^3$ which admits a collineation group $G$ isomorphic to $\text{PSU}_3(q)$, then all the involutions in $G$ are perspectivities.

**Proof.** Let $\Phi$ be the unital in $\Pi$ arising from the point orbit $\Delta$ of length $q^3 + 1$. All the involutions in $G$ are conjugate; so suppose some involution $\alpha$ in $G$ point-wise fixes a subplane $\Pi_0$ of order $q$. Then, by Lemma 3, $\alpha$ fixes exactly $q + 1$ points of $\Phi$, and so $q$ is odd. This is so since an involution in $G$ when considered as a permutation in the representation on the isotropic vectors of $\text{PG}_3(q^3)$ is either an elation or a homology. So if $\alpha$ fixes $q + 1$ such
vectors, $\alpha$ corresponds to a homology of $\text{PG}_2(q^2)$ and since $\alpha$ has order 2, 2 divides $q - 1$; so $q$ is odd. On the other hand, $G$ acts on the points of $\Phi$ as does $\text{PSU}_3(q)$ on the absolute points of $\text{PG}_2(q^2)$. So the centralizer of $\alpha$ in $G$, $C_G(\alpha)$ fixes a line of $\Phi$. But $C_G(\alpha)$ has exactly two orbits of length $q + 1$ and $q^2 - q$; so the $q + 1$ points of $\Phi$ fixed by $\alpha$ lie on this line fixed by $C_G(\alpha)$. This implies that $q$ is even. The contradiction shows that no involution in $G$ fixes a proper subplane point-wise and by [1] all the involutions in $G$ are perspectivities.

3. The Main Theorem

**Theorem.** If $\Pi$ is a projective plane of order $q^2$ and $G$ is a collineation group of $\Pi$ which is isomorphic to $\text{PSU}_3(q)$, then

1. $\Pi$ is a desarguesian plane, and

2. $G$ contains all the possible elations of $\Pi$ which commute with a suitable polarity of unitary type.

**Proof.** We separate the proof into two parts.

Case 1. $q$ even. Let $T$ be an elementary abelian 2-group of order $q$ in $G$; all $q^3 + 1$ such subgroups are conjugate in $G$. The elements in $T$ are elations of $\Pi$ since $G$ has no planar involutions. Let $\alpha$ be a nonidentity elation in $T$ with center at $x$; then since each element in $T$ commutes with $\alpha$, $T$ fixes $x$. The normalizer, $N$, of $T$ in $G$ has order $q^3(q^2 - 1)/d$ and so the index $|G : N|$ is equal to $q^3 + 1$. If $\gamma$ in $G$ does not normalize $T$, then $\gamma$ cannot fix $x$, for otherwise since $N$ is a maximal subgroup of $G$ [4], $N$ and $N\gamma$ generate $G$ and this implies that $G$ fixes $x$, which is not the case. Hence, $x$ is in an orbit of length $q^3 + 1$ and so is a point of the unital $\Phi$. $\alpha$ fixes no point of $\Phi$ other than $x$, so the axis of $\alpha$ is not a line of $\Phi$ and hence is the unique line on $x$, denoted $x^\circ$, which is not a line of the unital. This also implies that $T$ is the group of $(x, x^\circ)$-elations in $G$. The transitivity of $G$ on $\Delta$ implies that for each point $x$ in $\Phi$, there is a group $T'(x)$ of $(x, x^\delta)$-elations of order $q$ in $G$. Hence, there is a 1-1 involutory correspondence $\delta$ between the points $x$ of $\Phi$ and the lines $x^\delta$ of $\Pi$ not in $\Phi$ such that $x$ is incident with $x^\delta$.

We now show that $\delta$ is a polarity of $\Pi$. Let $a_1$, $a_2$ be two distinct points in $\Phi$, and let $a_1^\circ \cap a_2^\circ = l$, and $L = a_1a_2$. It must be that $a_i^\delta$ contains $l$ for each point $a_i$ in $\Phi \cap L$ since if $a_i^\delta \cap a_i^\delta = m$, for $i \neq 1, 2$, there exists an elation $\alpha$ in $T(a_i)$ such that $a_2^\alpha = a_2 \delta$. So $a_2^\delta = a_2^\delta$ and $m^\delta | a_2^\delta$. But $m$ is fixed by $\alpha$ since $m | a_1^\delta$, so $m | a_2^\delta$, hence $m = l$. Clearly $l$ is not a point in $\Phi$ since the only point of $\Phi$ on $a_2^\delta$ is $a_1$. If we denote $l^\delta = L$, for each such $l$, then $\delta$ yields a 1-1 correspondence between the points of $\Pi$ not in $\Phi$ with the lines
of $\Phi$, $\delta$ is defined so as to preserve incidence and so $\delta$ is a polarity of $\Pi$ with the $q^2 + 1$ points of $\Phi$ as absolute points. Also $\delta$ is centralized by $G$ and has $\Phi$ as the associated unital.

Now let $a_1, a_2$ denote two distinct points of $\Phi$ and $T_1, T_2$ the groups of $(a_1, a_1^\delta)$- and $(a_2, a_2^\delta)$-elations in $G$. Then the group $H$ generated by $T_1$ and $T_2$ is a subgroup of $G_L$, where $L$ is the line on $a_1$ and $a_2$, and $H$ is isomorphic to $\text{PSL}_2(q)$, [4]. Also $G_L$ is the product of $H$ with a cyclic group $D$ of order $(q + 1)/d$ which fixes the $q + 1$ points of $\Phi$ on $L$. Let $\Sigma$ denote the set of $q^2 - q$ points of $\Pi$ on $L$ which are not in $\Phi$. Then if $x$ is a point in $\Sigma$, $|H_x| \geq \frac{q(q^2 - 1)}{(q^2 - q)} = q + 1$, which is odd and so $|H_x| = q + 1$; so $H$ is primitive on $\Sigma$. Since $G$ leaves $\delta$ invariant, $H$ is imprimitive on $\Sigma$ where the blocks have length 2 and consist of the pairs of points $x, x^\delta \cap L$, $x \in \Sigma$, and $H$ is primitive on this block system. But $G_L$ is the product of $H$ and $D$ with $D$ normal in $G_L$; so either $D$ fixes all the blocks or $D$ is transitive on the $(q^2 - q)/2$ blocks. Since $D$ has order $(q + 1)/d$, $D$ fixes the blocks element-wise and since $D$ has odd order $D$ fixes each point of $\Sigma$ and hence each point of $L$. Also $D$ fixes $L^\delta$; so $D$ is the group $D(L)$ of $(L^\delta, L)$-homologies in $G$. Hence, we have a correspondence between each line $L$ of $\Phi$ with a group of $(L^\delta, L)$-homologies in $G$ of order $(q + 1)/d$.

We are at the position where every point $x$ in $\Phi$, and the line $x^\delta$ on $x$ and not in $\Phi$, are associated with an elementary abelian 2-group of elations $T(x)$ of order $q$ in $G$, and each line $L$ of $\Phi$, and point $L^\delta$ not in $\Phi$, are associated with a cyclic group $D(L)$, of order $(q + 1)/d$ of $(L^\delta, L)$-homologies in $G$. Now the only elations of $\Pi$ which commute with $\delta$ are $(x, x^\delta)$-elations, where $x$ is a point of $\Phi$, i.e., an absolute point of $\delta$. For each $x$ in $\Phi$, there are $q$ other points of $\Phi$ on each line of $\Phi$ which contains $x$; so $G$ contains all the possible $(x, x^\delta)$-elations of $\Pi$ which commute with $\delta$.

Let $\Pi$ denote the desarguesian plane $\text{PG}_2(q^2)$ on which $\text{PSU}_3(q)$ acts as a collineation group which leaves invariant a unitary polarity $\delta$. Let $\bar{\theta}$ be an isomorphism between $G$ and $G = \text{PSU}_3(q)$; then $\bar{\theta}$ induces a correspondence $\theta$ between $\Pi$ and $\Pi$ defined by:

For $x \in \Phi$, $\begin{cases} \theta(x) = \bar{x} \\ \theta(x^\delta) = \bar{x}^\delta \end{cases} \Rightarrow \bar{\theta}(T(x)) = \bar{T}(\bar{x});$

For $L \in \Phi$, $\begin{cases} \theta(L) = L \\ \theta(L^\delta) = L^\delta \end{cases} \Rightarrow \bar{\theta}(D(L)) = \bar{D}(L).$

Here the point-line pair $(x, x^\delta)$, with $x \in \Phi$ determines the group $T(x)$ and under $\bar{\theta}$, $T(x)$ corresponds to $\bar{T}(\bar{x})$, a group of elations in $G$ with center at $\bar{x}$. Each point-line pair $(L^\delta, L)$ with $L \in \Phi$ determines the group $D(L)$ and under $\theta$, $D(L)$ corresponds to $\bar{D}(L)$, a group of homologies in $G$ with center at $L^\delta$. For $q > 2$, the groups $D(L)$ are nontrivial; so $\theta$ is a 1-1 correspondence.
between the points and lines of $\Pi$ with the points and lines of $\overline{\Pi}$. We must show that $\theta$ preserves the incidence relation. The basic tool is that two $(x, X)$- and $(y, Y)$-perspectivities $\alpha$ and $\beta$ commute if and only if $x \mid Y$ and $y \mid X$. Since $G$ centralizes $\delta$, the perspectivity groups $G(x, x^\delta)$ and $G(y, y^\delta)$ centralize each other if and only if $x \mid y^\delta$ or $y \mid x^\delta$. If $X$ is a line not in $\Phi$, and $x$ is a point of $\Phi$, then $x \mid X$, $x^\delta = X$; so under $\theta$, $T(x)$ corresponds to $\theta(x)$ and so $\theta(x)$ fixes $X$ and $x$, where $X$ is a line not in $\Phi$, and $x$ is a point of $\Phi$, then $x \mid X$, $x^\delta = X$; so under $\theta$, $T(x)$ corresponds to $\theta(x)$ and so $\theta(x)$ fixes $X$ and $x$, where $X$ is a line not in $\Phi$. If $X$ is a line not in $\Phi$, and $x$ is a point of $\Phi$, then $x \mid X$, $x^\delta = X$; so under $\theta$, $T(x)$ corresponds to $\theta(x)$ and so $\theta(x)$ fixes $X$ and $x$. If $L$ is a line in $\Phi$ and $m$ is a point not in $\Phi$, then $m \mid L$ fixes $L$ and $L^\delta$ fixes $L$ and $L^\delta$ fixes $L$. So $\delta$ is an isomorphism and $\Pi$ is a desarguesian plane.

**Case 2.** $q$ odd. Let $\sigma_1$, $\sigma_2$ be two distinct involutions in $G$; these are homologies of $\Pi$ since $q$ is odd and $G$ contains no planar involutions. Let $C_G(\sigma_i)$, $i = 1, 2$, denote the centralizers of $\sigma_i$ in $G$. If $\sigma_1$ and $\sigma_2$ have the same axis $L$, then $L$ is fixed by $G$, the group generated by $C_G(\sigma_1)$ and $C_G(\sigma_2)$ [6], which is not the case. Likewise $\sigma_1$ and $\sigma_2$ have distinct centers. Hence, each involution in $G$ is associated with a distinct nonincident point-line pair. An involution $\sigma$ in $G$ corresponds to a homology in $PSU_3(q)$ acting on $PG_2(q^2)$ and so $\sigma$ fixes exactly $q + 1$ points of $\Phi$, and these points lie on the line $L$, so $L$ is a line of $\Phi$; likewise the center of $\sigma$ is a point not in $\Phi$, which we denote as $L^\delta$. This yields a 1-1 involutory correspondence $\delta$ between the lines of $\Phi$ and the points not in $\Phi$.

Let $Q$ denote a Sylow $p$-subgroup in $G$ of order $q^3$. $Q$ fixes a point $x$ in $\Phi$ and is regular (and transitive) on the points of $\Phi$ different from $x$. So $Q$ is transitive on the lines of $\Phi$ on $x$ and fixes the unique line of $\Pi$ on $x$ which does not lie in $\Phi$, denote this line as $x^\delta$. Let $T(x)$ denote the center of $Q$; $T(x)$ is an elementary abelian $p$-group of order $q$ which in its action on $\Phi$ fixes all the blocks on $x$. Hence, $T(x)$ fixes all the points on $x^\delta$ and $T(x)$ is a group of $(x, x^\delta)$-elations in $G$.

Now $\delta$ is a 1-1 involutory correspondence between the points and lines of $\Pi$ which we show is a polarity of $\Pi$. If $x$ is a point of $\Phi$ and $L$ is a line of $\Phi$, then if $x \mid L$ our definitions imply that $L^\delta \not= x^\delta$. Now $x \mid Y$, where $Y$ is not a line of $\Phi$ and $x^\delta \mid Y^\delta = x$. If $L$ and $M$ are two distinct lines of $\Phi$, let $\sigma_L$, $\sigma_M$ denote the unique involutory homologies with axes $L$ and $M$ respectively. Then $M^\delta \mid L$ fixes $M^\delta \mid L^\delta$ fixes $M^\delta \mid L^\delta$, so $\delta$ is a polarity of $\Pi$ and $G$ leaves $\delta$ invariant.

An elation of $\Pi$ which commutes with $\delta$ must be an $(x, x^\delta)$-elation for some point $x$ in $\Phi$. Since $T(x)$ has order $q$ and there are exactly $q$ points in $\Phi$, different from $x$, and, on each line of $\Phi$ which contains $x$, $G$ contains all the possible elations of $\Pi$ which commute with $\delta$.
Let \( \Pi, G, \theta, \) etc. have the same meaning as in Case 1. For each point-line pair \( x, X \) with \( x \in \Phi \) and \( x^\theta = X \), there corresponds a pair \( \bar{x}, \bar{X} \) in \( \Pi \) such that \( \bar{x}^\delta = \bar{X} \) and \( T(x) \) corresponds to \( T(\bar{x}) \) in \( G \). For each pair \( (l, L) \) in \( \Pi \) with \( l^\theta = L \), and \( L \) in \( \Phi \), there corresponds a pair \( \bar{l}, \bar{L} \) in \( \Pi \) such that \( \bar{l}^\delta = \bar{L} \) and \( \sigma_L \), the unique involutory homology in \( G \) with axis \( L \) corresponds to \( \bar{\sigma}_{\bar{L}} \) in \( G \). This yields the 1-1 correspondence between the points and line of \( \Pi \) with the points and lines of \( \Pi \). As before, if \( x \in \Phi \) and \( X \notin \Phi \), then \( x \mid X \Leftrightarrow X = x^\theta \Leftrightarrow \bar{x} \mid \bar{X} \). If \( L \) is a line of \( \Phi \) and \( x \) is a point of \( \Phi \), then \( x \mid L \) and \( L^\delta \mid x^\delta \Leftrightarrow T(x) \) fixes \( L \) and \( L^\delta \Leftrightarrow \sigma_L \) normalizes \( T(x) \Leftrightarrow \bar{\sigma}_L \) normalizes \( T(\bar{x}) \Leftrightarrow T(\bar{x}) \) fixes \( \bar{L} \Leftrightarrow \bar{x} \mid \bar{L} \) and \( \bar{L}^\delta \mid \bar{x}^\delta \). If \( L \) and \( M \) are distinct lines of \( \Phi \), then \( M^\delta \mid L \Leftrightarrow \sigma_L \) and \( \sigma_M \) commute \( \Leftrightarrow \bar{\sigma}_L \) and \( \bar{\sigma}_M \) commute \( \Leftrightarrow \bar{M}^\delta \mid \bar{L} \), and dually. So \( \Pi \) is a desarguesian plane.

Acknowledgment

I am grateful to W. Kantor and J. Yaqub for reading the original manuscript and making suggestions which resulted in eliminating an unnecessary hypothesis in the main theorem.

References

4. R. W. Hartley, Determination of the ternary collineation groups whose coefficients lie in the GF(2^a), *Ann. of Math.* 27 (1926), 140–158.