Martingale solutions and Markov selection of stochastic 3D Navier–Stokes equations with jump

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\textbf{A R T I C L E  I N F O}

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\textbf{A B S T R A C T}

In this paper, we study the existence of martingale solutions of stochastic 3D Navier–Stokes equations with jump, and following Flandoli and Romito (2008) \cite{7} and Goldys et al. (2009) \cite{8}, we prove the existence of Markov selections for the martingale solutions.

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\textbf{1. Introduction}

Let $D$ be a bounded open domain of $R^3$ with regular boundary $\partial D$, consider Newtonian fluid described by the stochastic 3-dimensional Navier–Stokes equation on $D$,

\begin{equation}
\frac{\partial u(t,x)}{\partial t} - \nu \Delta u(t,x) + (u(t,x) \cdot \nabla)u(t,x) = -\nabla p(t,x) + f(t,x) + G(u,\xi)(t,x),
\end{equation}

with the incompressibility condition

\[ \nabla \cdot u(t,x) = 0. \]
\[
\text{div } u(t, x) = 0, \quad t \in [0, \infty), \; x \in D, \tag{1.2}
\]
the boundary condition
\[
u(t, x) = 0, \quad t \in [0, \infty), \; x \in \partial D, \tag{1.3}
\]
and the initial condition
\[
u(0, x) = u_0(x), \quad x \in D. \tag{1.4}
\]

The fluid is described by the velocity field \( u = u(t, x) \) and the pressure field \( p = p(t, x) \). The parameter \( \nu > 0 \) is the kinematic viscosity. Here \( G \) is an operator acting on noise and solution. When the process \( \xi(t, x) \) is a Brownian motion, the stochastic equation (1.1) has been studied by many authors, see [3,5,6,8]. It is known that there exists a global solution of the martingale problem for this case; and also the Markov selections for the martingale solution, see [5,7,8].

Up to our knowledge, there have no results as the \( \xi(t, x) \) is a Lévy noise. In this paper, we prove that there exist Martingale solutions of stochastic 3D Navier–Stokes equations with jump, and then we prove that there exist Markov selections for the martingale solutions.

We consider the usual abstract form of Eqs. (1.1)–(1.4). Let \( \mathscr{D}^\infty \) be the space of infinitely differentiable 3-dimensional vector fields \( u(x) \) on \( D \) with compact support strictly contained in \( D \), satisfying \( \text{div} u(x) = 0 \). Denote by \( V_\alpha \) the closure of \( \mathscr{D}^\infty \) in the Sobolev space \( [H^\alpha(D)]^3 \), for \( \alpha \geq 0 \), and in particular
\[
H = V_0, \quad V = V_1.
\]

Denote by \( | \cdot |_H \) and \( \langle \cdot, \cdot \rangle_{H,H} \) the norm and inner product in \( H \). Identifying \( H \) with its dual space \( H' \), and let \( V_\alpha' \) the dual space of \( V_\alpha \), we have \( V_\alpha \subset H = H' \subset V_\alpha' \) with continuous injections. Denote the dual pairing between \( V_\alpha \) and \( V_\alpha' \) by \( \langle \cdot, \cdot \rangle_{V_\alpha, V_\alpha'} \).

Let \( D(A) = [H^2(D)]^3 \cap V \), and define the linear operator \( A : D(A) \subset H \to H \) as \( Au = -P \Delta u \), where \( P \) is the projection from \( [L^2(D)]^3 \) to \( H \). Since \( V \) coincides with \( D(A^{1/2}) \), we can endow \( V \) with the norm \( \| u \|_V = | A^{1/2}u |_H \). The operator \( A \) is positive selfadjoint with compact resolvent. Let \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) be the eigenvalues of \( A \), and \( e_1, e_2, \ldots \) be the corresponding eigenvectors, which form a complete orthonormal system in \( H \). We remark that \( \| u \|_V^2 \geq \lambda_1 | u |_H^2 \).

**Remark 1.1.** Note that \( D(A) = \{ u = \sum_{i=1}^{\infty} u_i e_i \in H : \sum_{i=1}^{\infty} \lambda_i^2 u_i^2 < \infty \} \), we may endow \( D(A) \) with the inner product
\[
\langle u, v \rangle_{D(A)} = \sum_{i=1}^{\infty} \lambda_i^2 u_i v_i,
\]
\( u_i = \langle u, e_i \rangle_H, \; v_i = \langle v, e_i \rangle_H \) for every \( u, v \in D(A) \). So \( D(A) \) is a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_{D(A)} \) and \( \{ \lambda_i^2 \}_{i \in \mathbb{N}} \) is a complete orthonormal system of \( D(A) \). For the dual space of \( D(A) \), \( D(A)' = \{ u = \sum_{i=1}^{\infty} u_i e_i : \sum_{i=1}^{\infty} \frac{u_i^2}{\lambda_i} < \infty \} \) and endow \( D(A)' \) with the inner product \( \langle u, v \rangle_{D(A)'} = \sum_{i=1}^{\infty} \frac{u_i v_i}{\lambda_i} \). \( D(A)' \) is a Hilbert space. For every \( u \in D(A), \; v \in D(A)' \), \( \langle u, v \rangle_{D(A), D(A)'} = \sum_{i=1}^{\infty} u_i v_i \), and if \( v \in H \), we have \( \langle u, v \rangle_{D(A), D(A)'} = \langle u, v \rangle_H \). It will be convenient to use fractional powers of the operator \( A \), as well as their domains \( D(A^\alpha) \) for \( \alpha \in \mathbb{R} \).
Define the bilinear operator $B(u, v) : V \times V \to (V \cap [L^2(D)]')$ as

$$\langle B(u, v), z \rangle = \int_D z(x) \cdot (u(x) \cdot \nabla) v(x) \, dx, \quad z \in V \cap [L^2(D)]'.$$

From the incompressibility condition, $\langle B(u, v), v \rangle = 0$, $\langle B(u, v), z \rangle = -\langle B(u, z), v \rangle$. By [11], there exists $\beta > 1$, $B$ can be extended to a continuous operator

$$B : H \times H \to D(A^{-\beta})$$

and

$$\|w, B(u, v)\| \leq C\|u\|_{H'} \|v\|_{H} \|w\|_{A^\beta}.$$  \hfill (1.5)

Eqs. (1.1)–(1.4) have the abstract form as a stochastic evolution equation:

$$\begin{cases}
du(t) + vAu(t) \, dt + B(u(t), u(t)) \, dt = f(t) \, dt + \int_{|x| \leq 1} F(u(t), x) \tilde{N}_p(dt, dx), \\
u(0) = u_0.
\end{cases}$$  \hfill (1.6)

In this article, we assume that

1. $u_0 \in H$ and $f \in L^2([0, \infty); \mathcal{V}')$.
2. $p = p(t)$, $t \in D_p$ is a stationary $\mathcal{F}_t$-Poisson point process of the class (QL) on a measurable space $K$, with compensator $t \lambda(U)$. $\lambda(dx)$ is the characteristic measure of $p$ satisfying $\int_K |x|_K^2 \wedge 1 \lambda(dx) < \infty$. $N_p(dt, dx)$ is the counting measure defined as follows:

$$N_p((0, t] \times U) = \# \{s \in D_p; \ s \leq t, \ p(s) \in U \}$$

for $t > 0$, $U \in \mathcal{B}(K)$, where $D_p$ is the domain of $p$, $\tilde{N}_p(dt, dx) = N_p(dt, dx) - dt \lambda(dx)$.

3. $F(\cdot, \cdot)$ is measurable function from $H \times K$ to $H$.

2. Preliminaries

Let $(E, r)$ denote a metric space. Denote by $\mathcal{B}$ the Borel $\sigma$-field of $(E, r)$. $Pr(E)$ the set of all probability measures on $(E, \mathcal{B})$. Let $D_E[0, \infty)$ be the space of right continuous functions from $[0, \infty)$ into $(E, r)$ having left limits, with the Skorokhod metric $d(\cdot, \cdot)$ (see in Chapter 3 of [4] for the details).

Lemma 2.1. Assume $\{x, y, x_n\} \subset D_E[0, \infty)$ and $\theta \in E \ f_T(y) = \sup_{t \in [0, T]} r(y(t), \theta)$. If $\lim_{n \to \infty} d(x_n, x) = 0$, then for any $T \in [t: x(t) = x(t)]$,

$$\lim_{n \to \infty} f_T(x_n) = f_T(x).$$

Proof. By [4], if $\lim_{n \to \infty} d(x_n, x) = 0$, then for any sequence $t_n \in [0, \infty)$, $t \geq 0$, and $\lim_{n \to \infty} t_n = t$,

$$\lim_{n \to \infty} r(x_n(t_n), x(t)) = r(x_n(t_n), x(t)).$$  \hfill (2.1)
and \( x \) has at most countably many points of discontinuity. We can select a countable subset \( Q \subset \{ s \in [0, T] : x(s) = x(s-) \} \) such that \( Q \) is dense in \([0, T]\) and \( T \in Q \). For every \( s \in Q \),
\[
\lim_{n \to \infty} r(x_n(s), x(s)) = 0.
\]
So for any \( \epsilon > 0 \),
\[
\epsilon + \lim_{n \to \infty} \inf_{n \in Q} \sup_{s \in Q} r(x_n(s), \theta) \geq \sup_{s \in Q} r(x(s), \theta).
\]
Since \( f_T(y) = \sup_{s \in Q} r(y(s), \theta) \), we have
\[
\epsilon + \lim_{n \to \infty} \inf_{n \in Q} f_T(x_n) \geq f_T(x).
\] (2.2)

On the other hand, from (2.1), for any fixed \( t \in [0, T] \), and any \( \epsilon > 0 \), there exists \( \delta_t > 0 \) and \( N_t \), such that for any \( |s - t| \leq \delta_t \) and \( n \geq N_t \), \( r(x_n(s), x(t)) \wedge r(x_n(s), x(t-)) \leq \epsilon \), thus \( r(x_n(s), \theta) \leq r(x(t), \theta) \vee r(x(t-), \theta) + \epsilon \). Therefore there exists \( \{ A_i = [t_i - s] \leq \delta_t, 1 \leq i \leq N \} \), for \( n \geq \max\{N_t\}_{i \in \{1, 2, \ldots, N\}} \),
\[
\sup_{s \in [0, T]} r(x_n(s), \theta) \leq \max_{1 \leq i \leq N} r(x(t_i), \theta) \vee r(x(t_i-), \theta) + \epsilon \leq \sup_{s \in [0, T]} r(x(t), \theta) + \epsilon.
\] (2.3)

This implies \( \lim_{n \to \infty} f_T(x_n) = f_T(x) \). \( \square \)

For \( \xi \in D_R[0, \infty) \) let
\[
\triangle \xi(s) = \xi(s) - \xi(s-),
\]
\[
U(\xi) = \{ u > 0 : |\triangle \xi(s)| = u \text{ for some } s \},
\]
\[
U^{u_0}_T(\xi) = \{ u > 0 : |\triangle \xi(s)| = u > u_0 \text{ for some } s \in [0, T] \}.
\]

Then \( U(\xi) \), the collection of all jump size of \( \xi \), is at most countable. Let \( U^c(\xi) \) be the complement of \( U(\xi) \). For \( u > 0 \), let
\[
t^0(\xi, u) = 0, \quad t^p(\xi, u) = \inf\{ t > t^{p-1}(\xi, u) : |\triangle \xi(t)| > u \},
\]
\( t^p(\xi, u) \) is the \( p \)-th jump time of \( \xi \) with the norm of jump size greater than \( u \). Because \( \xi \in D_R[0, \infty) \),
\[
\lim_{p \to \infty} t^p(\xi, u) = \infty.
\]
Set \( p_T(\xi) = \max\{ p : t^p(\xi, u) \leq T \} \), it is easy to see that \( p_T(\xi) < \infty \).

**Lemma 2.2.** Suppose \( g \) is a continuous function from \( R \) to \( R \) with \( g(x) = 0, x \in [-u, u] \) for some positive constant \( u \). For any fixed \( T > 0 \), set
\[
G_T(\xi) = \sum_{s \leq T} g(\triangle \xi(s)), \quad \xi \in D_R[0, \infty).
\]

If \( \xi \) is continuous at \( T \), then \( G_T(\cdot) \) is continuous at \( \xi \).

**Proof.** Suppose \( \lim_{n \to \infty} \hat{\xi}_n = \xi \) in \( D_R[0, \infty) \). Let \( 0 < t_1 < t_2 < \cdots < t_m < T \) be total points with \|\triangle \xi(t)\| \geq u, t \in [0, T] \). There exists \( \epsilon > 0 \), satisfying
\[
u - \epsilon \notin U(\xi), \quad [u - \epsilon, u] \cap U_{t}^{\epsilon/2}(\xi) = \emptyset.
\]
Lemma 2.4. valued Skorokhod space, see Lemmas 2.4 and 2.5. Also we prove the energy inequality.

Applying Theorem 15.30 in [16], we know that $t^i(\xi, u - \epsilon)$ and $\Delta \xi(t^i(\xi, u - \epsilon))$ is continuous in $D_R[0, \infty]$ at $\xi$. Note that $t^{n+1}(\xi, u - \epsilon) > T$, let $\delta = \frac{r^{n+1}(\xi, u - \epsilon - 1)}{2} \wedge T - r^n(\xi, u - \epsilon)$, there exists $N$, such that

$$|t^i(\xi_{m'}, u - \epsilon) - t^i(\xi, u - \epsilon)| \leq \frac{\delta}{2}, \quad m' > N, \; i = 1, 2, \ldots, n + 1.$$ 

So

$$G_T(\xi_{m'}) = \sum_{i=1}^{n} g(\Delta \xi_{m'}(t^i(\xi_{m'}, u - \epsilon))).$$

Since $t^i(\xi, u - \epsilon) = t_i$, $G_T$ is continuous at $\xi$. □

Lemma 2.3. Let $X$ be a Polish space, $P_n \xrightarrow{w} P$, $\varphi_n, \varphi : X \to R$ be measurable. If there exists $X' \in \mathcal{B}(X)$ with $P(X') = 1$ such that for $x \in X'$, $x_n \to x$ in $X$, $\varphi_n(x_n) \to \varphi(x)$ and $P_n[|\varphi_n|^{1+\epsilon}] \leq C$ for some $\epsilon$, $C > 0$, then

1. $P[|\varphi|] < \infty$ and $P_n[\varphi_n] \to P[\varphi]$,
2. $P[|\varphi|^{1+\epsilon}] \leq C$.

Proof. The proof of (1) is similar to the argument as in [8]. We only prove (2). By (1) we have

$$P[|\varphi|^{1+\delta}] = \lim_{n \to \infty} P_n[|\varphi|^{1+\delta}], \quad \text{for } \delta \in [0, \epsilon).$$

From Hölder's inequality

$$P_n[|\varphi|^{1+\delta}] \leq \left(P_n[|\varphi|^{1+\epsilon}]\right)^{(1+\delta)/(1+\epsilon)} \leq C^{(1+\delta)/(1+\epsilon)},$$

and by Fatou lemma

$$P[|\varphi|^{1+\epsilon}] \leq C. \quad \square$$

Remark 2.1. By [4], if $(E, r)$ is separable and complete, then $(D_E[0, \infty), d)$ is a Polish space.

To get our main results, we need to prove the tightness in vector valued Skorokhod space. The Aldous criterion for tightness is a sufficient condition for proving the tightness, refer to [9,10]. In [13,14], by using this criterion, one can get the tightness in Skorokhod space once the energy inequality is proved, thus martingale problem is formulated on vector valued Skorokhod space and with Gaussian as the limit measure. We use the same Aldous criterion to get the tightness in vector valued Skorokhod space, see Lemmas 2.4 and 2.5. Also we prove the energy inequality.

Lemma 2.4. (See [1]) Let $E$ be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. For an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ in $E$, define the function $r^2_N : E \to R^+$ by

$$r^2_N(x) = \sum_{k \geq N+1} \langle x, e_k \rangle^2, \quad N \in \mathbb{N}.$$
Let $D$ be a total and closed under addition subset of $E$. Then the sequence $\{X_n\}_{n \in \mathbb{N}}$ of stochastic processes with trajectories in $D_E[0, \infty)$ is tight if it is $D$-weakly tight and for every $\epsilon > 0$ and $t > 0$,

$$\lim_{N \to \infty} \lim_{n \to \infty} \sup \mathbb{P} \left( r_N^2(X_n(s)) > \epsilon \text{ for some } s \in [0, t] \right) = 0. \quad (2.4)$$

**Remark 2.2.** If there exist positive $C$ and a sequence $\lambda_n$ satisfying $\lambda_n \to \infty$ such that

$$\mathbb{E}^p \left( \sup_{s \in [0, t]} r_N^2(X_n(s)) \right) \leq \frac{C}{\lambda_N},$$

then (2.4) holds.

Let $\{X_n\}$ be a sequence of random elements of $D_R[0, T]$, and $\{\tau_n, \delta_n\}$ be such that:

(a) For each $n$, $\tau_n$ is a stopping time with respect to the natural $\sigma$-fields, and takes only finitely many values.

(b) For each $n$, the constant $\delta_n \in [0, T]$ satisfy $\delta_n \to 0$ as $n \to \infty$.

We introduce the following condition on $\{X_n\}$: for each sequence $\{\tau_n, \delta_n\}$ in (a), (b),

(A) $X_n(\tau_n + \delta_n) - X_n(\tau_n) \to 0$, in probability.

For $f \in D_R[0, T]$, let $J(f)$ denote the maximum of the jump $|f(t) - f(t^-)|$.

**Lemma 2.5.** (See [2].) Suppose that $\{X_n\}_{n \in \mathbb{N}}$ satisfies (A), and either $\{X_n(0)\}$ and $\{J(X_n)\}$ are tight on the line; or $\{X_n(t)\}$ is tight on the line for each $t \in [0, T]$, then $\{X_n\}$ is tight in $D_R[0, T]$.

3. Existence of martingale solution of Eq. (1.6)

We divide this section into additive Lévy and multiplicative noise parts. Instead of martingale representation theorems like wiener processes as in [6] etc., we use the Lévy–Khinchin formula for additive Lévy noise. For the multiplicative noise, this method fail to use and we use martingale character.

3.1. Additive Lévy noise

Eq. (1.6) has the following form:

$$du(t) + \left[ vAu(t) + B(u(t), u(t)) \right] dt = \sum_{i=1}^{\infty} \sigma_i \epsilon_i dL_i(t) \quad (3.1)$$

where $\{L_i(t)\}_{i \in \mathbb{N}}$ are independent Lévy processes defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$,

$$L_i(t) = \int_{0}^{t+} \int_{|x| \leq 1} x\tilde{N}_i(dt, dx), \quad \tilde{N}_i(dt, dx) = N_i(dt, dx) - dt\lambda(dx).$$

$\{N_i(dt, dx)\}_{i \in \mathbb{N}}$ are independent Poisson random measure with $\int_{|x| \leq 1} x^2\lambda(dx) < \infty$. 
3.2. Finite dimensional models

Let $H_n = \text{span}\{e_i, 1 \leq i \leq n\}$, $\pi_n$ is the orthogonal projection of $H$ on $H_n$, and $A_n$ is the restriction of $A$ to $H_n$. $B_n(\cdot, \cdot): H_n \times H_n \rightarrow H_n$ the continuous bilinear operator defined as

$$\langle B_n(u, v), w \rangle = \langle B(u, v), w \rangle$$

for every $u, v, w \in H_n$, $B_n(u, v) = \pi_n B(u, v)$.

Consider the equation on $H_n$:

$$dX^n_t + \left[ \nu A_n X^n_t + B_n\left(X^n_t, X^n_t\right) \right] dt = \sigma_i e_i dL_i(t). \quad (3.2)$$

Let $(X^n_t)_{t \geq 0}$ be the RCLL adapted solution of Eq. (3.2). In the following, $C(\cdot)$ means a positive constant with dependent only on the elements in the bracket.

**Theorem 3.1.** Assume $E|X^n_0|_H^2 < \infty$, then for every $T > 0$,

$$E\left( \sup_{t \in [0, T]} |X^n_t|_H^2 + 2 \nu \int_0^T \|X^n_s\|_V^2 \, ds \right) \leq C \left( E|X^n_0|_H^2, T, \sum_{i=1}^n \sigma_i^2, \int_{|x| \leq 1} x^2 \lambda(dx) \right), \quad (3.3)$$

$$E|X^n_T|_H^2 + 2 \nu \int_0^T \|X^n_s\|_V^2 \, ds = E|X^n_0|_H^2 + 2 \int_0^T \left( X^n(s-), \sigma_i e_i dL_i(s) \right)_H + \sum_{i=1}^n \int_0^T \int_{|x| \leq 1} \sigma_i^2 x^2 dN_i(ds, dx). \quad (3.4)$$

$$E|X^n_T|_H^2 + 2 \nu E\int_0^T \|X^n_s\|_V^2 \, ds = E|X^n_0|_H^2 + t \sum_{i=1}^n \sigma_i^2 \int_{|x| \leq 1} x^2 \lambda(dx). \quad (3.5)$$

**Proof.** From Eq. (3.2) and Itô formula (refer to [12]), we have

$$|X^n_t|_H^2 = |X^n(0)|_H^2 - 2 \nu \int_0^t \langle A_n X^n_s, X^n_s \rangle_H \, ds - 2 \int_0^t \langle B_n\left(X^n_s, X^n_s\right), X^n_s \rangle_H \, ds$$

$$+ 2 \int_0^t \left( X^n(s-), \sum_{i=1}^n \sigma_i e_i dL_i(s) \right)_H + \sum_{i=1}^n \int_0^t \int_{|x| \leq 1} \sigma_i^2 x^2 dN_i(ds, dx).$$

Since $\langle B_n(x, x), x \rangle = 0$ for every $x \in H_n$. 

\[ |X^n_t|^2_H + 2 \nu \int_0^t \| X^n_s \|^2_{V} \, ds = |X^n(0)|^2_H + 2 \int_0^t \left( X^n(s), \sum_{i=1}^n \sigma_i e_i \, dL_i(s) \right)_H + \sum_{i=1}^n \int_0^t \int_{|x| \leq 1} \sigma_i x^2 \, dN_i(ds, dx). \]

Set \( M^n_t = 2 \int_0^t \langle X^n(s), \sum_{i=1}^n \sigma_i e_i \rangle_H \, ds \), since for \( T > 0 \),

\[ \sum_{i=1}^n \mathbb{E} \left( \int_0^T \int_{|x| \leq 1} \langle X^n(s), \sigma_i e_i x \rangle_H^2 \lambda(dx) \, ds \right) < \infty, \quad (3.6) \]

\( M^n_t \) is a square integrable martingale.

Define the stopping time \( \tau_m = \inf \{ t \geq 0 : |X^n_t|^2_H \geq m \} \), then

\[ |X^n_{t \wedge \tau_m}|^2_H \leq |X^n(0)|^2_H + M^n_{t \wedge \tau_m} + \sum_{i=1}^n \int_0^{t \wedge \tau_m} \int_{|x| \leq 1} \sigma_i x^2 \, dN_i(ds, dx), \]

and thus

\[ E|X^n_{t \wedge \tau_m}|^2_H \leq E|X^n(0)|^2_H + t \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx). \]

From this inequality and the monotone convergence theorem, we get

\[ E \int_0^T |X^n_t|^2_H \, dt \leq T \left( E|X^n_0|^2_H + T \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx) \right). \]

Thus

\[ \sum_{i=1}^n E \left( \int_0^T \int_{|x| \leq 1} \langle X^n(s), \sigma_i e_i x \rangle_H^2 \lambda(dx) \, ds \right) = \sum_{i=1}^n \left( \int_0^T \langle X^n(s), e_i \rangle_H^2 \cdot \int_{|x| \leq 1} \sigma_i^2 x^2 \lambda(dx) \, ds \right) \leq E \left( \int_0^T |X^n(s)|^2_H \, ds \right) \cdot \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx) < \infty, \]

so \( M^n_t \) is now a square integrable martingale. The (3.4) and (3.5) are follows.

For the inequality (3.3), since

\[ |X^n_t|^2_H \leq |X^n_0|^2_H + |M^n(t)| + \sum_{i=1}^n \int_0^t \int_{|x| \leq 1} \sigma_i^2 x^2 N_i(ds, dx), \]
we have

$$\sup_{t \in [0, T]} |X_t^n|_H^2 \leq |X_0^n|_H^2 + 1 + \sup_{t \in [0, T]} |M^n(t)|^2 + \sum_{i=1}^n \int_0^T \int_{|x| \leq 1} \sigma_i^2 x^2 N_i(ds, dx)$$

and

$$E \sup_{t \in [0, T]} |X_t^n|_H^2 \leq E |X_0^n|_H^2 + 1 + E \sup_{t \in [0, T]} |M^n(t)|^2 + T \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx)$$

$$\leq E |X_0^n|_H^2 + 1 + C E \left( \int_0^T |X^n(s)|_H^2 ds \right) \cdot \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx)$$

$$+ T \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx)$$

$$\leq E |X_0^n|_H^2 + T \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx)$$

$$+ CT \left( E |X^n(0)|_H^2 + T \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx) \right) \cdot \left( \sum_{i=1}^n \sigma_i^2 \right) \int_{|x| \leq 1} x^2 \lambda(dx)$$

$$< \infty.$$  

(3.5) implies the second part of the bound (3.3). □

**Corollary 3.1.** Let $\tau \geq 0$ be a stopping time and $(X_t^n)_{t \geq 0}$ be a càdlàg adapted processes that P-a.s. satisfy (3.2) for $t \in [0, \tau(\omega)]$. Assume $E |X_0^n|_H^2 < \infty$. Then for $T > 0$,

$$E \left( \sup_{t \in [0, T]} |X_{t \wedge \tau}^n|_H^2 + 2 \int_0^T \|X_{s \wedge \tau}^n\|_V^2 ds \right) \leq C \left( E |X_0^n|_H^2, T, \sum_{i=1}^n \sigma_i^2, \int_{|x| \leq 1} x^2 \lambda(dx) \right). \quad (3.7)$$

**Lemma 3.1.** Suppose $(X_t^{(1)})$ and $(X_t^{(2)})$ be two solutions of Eq. (3.2) on interval $[0, T]$. Set $\Delta_t = X_t^{(1)} - X_t^{(2)}$ and $C_{n, B}$ be a constant such that $(B_n(x, y), x)_H \leq C_{n, B} |x|_H^2 |y|_H$, $x, y \in H_n$. Then

$$|\Delta_t|_H \leq |\Delta_0|_H \exp \left\{ 2C_{n, B} \int_0^t |X_s^{(2)}|_H ds \right\}.$$  

The proof is similar to the case of white noise as in [5].
Theorem 3.2. For every $\mathcal{F}_0$-measurable $X^n_0 : \Omega \to H_n$, there exists a unique càdlàg adapted solution $(X^n_t)_{t \geq 0}$ of Eq. (3.2) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. If the initial conditions $x^n$ converge to $x$ in $H_n$, the corresponding solutions converge $\mathbb{P}$-a.s. uniformly in time on bounded intervals. And $(X^n_t)_{t \geq 0}$ has Feller property.

Proof. Step 1 (Existence for bounded initial value $X^n_0$). Assume that $|X^n_0|_H \leq C$ for some positive constant $C$. For any $m > C$, let $B^n_m(\cdot) : H_n \to H_n$ be a Lipschitz continuous function such that $B^n_m(x) = B_n(x, x)$ for every $|x|_{H_n} \leq m$.

Consider the equation

$$dY^{(m)}_t = \left[-\nu A_n Y^{(m)}_t - B^n_m(Y^{(m)}_t, Y^{(m)}_t)\right] dt + \sum_{i=1}^n \sigma_i e_i dL_i(t).$$

It has globally Lipschitz coefficients, so there exists a unique càdlàg adapted solution $(Y^{(m)}_t)_{t \geq 0}$, this is a classical result which can be obtained by contraction principle.

Defined $\tau_m = \inf\{t \geq 0 : |Y^{(m)}_t|_H \geq m\} \wedge T$. Up to $\tau_m$, the solution $Y^{(m)}_t$ is also a solution of the original equation (3.2). Since $|X^n_0|_H < C$, we have

$$E\left(\sup_{t \in [0,T]} |Y^{(m)}_{t \wedge \tau_m}|^2_H\right) \leq C\left(E|X^n_0|^2_H, T, \sum_{i=1}^n \sigma_i^2, \int_{|x| \leq 1} x^2 \lambda(dx)\right).$$

In particular,

$$E\left(1_{\{\tau_m < T\}} \sup_{t \in [0,T]} |Y^{(m)}_{t \wedge \tau_m}|^2_H\right) \leq C\left(E|X^n_0|^2_H, T, \sum_{i=1}^n \sigma_i^2, \int_{|x| \leq 1} x^2 \lambda(dx)\right),$$

which implies

$$P(\tau_m < T) \leq \frac{1}{m^2} C\left(E|X^n_0|^2_H, T, \sum_{i=1}^n \sigma_i^2, \int_{|x| \leq 1} x^2 \lambda(dx)\right).$$

If $N > m$, $\tau_N \geq \tau_m$, $P(Y^{(N)}_t = Y^{(m)}_t, t \in [0, \tau_m]) = 1$. Let $\tau_\infty = \sup_{m \geq C} \tau_m$, we can define a process $Y^{(\infty)}_t$ for $t \in [0, \tau_\infty)$ uniquely, which equal to $Y^{(m)}_t$ on $[0, \tau_m]$ for every $m$. Hence $Y^{(\infty)}_t$ is a solution on $[0, \tau_\infty)$. Since for any $m$,

$$P(\tau_\infty < T) \leq P(\tau_m < T) \leq \frac{C}{m^2},$$

$P(\tau_\infty < T) = 0$. Thus $Y^{(\infty)}_t$ is a solution for $t \in [0, T - \epsilon]$ for any small $\epsilon > 0$, which shows that there exists a global solution.

Step 2 (Existence for general initial value $X^n_0$). For general case, let $\Omega_m = \{ |X^n_0|^2_H \leq m \}$. Define $X^n_0$ as $X^n_0$ on $\Omega_m$. Let $(Y^{(m)}_t)_{t \leq 0}$ be the unique solution of equation (3.2) with initial condition $X^n_0$. If $N > m$, then

$$P(\Omega_m \cap \{Y^{(N)}_t = Y^{(m)}_t \text{ for every } t \geq 0\}) = P(\Omega_m).$$
We may then uniquely define a process $Y_t^{(\infty)}$ on $\Omega' = \bigcup_n \Omega_n$ as $Y_t^{(\infty)} = Y_t^{(n)}$ on $\Omega_n$, it is clear that $Y_t^{(\infty)}$ solves Eq. (3.2) on $\Omega'$. But $P(\Omega') = 1$, hence we have proved the existence of a global solution. Below we will let $(Y_t^{x})_{t \geq 0}$ be the unique solution with initial condition $x \in H_n$.

Step 3. Uniqueness and continuous of the solution are follows from Lemma 3.1, which also implies his implies Markov property and the Feller property. \hfill \Box

### 3.3. Solution to the martingale problem

Let $\Omega = D_{D(A)'}[0, \infty)$, denote $T$ the Skorokhod topology of $\Omega$, and $T_t$ the Skorokhod topology of $D_{D(A)'}[0, t]$, let $\mathcal{F} = \sigma(T)$, $\mathcal{F}_t = \sigma(T_t)$.

**Definition 3.1.** Given a probability measure $\mu_0$ on $H$, a probability measure $P$ on $(\Omega, \mathcal{F})$ is called a solution of the martingale problem associated to Eq. (3.1) with initial law $\mu_0$, if

1. for every $T > 0$,
   $$P \left( \sup_{t \in [0, T]} |\xi_t|_H^2 + \int_0^T \|\xi_s\|_Y^2 \, ds < \infty \right) = 1. \quad (3.8)$$

2. for every $\varphi \in \mathcal{D}^\infty$ the process $M^\varphi_t$ defined $P$-a.s. on $(\Omega, \mathcal{F})$ as
   $$M^\varphi_t (\xi) := \langle \xi_t - \xi_0, \varphi \rangle_H + \int_0^t \mathcal{L}_\varphi \xi_s \, ds - \int_0^t \langle B(\xi_s, \varphi), \xi_s \rangle_H \, ds. \quad (3.9)$$

$(M^\varphi_t, \mathcal{F}_t, P)$ is a Lévy process. Further more, $(M^\varphi_t)_{t \geq 0}$ are independent Lévy processes defined on the complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, and the characteristic function is

$$E^P \exp \left\{ iu \left( \frac{M^\varphi_{t_2} - M^\varphi_{t_1}}{\sigma_i} \right) \right\} = \exp \left\{ (t_2 - t_1) \int_{|x| \leq 1} \left( e^{iux} - 1 - iux \right) \lambda(dx) \right\} \quad (3.10)$$

for $t_2 \geq t_1 \geq 0$. 

3. $\mu_0 = \pi_0 P$.

**Theorem 3.3.** Assume $\sigma^2 = \sum_{i=1}^{\infty} \sigma_i^2 < \infty$. Let $\mu$ be a probability measure on $H$ such that $m_2 = \int_H |x|^2 \mu(dx) < \infty$. Then there exists at least one solution to the martingale problem (3.1) with initial condition $\mu$.

**Proof.** Let $(W, (\mathcal{W}_t)_{t \geq 0}, Q, (L_i(t))_{t \geq 0, i \in \mathbb{N}})$ be a stochastic basis supporting, $u_0 : W \to H$ is $\mathcal{W}_0$-measurable random variable with law $\mu$. Let $X^0_0 = \pi_0 u_0$.

For every $n$, there exists a unique càdlàg adapted solution $(X_n^\varphi)_{t \geq 0}$ of Eq. (3.2) in $H_n$, with initial condition $X^\varphi_0 = (X_n^\varphi)_{t \geq 0}$ is a càdlàg adapted process in $H_n$. Since $H_n \subset D(A)'$. It defines a probability measure $P_n$ on $D_{D(A)'}[0, \infty)$.

Step 1 (Tightness). Let $D = D(A)$. From Lemma 2.4, we have to prove that $X^n$ is $D$-weakly tight and for every $\epsilon > 0$, $t > 0$,

$$\lim_{N \to \infty} \lim_{n \to \infty} \sup P (|X^n(s)|^2 > \epsilon) = 0.$$
First ($D$-weakly tightness).
For every $\varphi \in D(A)$, we prove that $\{\langle X^n(\cdot), \varphi \rangle_{D(A)'}, D(A) \}, t \in [0, T]\}$ is tight for every $T > 0$.
For $\tau_n$, $\delta_n$ satisfies (a), (b), since

$$\langle X^n(\tau_n + \delta_n) - X^n(\tau_n), \varphi \rangle_{D(A)'}, D(A) \rangle = \lim_{n \to \infty} \| \varphi \|_{D(A)'} \langle X^n(\tau_n + \delta_n) - X^n(\tau_n), \varphi \rangle_{D(A)}$$

$$= \sum_{i=1}^{n} \sigma_i h_i \left[ L_i(\tau_n + \delta_n) - L_i(\tau_n) \right], \varphi \rangle_{H} - \int_{\tau_n}^{\tau_n + \delta_n} \| [vA_n X^n(s) + B_n(X^n(s), X^n(s)), \varphi]_{H} \| ds,$$

we have

$$Q \left( \| \langle X^n(\tau_n + \delta_n) - X^n(\tau_n), \varphi \rangle_{D(A)'}, D(A) \rangle \| \geq \varepsilon \right)$$

$$\leq Q \left( \int_{\tau_n}^{\tau_n + \delta_n} \| vA_n X^n(s), \varphi \|_{H} ds \geq \frac{\varepsilon}{3} \right) + Q \left( \int_{\tau_n}^{\tau_n + \delta_n} \| B_n(X^n(s), X^n(s)), \varphi \|_{H} ds \geq \frac{\varepsilon}{3} \right)$$

$$+ Q \left( \left\| \sum_{i=1}^{n} \sigma_i e_i \left[ L_i(\tau_n + \delta_n) - L_i(\tau_n) \right], \varphi \right\|_{H} \geq \frac{\varepsilon}{3} \right)$$

$$= I_1 + I_2 + I_3.$$

Since $\lim_{n \to \infty} \delta_n = 0$, $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$,

$$I_1 \leq Q \left( \int_{\tau_n}^{\tau_n + \delta_n} \| X^n(s) \|_{H}^2 + \| A \varphi \|_{H}^2 ds \geq \frac{2\varepsilon}{3\nu} \right)$$

$$\leq Q \left( \sup_{0 \leq s \leq T} \| X^n(s) \|_{H}^2 + \| A \varphi \|_{H}^2 \delta_n \geq \frac{2\varepsilon}{3\nu} \right) \to 0, \text{ as } n \to \infty,$$

$$I_3 \leq Q \left( \sum_{i=1}^{\infty} \sigma_i^2 \left( \int_{\tau_n}^{\tau_n + \delta_n} \| L_i(\tau_n + \delta_n) - L_i(\tau_n) \|_{H}^2 + \| \varphi \|_{H}^2 \right) \geq \left( \frac{\varepsilon}{3} \right)^2 \right) \to 0, \text{ as } n \to \infty.$$

Since for every $\xi \in V$, $\int_{0}^{T} |\langle B(u_s, \xi), u_s \rangle| ds \leq C(\xi) \int_{0}^{T} \| u_s \|_{H}^{1/2} \| u_s \|_{V}^{3/2} ds$,

$$I_2 \leq Q \left( (C(\varphi) \sup_{s \in [0, T]} \| X^n(s) \|_{H}^{1/2} \int_{\tau_n}^{\tau_n + \delta_n} \| X^n(s) \|_{V}^{3/2} ds \geq \frac{\varepsilon}{3} \right)$$

$$\leq Q \left( (C(\varphi) \sup_{s \in [0, T]} \| X^n(s) \|_{H}^{1/2} \left( \int_{\tau_n}^{\tau_n + \delta_n} \| X^n(s) \|_{V}^{2} ds \right)^{3/4} \delta_n^{1/4} \geq \frac{\varepsilon}{3} \right)$$

$$\leq Q \left( (C(\varphi) \left( \sup_{s \in [0, T]} \| X^n(s) \|_{H}^2 + \int_{\tau_n}^{\tau_n + \delta_n} \| X^n(s) \|_{V}^{2} ds \right) \delta_n^{1/4} \geq \frac{\varepsilon}{3} \right) \to 0, \text{ as } n \to \infty,$$

so $\langle X^n(\cdot), \varphi \rangle_{D(A)'}, D(A) \rangle$ satisfies (A).
Since
\[ Q\left(\phi_0^2\right) \leq \phi_H^2 Q\left(\phi_0^2\right) \leq m_2 \phi_H^2 \]
and
\[ J\left(\phi_0^2\right) \leq \max_{i \in N} \left|\sigma_i\right| \cdot \phi_H. \]

Thus \(J(\phi_0^2)\) and \(\phi_H^2\) are tight on line. Thus \(\phi(\cdot), \phi(\cdot)\) is tight in
\(D[0, T]\) by Lemma 2.5.

Second. Let \(h_k = \lambda_i e_i\), which forms a complete orthonormal system of \(D(A)^\prime\). Since
\[
E^Q\left[\sup_{t \in [0, T]} \phi_0^2\right] = E\left[\sup_{s \in [0, t]} \sum_{k \geq N + 1} \phi_k^2\right] \leq \frac{E^Q\left[\sup_{s \in [0, T]} \phi_0^2\right]}{\lambda_{N+1}^2} \leq \frac{C}{\lambda_{N+1}^2},
\]
(2.4) is proved by Lemma 2.4.

Denote \(P_n = Q \circ (\phi)^{-1}\), since \(\{\phi\}\) is tight in \(D(A)^\prime[0, \infty)\), there exist \(n_k\) such that \(p_{n_k} \xrightarrow{w} P\) in
\(D(A)^\prime[0, \infty)\).

Step 2 (P is a martingale solution).
We need to check the properties (1)–(3) in Definition 3.1.

(1) of Definition 3.1. Given \(R, N > 0\),
\[
f_{N, R}(\xi) = \sup_{t \in [0, T]} \sum_{n=1}^N \left|\xi(t), e_n\right|_H^2 \wedge R = \sup_{t \in [0, T]} \sum_{n=1}^N \lambda_n^2 \left|\xi(t), h_n\right|_{D(A)^\prime} \wedge R.
\]

By [5], we know that, if \(X\) is process with sample paths in \(D(E)[0, \infty)\), \(E\) is a metric space, then the
complement in \([0, \infty)\) of \(D(X) = \{t \in [0, \infty): P(X(t) = X(t-)) = 1\}\) is at most countable. Let \(Z = \{t \in [0, \infty): P(\xi(t) = \xi(t-)) = 1\}\). For every \(T \in Z\), denote \(\Omega_T = \{\xi \in D(A)^\prime[0, \infty): \xi(T) = \xi(T^-)\}\), then
\(P(\Omega_T) = 1\). By Lemmas 2.1, 2.3 and Theorem 3.1,
\[
\lim_{k \to \infty} P_{n_k} f_{N, R}^T = P f_{N, R}^T \leq C\left(m_2, T, \sum_{i=1}^N \sigma_i^2, \int_{|x| \leq 1} x^2 \lambda(dx)\right),
\]
so we have (1).

(2) of Definition 3.1. We just prove that \(\{M^e_i\}_{i \in N}\) are independent Lévy processes.

(i) Obviously, \(M^e_i(0) = 0\), \(P\)-a.e.

(ii) We prove (3.10).

Choose \(t_1, t_2 \in Z\), let \(\Omega_{t_1, t_2} = \{\xi \in D(A)^\prime[0, \infty): \xi(t_i) = \xi(t_i^-), i = 1, 2\}\), then \(P(\Omega_{t_1, t_2}) = 1\). Let
\[
F_{t_2}^{e_i}(\xi) = \exp\left\{iu \frac{M^e_i(t_2)(\pi_m \xi) - M^e_i(t_1)(\pi_m \xi)}{\sigma_i}\right\}.
\]
Suppose $\lim_{n \to \infty} \xi_n = \xi$, $\xi_n \in D_{D(A)}[0, \infty)$, $\xi \in \Omega_{t_1, t_2}$, it is easy to check that $\lim_{n \to \infty} F^\xi_m(\xi_n) = F^\xi_m(\xi)$. By Lemma 2.3,

$$\lim_{k \to \infty} E^P_{P^k} F^\xi_m(\xi) = E^P F^\xi_m(\xi).$$

Since

$$E^P_{P^k}\left[ \int_{t_1}^{t_2} \left| B(\pi_m \xi_s, \pi_m \xi_s), e_1 \right| ds - \int_{t_1}^{t_2} \left| B(\xi_s, \xi_s), e_1 \right| ds \right] \leq E^P_{P^k}\left[ \int_{t_1}^{t_2} \left| B(\pi_m \xi_s - \xi_s, \pi_m \xi_s), e_1 \right| ds \right] + E^P_{P^k}\left[ \int_{t_1}^{t_2} \left| B(\xi_s, \pi_m \xi_s - \xi_s), e_1 \right| ds \right]$$

$$\leq C \| e_1 \| A \Bigg[ \int_{t_1}^{t_2} \left| \pi_m \xi_s - \xi_s \right| \| \xi_s \| H ds \Bigg]$$

$$\leq C \| e_1 \| A \Bigg[ E^P_{P^k}\left[ \int_{t_1}^{t_2} \left| \pi_m \xi_s - \xi_s \right|^2 H ds \right] \Bigg]^{1/2} \Bigg[ E^P_{P^k}\left[ \int_{t_1}^{t_2} \left| \xi_s \right|^2 H ds \right] \Bigg]^{1/2}$$

$$\leq C \| e_1 \| A \| e_1 \| A \Bigg[ \int_{t_1}^{t_2} \left| \xi_s \right|^2 H ds \Bigg]^{1/2}$$

we have for every $\varepsilon > 0$, there exists $M$, for $m > M$,

$$|E^P_{P^k}\left[ e^{iu(M^\xi_{t_2} - M^\xi_t) - M^\xi_t(\tau_m(\xi))} \right] - E^P_{P^k}\left[ e^{iu(M^\xi_{t_2} - M^\xi_t) - M^\xi_t(\tau_m(\xi))} \right]|$$

$$\leq E^P_{P^k}\left[ e^{iu(M^\xi_{t_2} - M^\xi_t) - M^\xi_t(\tau_m(\xi))} \right] - e^{iu(M^\xi_{t_2} - M^\xi_t) - M^\xi_t(\tau_m(\xi))} ds$$

$$\leq \varepsilon,$$

then we get $\lim_{(k,m) \to \infty} E^P_{P^k} e^{iu(M^\xi_{t_2} - M^\xi_t) - M^\xi_t(\tau_m(\xi))} = \exp\{[(t_2 - t_1) \int_{[0]}(e^{iux} - 1 - iux)\lambda(dx)]\}$.

Note that

$$\lim_{m \to \infty} E^P F^\xi_m(\xi) = E^P F^\xi(\xi),$$

(2) is proved. By the same argument, it can be proved that $\{M^\xi_t\}_{t \in \mathbb{R}}$ are independent.

(3) of Definition 3.1. Since $\lim_{k \to \infty} P^k(\varphi) = P(\varphi)$, $\varphi \in C_b(D_{D(A)}[0, \infty))$. $\varphi(\xi) = e^{i(u,\xi(0))} \in C_b(D_{D(A)}[0, \infty))$, for every $u \in D(A)$, $\lim_{k \to \infty} P^k(\varphi(\xi)) = P\varphi(\xi)$, hence $\Pi_0 P = \mu$. $\square$
Remark 3.1. In fact, $P_n$ is tight in $L^2(0; T; H)$ also. Let $K$ be a separable Hilbert space, $W^{\alpha, p}(0; T; K)$ be the space (cf. [5]) of all measurable functions $f : [0, T] \to K$ such that

$$
\|f\|_{W^{\alpha, p}(0; T; K)}^p := \int_0^T |f(t)|_K^p dt + \int_0^T \frac{|f(t) - f(r)|_K^p}{|t - r|^{1 + \alpha p}} dt dr < \infty.
$$

Note that for $p = 2$, $0 < \alpha < \frac{1}{2}$,

$$
E^Q \left\| \sum_{i=1}^n \sigma_i e_i L_i(\cdot) \right\|^2_{W^{\alpha, 2}(0; T; H)} = E^Q \left( \sum_{i=1}^n |\sigma_i e_i L_i(t)|^2_H \right)^2 dt + E^Q \sum_{i=1}^n |\sigma_i e_i dL_i(s)|^2_H dt dr
$$

$$
\leq \sigma^2 T^2 \int_{|x| \leq 1} x^2 \lambda(dx) + \sigma^2 \sum_{i=1}^n \int_{|x| \leq 1} x^2 \lambda(dx) \int_0^T \frac{1}{|t - r|^{2\alpha}} dt dr < \infty.
$$

Let $J^n_t = -\int_0^t [v A_n X^n_s + B_n(X^n_s, X^n_s)] ds$, we have (cf. [5])

$$
\|J^n_t\|^2_{W^{1, 2}(0; T; D(A^{-\gamma}))} \leq C_v \int_0^T \|X^n_s\|^2_V ds + C \sup_{x \in [0, T]} \|X^n_s\|^2_H \int_0^T \|X^n_s\|^2_V ds,
$$

for $\gamma \in (3/2, 2)$, therefore

$$
E^Q \left[ \|\xi\|_{W^{\alpha, 2}([0, T]; D(A^{-\gamma}))}^2 \right] \leq C \left( v, m_2, T, \int_{|x| \leq 1} x^2 \lambda(dx), \sigma^2 \right)
$$

for every $\alpha \in (0, 1/2)$. From Theorem 4.6 in [5], the family of measures $\{P_n\}$ is tight in $L^2(0; T; H)$.

4. General case

For the multiplicative noise, in addition to the hypotheses of Section 1, we assume that there exists $0 < p < 2$ such that for every $u, u_1, u_2 \in H$ and $x \in K$,

$$
\int_{|x| \leq 1} |F(u_1, x) - F(u_2, x)|^2_H \lambda(dx) \leq C_1 \{ |u_1 - u_2|_H^2 \wedge |u_1 - u_2|_H^p \}, \tag{4.1}
$$

$$
\int_{|x| \leq 1} |F(u, x)|_H^2 \lambda(dx) \leq C_2 (1 + |u|_H^2). \tag{4.2}
$$

Let $\Omega_\alpha = D_{V_\alpha} [0, \infty)$, $\mathcal{T}_\alpha$ the Skorokhod topology of $\Omega_\alpha$, $\mathcal{T}_{\alpha}'$ the Skorokhod topology of $D_{V_\alpha} [0, t]$, and $\mathcal{F}_t^\alpha$ the $\sigma$-algebra generated by $\mathcal{T}_{\alpha}'$. And we assume $v = 1$. 
Definition 4.1. Given a probability measure $\mu_0$ on $H$, a probability measure $P$ on $(\Omega_\alpha, F)$ is called a solution of the martingale problem associated to Eq. (1.6) with initial law $\mu_0$, if

1. for every $T > 0$,

$$P \left( \sup_{t \in [0, T]} |\xi_t|^2_H + \int_0^T \|\xi_s\|^2_V \, ds < \infty \right) = 1.$$ 

2. for every $\varphi \in V_\alpha$ the process $\langle \xi_t, \varphi \rangle_{V, V}'$ is a semi-martingale, $(M^\varphi_t, F^\varphi_t, P)$ is a square integrable càdlàg martingale. Here $M^\varphi_t$ is defined $P$-a.s. on $(\Omega_\alpha, F)$ as

$$M^\varphi_t(\xi) = \langle \xi_t - \xi_0, \varphi \rangle_H + \int_0^t \langle \xi_s, A\varphi \rangle_H \, ds - \int_0^t \langle B(\xi_s, \varphi), \xi_s \rangle_H \, ds - \int_0^t \langle \varphi, f(s) \rangle_{V, V}' \, ds,$$

satisfies

$$M^\varphi_t(\xi) = \sum_{0 \leq s \leq t} \Delta(\xi_s, \varphi)_H - \int_0^t \int_{|x| \leq 1} \langle F(\xi_s, x), \varphi \rangle_H \lambda(dx) \, ds.$$ 

3. $\mu_0 = \pi_0 P$.

In the following of the paper, we denote $(\Omega, F, F_t)$ as $(\Omega_\alpha, F_\alpha, F^\alpha_t)$.

Remark 4.1. $\sum_{0 \leq s \leq t} \Delta(\xi_s, \varphi)_H - \int_0^t \int_{|x| \leq 1} \langle F(\xi_s, x), \varphi \rangle_H \lambda(dx) \, ds$ may have no sensible in path, in Definition 4.1, but it has meaning in mean. This can be seen from following. Let $\epsilon_n \downarrow 0$, define a continuous function $g_n$ on $R$,

$$g_n(x) = \begin{cases} x, & |x| \geq 2\epsilon_n, \\ 0, & |x| \leq \epsilon_n. \end{cases}$$

Let

$$G_n(\xi)(t) = \sum_{0 \leq s \leq t} g_n(\Delta(\xi_s, \varphi)_H) - \int_0^t \int_{|x| \leq 1} g_n(\langle F(\xi_s, x), \varphi \rangle_H) \lambda(dx) \, ds.$$

$(G_n, F^\alpha_t, P)$ is a square integrable càdlàg martingale, and there exists $G$, $\lim_{n \to \infty} G_n = G$ in $(\Omega, P) \times L^2([0, T]; R)$, so $G$ is a square integrable càdlàg martingale, and we denote $G(\xi)(t) = \sum_{0 \leq s \leq t} \Delta(\xi_s, \varphi)_H - \int_0^t \int_{|x| \leq 1} \langle F(\xi_s, x), \varphi \rangle_H \lambda(dx) \, ds$.

Theorem 4.1. Assume (4.1)–(4.2). For any probability measure $\mu$ on $H$ with $m_2 = \int_H |x|^2_H \mu(dx) < \infty$, there exists a martingale solution of Eq. (1.6).

The proof of this theorem is based on a classical Galerkin approximation scheme. Let $u_n(t)$ be the càdlàg adapted solution of the following equation (4.3) in $H_n$. 


\begin{equation}
\begin{aligned}
\left\{ \begin{array}{l}
du_n(t) + A_n u_n(t) \, dt + B_n \left( u_n(t), u_n(t) \right) \, dt = f_n(t) \, dt + \int_{|x| \leq 1} F_n \left( u_n(t^-), x \right) \tilde{N}_p(dt, dx), \\
u_0^n \in H_n,
\end{array} \right.
\end{aligned}
\tag{4.3}
\end{equation}

where $f_n(t) = \pi_n f(t)$, $G_n(u) = \pi_n G(u)$, $F_n(u, x) = \pi_n F(u, x)$.

**Lemma 4.1.** Assume $E|u_n(0)|^2_H < \infty$, then for every $T > 0$,

$$
E \sup_{0 \leq t \leq T} |u_n(t)|^2_H + E \int_0^T \|u_n(s)\|^2_V \, ds \leq C \left( T, \int_0^T \|f_n(s)\|^2_V \, ds, E|u_n(0)|^2_H \right)
$$

for some positive $C(\cdot)$.

**Proof.** Apply Itô’s formula on (4.3), and note that $\langle x, B_n(x, x) \rangle_H = 0$,

$$
|u_n(t)|^2_H = |u_n(0)|^2_H - 2 \int_0^t \langle u_n(s), A_n u_n(s) \rangle_H \, ds - 2 \int_0^t \langle u_n(s), B_n(u_n(s), u_n(s)) \rangle_H \, ds
$$

$$
+ 2 \int_0^t \langle u_n(s), f_n(s) \rangle_{V, V} \, ds
$$

$$
+ \int_0^t \int_{|x| \leq 1} |u_n(s^-) + F_n(u_n(s^-), x)|^2_H - |u_n(s^-)|^2_H \tilde{N}(ds, dx)
$$

$$
+ \int_0^t \int_{|x| \leq 1} |u_n(s) + F_n(u_n(s), x)|^2_H - |u_n(s)|^2_H - 2\langle u_n(s), F_n(u_n(s), x) \rangle_H \lambda(dx) \, ds,
$$

$$
|u_n(t)|^2_H + 2 \int_0^t \|u_n(s)\|^2_V \, ds = |u_n(0)|^2_H + 2 \int_0^t \langle u_n(s), f_n(s) \rangle_{V, V} \, ds
$$

$$
+ \int_0^t \int_{|x| \leq 1} |u_n(s^-) + F_n(u_n(s^-), x)|^2_H - |u_n(s^-)|^2_H \tilde{N}(ds, dx)
$$

$$
+ \int_0^t \int_{|x| \leq 1} |F_n(u_n(s), x)|^2_H \lambda(dx) \, ds.
$$

Since $2 \int_0^t \langle u_n(s), f_n(s) \rangle_{V, V} \, ds \leq \int_0^t \|u_n(s)\|^2_V \, ds + \int_0^t \|f_n(s)\|^2_V \, ds$, we have

$$
|u_n(t)|^2_H + \int_0^t \|u_n(s)\|^2_V \, ds \leq |u_n(0)|^2_H + \int_0^t \|f_n(s)\|^2_V \, ds
$$
\[
\begin{align*}
&+ \int_{0}^{t} \int_{|x| \leq 1} |F_{n}(u_{n}(s)-, x)|_{H}^{2} + 2|u_{n}(s)-, F_{n}(u_{n}(s)-, x)|_{H} \tilde{N}(ds, dx) \\
&+ \int_{0}^{t} \int_{|x| \leq 1} |F_{n}(u_{n}(s), x)|_{H}^{2} \lambda(dx) ds.
\end{align*}
\]

By Burkholder–Davis–Gundy inequality,

\[
E \sup_{t \in [0, T]} \left| \int_{0}^{t} \int_{|x| \leq 1} |F_{n}(u_{n}(s)-, x)|_{H}^{2} + 2|u_{n}(s)-, F_{n}(u_{n}(s)-, x)|_{H} \tilde{N}(ds, dx) \right| \\
\leq E \sup_{t \in [0, T]} \left| \int_{0}^{t} \int_{|x| \leq 1} |F_{n}(u_{n}(s)-, x)|_{H}^{2} \tilde{N}(ds, dx) \right| \\
+ E \sup_{t \in [0, T]} \left| \int_{0}^{t} \int_{|x| \leq 1} 2|u_{n}(s)-, F_{n}(u_{n}(s)-, x)|_{H} \tilde{N}(ds, dx) \right| \\
\leq 2E \left[ \sup_{t \in [0, T]} |u_{n}(s)|_{H} \left( \int_{0}^{T} \int_{|x| \leq 1} |F_{n}(u_{n}(s)-, x)|_{H}^{2} N_{p}(ds, dx) \right)^{1/2} \right] \\
+ E \left[ \int_{0}^{T} \int_{|x| \leq 1} |F_{n}(u_{n}(s)-, x)|_{H}^{4} N_{p}(ds, dx) \right]^{1/2} \\
\leq \frac{1}{4} E \left( \sup_{t \in [0, T]} |u_{n}(s)|_{H}^{2} \right) + 5E \left[ \int_{0}^{T} \int_{|x| \leq 1} |F_{n}(u_{n}(s)-, x)|_{H}^{2} N_{p}(ds, dx) \right].
\]

So

\[
\frac{3}{4} E \sup_{t \in [0, T]} |u_{n}(t)|_{H}^{2} \leq \left| u_{n}(0) \right|_{H}^{2} + \int_{0}^{T} |f(s)|_{V}^{2} ds + 6E \int_{0}^{T} \int_{|x| \leq 1} |F_{n}(u_{n}(s), x)|_{H}^{2} \lambda(dx) ds
\]
\[
\begin{aligned}
&\leq |u_n(0)|^2_H + \int_0^T |f(s)|^2_{V'} \, ds + 6CE \int_0^T \left(1 + |u_n(s)|^2_H \right) \, ds \\
&\leq |u_n(0)|^2_H + \int_0^T |f(s)|^2_{V'} \, ds + 6CT + 6C \int_0^T E \left( \sup_{s' \in [0,s]} |u_n(s)|^2_H \right) \, ds.
\end{aligned}
\]

By Gronwall lemma,

\[
E \sup_{s \in [0,T]} |u_n(s)|^2_H \leq C \left(T, |u_n(0)|^2_H, \int_0^T |f_n(s)|^2_{V'} \, ds \right).
\]

Since

\[
E \int_0^T \left\| u_n(s) \right\|_{V'}^2 \, ds \leq |u_n(0)|^2_H + \int_0^T \left\| f(s) \right\|_{V'}^2 \, ds + E \int_0^T \int_{|x| \leq 1} |F_n(u_n(s), x)|^2_{H^\lambda} \, (dx) \, ds
\]

\[
\leq |u_n(0)|^2_H + \int_0^T \left\| f(s) \right\|_{V'}^2 \, ds + E \int_0^T C(1 + |u_n(s)|^2_H) \, ds
\]

\[
\leq |u_n(0)|^2_H + \int_0^T \left\| f(s) \right\|_{V'}^2 \, ds + CT + CTE \sup_{s \in [0,T]} |u_n(s)|^2_H.
\]

(4.4) is proved. \(\square\)

**Corollary 4.1.** Let \(\tau \geq 0\) be a stopping time and \((u_n(t))_{t \geq 0}\) a càdlàg adapted process that P-a.s. satisfies Eq. (4.3) for \(t \in [0, \tau(\omega))\). Assume \(E \left| u_n(0) \right|^2_H < \infty\). Then, for every \(T > 0\),

\[
E \left( \sup_{t \in [0,T]} \left| u_n(t \wedge \tau) \right|^2_H \right) \leq C \left(T, \int_0^T \left\| f_n(s) \right\|_{V'}^2 \, ds, E \left| u_n(0) \right|^2_H \right).
\]

**Lemma 4.2.** For every \(F_0\)-measurable \(u_n(0) : \Omega \to H_n\), there exists a unique càdlàg adapted solution \((u_n(t))_{t \geq 0}\) of Eq. (4.3) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\).

**Proof.** Step 1 (Existence for bounded initial condition \(u_n(0)\)). Assume that \(|u_n(0)|^2_H \leq C\) for some constant \(C > 0\). For any \(m > C\), let \(B^m_n(\cdot) : H_n \to H_n\) be a Lipschitz continuous function such that \(B^m_n(x) = B_n(x, x)\) for every \(|x|_{H_n} \leq m\). Consider the equation

\[
du_n(t) + A_n u_n(t) \, dt + B^m_n(u_n(t), u_n(t)) \, dt = f_n(t) \, dt + \int_{|x| \leq 1} F_n(u_n(t-), x) N_p \, (dt, dx)
\]
with initial condition \( u_n(0) \). It has globally Lipschitz coefficients, so there exists a unique càdlàg adapted solution \((u_n^{(m)}(t))_{t \geq 0}\) by the standard classical argument. Let \( \tau_m \) be defined as

\[
\tau_m = \inf \{ t \geq 0 : |u_n^{(m)}|_{H_n} \geq m \} \wedge T,
\]

the solution \((u_n^{(m)}(t))_{t \in [0, \tau_m)}\) is also a solution of Eq. (4.3). (Just note that for \( u_n^{(m)}(\tau_m) \) is decided by \( t \in [0, \tau_m) \).) Therefore, by Corollary 4.1, we have

\[
E \left( \sup_{t \in [0, T]} \left| u_n^{(m)}(t \wedge \tau_m) \right|^2_{H_n} \right) \leq C \left( T, E \left| u_n(0) \right|^2_{H_n}, \int_0^T \left\| f(s) \right\|_{V'}^2 ds \right).
\]

In particular

\[
E \left( 1_{\tau_m < T} \left| u_n^{(m)}(\tau_m) \right|^2_{H_n} \right) \leq C \left( T, E \left| u_n(0) \right|^2_{H_n}, \int_0^T \left\| f(s) \right\|_{V'}^2 ds \right),
\]

which implies

\[
P(\tau_m < T) \leq \frac{1}{m^2} C \left( T, E \left| u_n(0) \right|^2_{H_n}, \int_0^T \left\| f(s) \right\|_{V'}^2 ds \right).
\]

Since \( |u_n^{(m)}(T \wedge \tau_m)|_{H_n}^2 \geq m^2 \) on \( \{ \tau_m < T \} \), \( \tau_m \geq \tau_m \) as \( M > m \) and

\[
P(\tau_n^{(M)}(t) = u_n^{(m)}(t), t \in [0, \tau_m)) = 1,
\]

therefore, if \( \tau_\infty := \sup_{m \geq C} \{ \tau_m \} \), we may uniquely define a process \( u_n^{(\infty)}(t) \) for \( t \in [0, \tau_\infty) \), which equal to \( u_n^{(m)}(t) \) on \( [0, \tau_m) \) for every \( m \). Hence \( u_n^{(\infty)}(t) \) is a solution on \([0, \tau_\infty)\). Since

\[
P(\tau_\infty < T) \leq P(\tau_m < T) \leq \frac{C}{m^2},
\]

for every \( m \), \( P(\tau_\infty < T) = 0 \). Thus \( u_n^{(\infty)} \) is a solution for \( t \in [0, T - \epsilon] \) for every small \( \epsilon > 0 \). Since \( T \) is arbitrary, we have proved global existence.

**Step 2 (Existence for general initial condition \( u_n(0) \)).** Let \( \Omega_m \in \mathcal{F} \) be defined as \( \Omega_m = \{ |u_n(0)|_{H_n}^2 \leq m \} \). Define \( u_n^{(m)}(0) \) as \( u_n(0) \) on \( \Omega_m \), 0 otherwise. Let \((u_n^{(m)}(t))_{t \geq 0}\) be the unique solution of Eq. (4.3) with initial condition \( u_n^{(m)}(0) \). If \( M > m \), then

\[
P(\Omega_m \cap (u_n^{(M)}(t) = u_n^{(m)}(t) \text{ for every } t \geq 0)) = P(\Omega_m).
\]

We may then uniquely define a process \( u_n \) on \( \Omega' = \bigcup_m \Omega_m \) as \( u_n(t) = u_n^{(m)}(t) \) on \( \Omega_m \), it is clear that \( u_n \) solves Eq. (2.4) on \( \Omega' \). Since \( P(\Omega') = 1 \), we have get a global solution. \( \square \)
5. Proof of Theorem 4.1

Proof. Let \((W, \mathcal{W}, (\mathcal{W}_t)_{t \geq 0}, Q, p)\) be a stochastic basis supporting, a \(\mathcal{W}_0\)-measurable random variable \(u_0 : W \to H\) with law \(\mu, p\) is a stationary Poisson point process defined as in Section 1. Let \(X^n_0 = \pi_n u_0\).

For every \(n\), there exists a unique \(\text{càdlàg}\) adapted solution \((X^n(t))_{t \geq 0}\) of Eq. (4.3) in \(H_n\), with initial condition \(X^n_0\). Since \(H_n \subset H\), \((X^n(t))_{t \geq 0}\) can be viewed as \(\text{càdlàg}\) adapted process in \(V_{\alpha'}\), so it defines a probability measure \(P_n\) on \(D_{V_{\alpha'}}[0, \infty)\).

**Step 1 (Tightness in \(\Omega \cap L^2([0, T]; H)\)).** From Lemma 2.4, choose \(\alpha \geq 1\), let \(D = V_{\alpha}\), we prove \(X^n\) is \(D\)-weakly tight and for every \(\epsilon > 0\) and \(t > 0\),

\[
\lim_{N \to \infty} \lim_{n \to \infty} \sup_{s \in [0, t]} P(r^2_N(X^n(s)) > \epsilon) = 0.
\]

First, \(D\)-weakly tightness: by using Lemma 2.5, we only need to prove that for every \(\varphi \in D\), \((\langle X^n(.), \varphi \rangle_{D', D})\) is tight in \(D_R[0, T]\).

For every \(T > 0\), by Lemma 4.1

\[
E^Q \left[ \sup_{t \in [0, T]} (\langle X^n(t), \varphi \rangle^2_{D', D}) \right] \leq \|\varphi\|^2_{H} E \left[ \sup_{t \in [0, T]} |X^n(t)|^2_H \right] \leq \|\varphi\|^2_{H} C \left( T, \int_0^T \|f(s)\|^2_{V'} ds, m_2 \right), \quad (5.1)
\]

so \((\langle X^n(.), \varphi \rangle_{D', D})\) is tight on the line for each \(t \in [0, T]\).

For any \(\tau_n, \delta_n\) satisfy (a), (b), we have

\[
\left| \langle X^n(\tau_n + \delta_n) - X^n(\tau_n), \varphi \rangle_{D', D} \right|
\leq \int_{\tau_n}^{\tau_n+\delta_n} |(\varphi, B_n(X^n(s), X^n(s)))|_H ds + \int_{\tau_n}^{\tau_n+\delta_n} |(\varphi, AX^n(s))|_H ds + \int_{\tau_n}^{\tau_n+\delta_n} |(\varphi, f_n(s))|_{V', V'} ds
\]

\[
+ \int_0^T \int_{|x| \leq 1} \left[ 1_{(\tau_n, \tau_n+\delta_n)} |F_n(X^n(s-), x), \varphi \rangle_{H}, \tilde{N}_p(ds, dx) \right]
= I_1 + I_2 + I_3 + I_4.
\]

Since \(\lim_{n \to \infty} \delta_n = 0\) and \(|\langle B(u, v), w \rangle| \leq C|u|^{1/4}_H |u|^{3/4}_V |v| |v|^{1/4}_V |w|^{3/4}_V\),

\[
E^Q [I_2] \leq \|\varphi\|_{V'} \left[ \int_0^T \|X^n(s)\|^2_V ds \right]^{1/2} |\delta_n|^{1/2} \to 0, \quad \text{as } n \to \infty,
\]

\[
E^Q [I_1] \leq C_{\varphi} E^Q \left[ \int_{\tau_n}^{\tau_n+\delta_n} |X^n(s)|^{1/2}_H \|X^n(s)\|^{3/2}_V ds \right]
\leq C_{\varphi} \left[ E^Q \sup_{s \in [0, T]} |X^n(s)|^{2}_H \right]^{1/4} \left[ E^Q \int_0^T \|X^n(s)\|^2_V ds \right]^{3/4} \sigma_n^{1/4}
\to 0, \quad \text{as } n \to \infty.
\]
It's easy to see that $E^Q [I_3] \to 0$, as $n \to \infty$. Since

$$E^Q \left[ \int_0^T \int_{|x| \leq 1} 1_{(\tau_n, \tau_n + \delta_n]} (F_n(X^n(s-), x), \varphi)_{\bar{H}}^2 \bar{N}_p (ds, dx) \right]^2$$

$$= E^Q \left[ \int_0^T \int_{|x| \leq 1} 1_{(\tau_n, \tau_n + \delta_n]} (F_n(X^n(s-), x), \varphi)_{\bar{H}} \lambda(\cdot) (dx) \right]$$

$$\leq E^Q \left[ \int_0^T 1_{(\tau_n, \tau_n + \delta_n]} |\varphi|_{\bar{H}}^2 C(\|X^n(s)\|_{\bar{H}}^2 + 1) \right]$$

$$\leq C|\varphi|_{\bar{H}}^2 E^Q \left[ \sup_{s \in [0, T]} \|X^n(s)\|_{\bar{H}}^2 + 1 \right] \cdot \delta_n,$$

we get $I_4 \to 0$, in probability. Thus \{(X^n(t), \varphi)'_{D', D}, t \in [0, T]\} satisfy (A), and by Lemma 2.5, \{(X^n(\cdot))\} is $D$-weakly tight. Finally, since for $h_k = \lambda_k^{a/2} e_k$,

$$E^Q \left[ \left( \sup_{s \in [0, T]} r^2_{D}(X^n(s)) \right) \right] = E^Q \left[ \sup_{s \in [0, T]} \left( \sum_{k \geq N+1} \langle X^n(s), h_k \rangle_{D'}^2 \right) \right]$$

$$\leq E^Q \left[ \sup_{s \in [0, T]} \|X^n(s)\|_{D'}^2 \right]$$

$$\leq \frac{C_\delta}{\lambda_k^a N + 1},$$

$\{X^n\}$ is tight in $D_D[0, \infty)$.

Next, I will prove that $\{X^n\}$ is tight in $L^2([0, T]; H)$.

Since

$$X_n(t) = X_n(0) - \int_0^t A_n X_n(s) \, ds - \int_0^t B_n (X_n(s), X_n(s)) \, ds$$

$$+ \int_0^t f_n(s) \, ds + \int_0^{t'} F_n (X_n(s-), x) \bar{N}_p (ds, dx)$$

$$= I_5(t) + I_6(t) + I_7(t) + I_8(t) + I_9(t).$$

For $0 < \theta < 1/2$,

$$E^Q \left[ \|I_9(\cdot)\|_{W^{0, 2}([0, T]; H)}^2 \right] = E^Q \left[ \int_0^T |I_9(t)|^2_{D'} \, dt \right] + E^Q \left[ \int_0^T \int_T^T \frac{|I_9(t) - I_9(s)|^2_{D'}}{|t - s|^{1+2\theta}} \, dt \, ds \right]$$

$$= \int_0^T E^Q \left[ |I_9(t)|^2_{D'} \right] \, dt + \int_0^T \int_0^T \frac{E^Q [ |I_9(t) - I_9(s)|^2_{D'} ]}{|t - s|^{1+2\theta}} \, dt \, ds$$
\[
E^Q \left[ \int_0^T \int \int_{|x| \leq 1} \left| F_n(X_n(s), x) \right|^2_{H^2} \lambda(dx) \right] dt \\
+ \int_0^T \int \int_{|x| \leq 1} \frac{E^Q \left[ \int_{s}^{t} \frac{\left| F_n(X_n(z), x) \right|^2_{H^2}}{|t - s|^{1 + 2\beta}} dz \right]}{dt} \right] dt \right] ds \\
\leq C(T, \theta) \left\{ \int_0^T E^Q \left[ \left| X_n(s) \right|^2_{H^2} \right] ds + \int_0^T \int_{s}^{t} E^Q \left[ \left| X_n(z) \right|^2_{H^2} \right] dz \right] dt \right\} \\
\leq C \left( T, \theta, \int \| f(s) \|_{V'}^2 ds, m_2 \right).
\]

Since \( E^Q \left[ \| I_5(t) \|_{H^2}^2 \right] \leq m_2 \) and \( E^Q \left[ \| I_8(.) \|_{W^{0,2}(0, T); V')}^2 \right] \leq C_4 \), by Remark 3.1 and Lemma 4.1, for \( \gamma \in (3/2, 2) \),

\[
E^Q \left[ \| I_6(.) + I_7(.) \|_{W^{1,2}(0, T); D(A^{-\gamma})} \right] \leq C \left( m^2, T, \int \| f(s) \|_{V'}^2 ds \right).
\]

Therefore

\[
E^P_n \left[ \| \xi(.) \|_{W^{0,2}(0, T); D(A^{-\gamma})} \right] \leq C \left( m^2, T, \int \| f(s) \|_{V'}^2 ds \right).
\]

By [5], Theorem 4.6, \( P_n \) is tight in \( L^2((0, T]; H) \). Hence there exists a probability measure \( P \) on \( \Omega \cap L^2_{\text{loc}}((0, \infty); H) \), which is the weak limit of a sub-sequence \( \{P_{n_k}\} \).

Step 2 (Prove \( P \) is a martingale solution). (1) and (3) can be proved by the same argument as in the proof of Theorem 3.3.

For checking (2), let \( g(.) \) be a continuous function from \( R \) to \( R \),

\[
g(x) = \begin{cases} 
  x, & |x| \geq 1, \\
  0, & |x| \leq 1/2. 
\end{cases}
\]

with \( |g(x)| \leq |x| \) and \( |g'(x)| \leq C \).

For any \( \varphi \in D \), choose \( t \in Z \) (using in Theorem 3.3), let

\[
Y^m_n(\xi)(t) = \sum_{s \in [0, t]} \frac{1}{m} g(m \cdot \langle \Delta \xi(s), \varphi \rangle_{D', D}) - \int_0^t \int \frac{1}{m} g(m \cdot \langle F_n(\xi(s), x), \varphi \rangle) \lambda(dx) ds,
\]

and

\[
Y^m(\xi)(t) = \sum_{s \in [0, t]} \frac{1}{m} g(m \cdot \langle \Delta \xi(s), \varphi \rangle_{D', D}) - \int_0^t \int \frac{1}{m} g(m \cdot \langle F(\xi(s), x), \varphi \rangle) \lambda(dx) ds.
\]
First, \((X^m, \mathcal{F}_s, P)\) is a square integrable martingale. Set \(\Omega^0_t = \{\xi \in \Omega: \xi(t) = \xi(t-), \xi \in \Omega \cap L^2_{\text{loc}}([0, \infty); H)\}\), by Lemma 2.2,

\[
\lim_{n \to \infty} \sum_{s \in [0,t]} \frac{1}{m} g(m \cdot \langle \Delta \xi_n(s), \varphi \rangle_{D', D}) = \frac{1}{m} g(m \cdot \langle \Delta \xi(s), \varphi \rangle_{D', D}).
\]

Since

\[
\int_0^t \int_{|x| \leq 1} \frac{1}{m} g(m \cdot \langle F_n(\xi_n(s), x), \varphi \rangle) \lambda(dx) ds \leq 2m \int_0^t \int_{|x| \leq 1} \|F_n(\xi_n(s), x), \varphi\|_H^2 \lambda(dx) ds
\]

\[
\leq 2mC|\varphi|_H^2 \left[ \int_0^t \|\xi_n(s)\|_H^2 ds + t \right].
\]

\(Y^m_n(t)\) is sensible \(P_n\)-a.s.

Set

\[
A_n^s = \left\{ x: |x| \leq 1 \text{ and } \|F_n(\xi_n(s), x), \varphi\| \geq \frac{1}{2m} \right\},
\]

\[
A^s = \left\{ x: |x| \leq 1 \text{ and } \|F(\xi(s), x), \varphi\| \geq \frac{1}{2m} \right\},
\]

we have

\[
\lambda(A_n^s) = \int_{|x| \leq 1} I_{A_n^s}(x) \lambda(dx) \leq 4m^2 \int_{|x| \leq 1} \|F_n(\xi_n(s), x), \varphi\|_H^2 \lambda(dx)
\]

\[
\leq Cm^2|\varphi|_H^2 \left( \|\xi_n(s)\|_H^2 + 1 \right),
\]

and

\[
\int_0^t \int_{|x| \leq 1} \left| g(m \cdot \langle F_n(\xi_n(s), x), \varphi \rangle) - g(m \cdot \langle F(\xi(s), x), \varphi \rangle) \right| \lambda(dx) ds
\]

\[
= \int_0^t \int_{A_n^s \cup A_n^s} \left| g(m \cdot \langle F_n(\xi_n(s), x), \varphi \rangle) - g(m \cdot \langle F(\xi(s), x), \varphi \rangle) \right| \lambda(dx) ds
\]

\[
\leq C \int_0^t \int_{|x| \leq 1} \left| \langle F_n(\xi_n(s), x), \varphi \rangle - \langle F(\xi(s), x), \varphi \rangle \right| \lambda(dx) ds
\]

\[
\leq C \left[ \int_0^t \int_{|x| \leq 1} \left| \langle F_n(\xi_n(s), x), \varphi \rangle - \langle F(\xi(s), x), \varphi \rangle \right|^2 \lambda(dx) ds \right]^{1/2} \left[ \int_0^t \lambda(A^s \cup A_n^s) ds \right]^{1/2}
\]
\[
\begin{align*}
&\leq C m |\varphi|_H \cdot \left[ \int_0^t \int_{|x| \leq 1} \left| \left| F(\xi_n(s), x) - F(\xi(s), x), \pi_n \varphi \right| \right|^2 + \left| \left| F(\xi(s), x), \pi_n^c \varphi \right| \right|^2 \lambda(dx) ds \right]^{1/2} \\
&\quad \cdot \left[ \int_0^t |\xi_n(s)|_H^2 ds + \int_0^t |\xi(s)|_H^2 ds + t \right]^{1/2} \\
&\leq C m |\varphi|_H \cdot \left[ \int_0^t \int_{|x| \leq 1} \left| \left| F(\xi_n(s), x) - F(\xi(s), x) \right| \right|^2 \lambda(dx) ds \right]^{1/2} \\
&\quad + \left| \pi_n^c \varphi \right|_H^2 \cdot \left[ \int_0^t \int_{|x| \leq 1} \left| \left| F(\xi(s), x) \right| \right|^2 \lambda(dx) ds \right]^{1/2} \cdot \left[ \int_0^t |\xi_n(s)|_H^2 ds + \int_0^t |\xi(s)|_H^2 ds + t \right]^{1/2} \\
&\leq C m |\varphi|_H \left[ \int_0^t \int_{|x| \leq 1} \left| \left| \xi_n(s) - \xi(s) \right| \right|^2 \lambda(dx) ds + \left| \pi_n^c \varphi \right|_H^2 \cdot \left[ \int_0^t \int_{|x| \leq 1} \left| \left| \xi(s) \right| \right|^2 + 1 \right] ds \right]^{1/2} \\
&\quad \cdot \left[ \int_0^t |\xi_n(s)|_H^2 ds + \int_0^t |\xi(s)|_H^2 ds + t \right]^{1/2}.
\end{align*}
\]

Thus
\[
\lim_{n \to \infty} \int_0^t \int_{|x| \leq 1} \frac{1}{m} g(m \cdot \{F_n(\xi_n(s), x), \varphi\}) \lambda(dx) ds = \int_0^t \int_{|x| \leq 1} \frac{1}{m} g(m \cdot \{F(\xi(s), x), \varphi\}) \lambda(dx) ds.
\]

Since \((Y_n^m(\xi)(t), P_n)\) is a square integrable càdlàg martingale, and for \(t_1 < t_2 \in Z\),
\[
E P_n \left| Y_n^m(\xi)(t_2) - Y_n^m(\xi)(t_1) \right|^2 = E \left[ \int_{t_1}^{t_2} \int_{|x| \leq 1} \frac{1}{m} g(m \cdot \{F_n(X_n(s-), x), \varphi\}) \tilde{N}_p(dx, ds) \right]^2
\]
\[
\leq E Q \left[ \int_{t_1}^{t_2} \int_{|x| \leq 1} \left[ \frac{1}{m} g(m \cdot \{F_n(X_n(s), x), \varphi\}) \right]^2 \lambda(dx) ds \right]
\]
\[
\leq |\varphi|_H^2 E Q \left[ \int_{t_1}^{t_2} \int_{|x| \leq 1} \left| F_n(X_n(s), x) \right|_H^2 \lambda(dx) ds \right]
\]
\[
\leq C |\varphi|_H^2 E Q \int_{t_1}^{t_2} \left( \left| X_n(s) \right|_H^2 + 1 \right) ds
\]
\[
\leq C \left( t_2 - t_1, \int_{t_1}^{t_2} \| f(s) \|_{C^0}^2, ds, E \left| X(0) \right|_H^2 \right) |\varphi|_H^2.
\]
Let $t_1 = 0$ and using Lemma 2.5,

$$E^P|Y^m(ξ)(t_2)|^{1+\varepsilon} = \lim_{n \to \infty} E^P_n|Y^m_n(ξ)(t_2)|^{1+\varepsilon}$$

$$\leq \sup_n [E^P_n|Y^m_n(ξ)(t_2)|^2]^{(1+\varepsilon)/2}$$

$$\leq \left\{ c(t_2, \int_0^{t_2} \| f(s) \|_{V}, ds, E|u(0)|_H^2) |\varphi|^2_H \right\}^{(1+\varepsilon)/2}$$

$$< \infty,$$

for every $0 < \varepsilon < 1$. So $E^P|Y^m(ξ)(t_2)|^2 < \infty$. Since for every $\mathcal{F}_t$-measurable bounded random variable $Z$,

$$E^P\{[Y^m(t_2) - Y^m(t_1)] \cdot Z\} = \lim_{n \to \infty} E^P_n\{[Y^m_n(t_2) - Y^m_n(t_1)] \cdot Z\} = 0,$$

$(Y^m, \mathcal{F}_t, P)$ is a square integrable càdlàg martingale.

Second, we prove that $(Y^m, P)$ is a Cauchy sequence in $L^2(\Omega \times [0, T]; R)$, and denote $\lim_{m \to \infty} Y^m = Y^\psi$.

By Skorokhod embedding theorem, there exists a stochastic basis $(\Omega', \mathcal{F}', \{\mathcal{F}_t\}_{t \geq 0}, P')$ and, on this basis, $\Omega \cap L^2_{loc}([0, \infty); H)$-valued random variables $X', X'_k$, $k \geq 1$, such that $X'_k$ has the same law of $P_{n_k}$ on $\Omega \cap L^2_{loc}([0, \infty); H)$, and $X'_k \to X'$ in $\Omega \cap L^2_{loc}([0, \infty); H)$ $P'$-a.s.

Let

$$G_{m_1, m_2, n}(ξ)(t) = Y^m_{n_k}(ξ)(t) - Y^m_n(ξ)(t), \quad G_{m_1, m_2}(ξ)(t) = Y^{m_2}(ξ)(t) - Y^{m_1}(ξ)(t).$$

Then

$$E^P \int_0^T |G_{m_1, m_2}(ξ)(t)|^2 dt = \int_0^T E^P \left[ |G_{m_1, m_2}(Y')(t)|^2 \right] dt$$

$$\leq \int_0^T \liminf_{n \to \infty} E^P \left[ |G_{m_1, m_2, n}(Y')(t)|^2 \right] dt$$

and

$$E^P \left[ G_{m_1, m_2, n}(Y')(t) \right]^2$$

$$= E^Q \left[ \int_0^t \int \frac{1}{m_1} g(m_1 \cdot \langle F_n(X_n(s), X_1), \varphi \rangle) - \frac{1}{m_2} g(m_2 \cdot \langle F_n(X_n(s), X_1), \varphi \rangle) \tilde{N}_p(dx, ds) \right]^2$$

$$= E^Q \left[ \int_0^t \int \frac{1}{m_1} g(m_1 \cdot \langle F_n(X_n(s), X_1), \varphi \rangle) - \frac{1}{m_2} g(m_2 \cdot \langle F_n(X_n(s), X_1), \varphi \rangle) \left[ \lambda(dx) ds \right] \right]^2.$$
\[ E^{P'} \left[ \int_{0}^{t} \int_{|x| \leq 1} \left| \frac{1}{m_1} g(m_1 \cdot \{F_n(X'_n(s), x), \varphi\}) - \frac{1}{m_2} g(m_2 \cdot \{F_n(X'_n(s), x), \varphi\}) \right|^2 \lambda(dx) \, ds \right]. \]

Note that

\[ E^{P'} \left[ \int_{0}^{t} \int_{|x| \leq 1} \left| \frac{1}{m_1} g(m \cdot \{F(X'(s), x), \varphi\}) - \frac{1}{m_2} g(m \cdot \{F(X'_n(s), x), \varphi\}) \right|^2 \lambda(dx) \, ds \right] \]

\[ \leq C_{\varphi} E^{P'} \left[ \int_{0}^{t} \left| X'(s) - X'_n(s) \right|^2 \lambda(dx) \, ds \right] + \left| \pi_n \varphi \right|^2 E^{P'} \left[ \int_{0}^{t} \left| F(X'(s), x) \right|^2 \lambda(dx) \, ds \right], \]

\[ \lim_{n \to \infty} \int_{0}^{t} \left| X'(s) - X'_n(s) \right|^2 \lambda(dx) \, ds = 0, \quad P'-a.s., \]

\[ E^{P'} \int_{0}^{t} \left| X'(s) - X'_n(s) \right|^2 \lambda(dx) \, ds \leq C \left( m^2, t, \int \left( \| f(s) \|_{V'} \right)^2 \, ds \right), \]

by Lemma 2.5,

\[ (5.2) = E^{P'} \left[ \int_{0}^{t} \int_{|x| \leq 1} \left| \frac{1}{m_1} g(m_1 \cdot \{F(X'(s), x), \varphi\}) - \frac{1}{m_2} g(m_2 \cdot \{F(X'(s), x), \varphi\}) \right|^2 \lambda(dx) \, ds \right] \]

\[ \leq C_{\varphi} \int_{0}^{t} E^{P'} \left[ \int_{0}^{t} \int_{|x| \leq 1} \left| F(X'(s), x) \right|^2 \lambda(dx) \, ds \right] dt \]

\[ \to 0 \quad \text{as} \quad (m_1, m_2) \to \infty. \]

**Third,** we prove that \( E^{\psi_{\mu}(M^t \cdot Y_{\psi}(\cdot))} = 1. \)

By using the same method as above, we have for fixed \( n, \ (Y^m_n, P_n) \) is Cauchy sequence in \( L^2(\Omega \times [0, T]; R) \) and denote \( m^2 Y = \lim_{n \to \infty} Y^m_n. \) For any fixed \( n > i, \) let

\[ M_n^m(\xi)(t) = \langle \xi_t - \xi_0, e_i \rangle_H + \int_{0}^{t} \langle \xi_{s}, A e_i \rangle_H \, ds - \int_{0}^{t} \langle B(\pi_m \mathbf{\xi}, e_i), \pi_m \mathbf{\xi} \rangle_H \, ds - \int_{0}^{t} \langle e_i, f_n(s) \rangle_{V', V} \, ds, \]

\[ M^m(\xi)(t) = \langle \xi_t - \xi_0, e_i \rangle_H + \int_{0}^{t} \langle \xi_{s}, A e_i \rangle_H \, ds - \int_{0}^{t} \langle B(\mathbf{\xi}, e_i), \mathbf{\xi} \rangle_H \, ds - \int_{0}^{t} \langle e_i, f_n(s) \rangle_{V', V} \, ds, \]

\[ M_m(\xi)(t) = \langle \xi_t - \xi_0, e_i \rangle_H + \int_{0}^{t} \langle \xi_{s}, A e_i \rangle_H \, ds - \int_{0}^{t} \langle B(\pi_m \mathbf{\xi}, e_i), \pi_m \mathbf{\xi} \rangle_H \, ds - \int_{0}^{t} \langle e_i, f(s) \rangle_{V', V} \, ds, \]
\[ |E^P e^{iu[M_n^i - Y^i(t)]} - 1| \]
\[ = \lim_{m \to \infty} |E^P e^{iu[M_m(t) - Y^i(t)]} - 1| \]
\[ = \lim_{m \to \infty} \lim_{k \to \infty} |E^P e^{iu[M_m(t) - Y^i(t)]} - 1| \]
\[ = \lim_{m \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} |E^P_n e^{iu[M^i_n(t) - Y^i_n(t)]} - 1| \]
\[ = \lim_{m \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} \left[ |E^P_n e^{iu[M^i_n(t) - Y^i_n(t)]} - E^P_n e^{iu[M^i(t) - Y^i(t)]}| \right] \]
\[ \leq \lim_{m \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} |E^P_n \left[ e^{iu[\int_0^t (B(\xi(s), e_i), \epsilon_i) \eta(s) \, ds] + iu[Y_n(t) - Y^i_n(t)]} - 1 \right]| \]
\[ \leq \lim_{m \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} \left[ |E^P_n e^{iu[Y_n(t) - Y^i_n(t)]} - 1| + |E^P_n e^{iu[Y(t) - Y^i(t)]} - 1| \right]. \]

We only need to prove the second part equals 0, the first part is similar to prove as Theorem 3.3. Since

\[
\lim_{n \to \infty} E^P \left[ \int_0^t \int_{|x| \leq 1} \left| \frac{1}{m_1} g(m_1 \cdot \{ F_n(X'_n(s), x), \varphi \}) - \frac{1}{m_2} g(m_2 \cdot \{ F_n(X'_n(s), x), \varphi \}) \right|^2 \lambda(dx) \, ds \right] = 0.
\]

we have

\[
\lim_{k \to \infty} \lim_{n \to \infty} E^P_n |Y_n(t) - Y^i_n(t)|^2 \leq \lim_{k \to \infty} E^P \left[ \int_0^t \int_{|x| \leq 1} |F(\xi(s), x)|^2 \lambda(dx) \, ds \right] = 0,
\]

thus \( \lim_{m \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} E^P_n |e^{iu[Y(t) - Y^i(t)]} - 1| = 0. \) \( \square \)

6. Markov selection

We start by giving a few definitions and notations. Let \( V \subseteq H \subseteq V' \) be a Gelfand triple of separable Hilbert spaces with continuous injections. Set \( \Omega = D([0, \infty); V') \). Denote by \( B \) the Borel \( \sigma \)-field of \( \Omega \) with Skorokhod topology and by \( Pr(\Omega) \) the set of all probability measures on \( (\Omega, B) \). Define the canonical process \( \xi : \Omega \to V' \) as \( \xi_0(\omega) = \omega(t) \).

6.1 Preliminaries on the state space

For each \( t \geq 0 \), let \( \Omega_t = D([0, t]; V') \) (resp. \( \Omega^t = D([t, \infty); V') \)). Denote by \( B_t \) (resp. \( B^t \)) be the Borel \( \sigma \)-field of \( \Omega_t \) (resp. \( \Omega^t \)) with Skorokhod topology respectively. Define for each given \( t > 0 \), the map \( \Phi_t : \Omega \to \Omega^t \) as

\[ \Phi_t(\omega)(s) = \omega(s - t), \quad s \geq t. \]
Denote \( \mathcal{C}_T \) the \( \sigma \)-field generated by simple cylindrical subsets of \( \Omega_T \). Recall that
\[
\mathcal{C}_T = \sigma \left( \xi_t^{-1} (B(V)) \mid t \in [0, T] \right).
\]

**Lemma 6.1.** (See [1].) If \( V' \) is a separable metric space, then
\[
\mathcal{C}_T = B_T.
\]
Similarly, we can define \( \mathcal{C} = \sigma (\xi^{-1} (B(V))) \mid t \in [0, \infty) \) (resp. \( \mathcal{C}_T = \sigma (\xi_t^{-1} (B(V))) \mid t \in [T, \infty) \)), and we have
\[
\mathcal{C} = B \quad (\text{resp. } \mathcal{C}_T = B_T).
\]

**Lemma 6.2.** The set \( L_{\infty}^\text{loc}([0, \infty); H) \cap \Omega \) is a Borel set in \( \Omega \). Moreover,
\[
L_{\infty}^\text{loc}([0, \infty); H) \cap \Omega = D([0, \infty); \mathcal{H}_\sigma) \cap \Omega
\]
where \( \mathcal{H}_\sigma \) denotes the space \( \mathcal{H} \) endowed with the weak topology. Finally, the set \( L_{\infty}^2([0, \infty); \mathcal{V}) \cap \Omega \) is Borel in \( \Omega \) as well.

**Lemma 6.3.** Let \( P \in \text{Pr}(\Omega) \) be such that
\[
P \left( D([0, \infty); \mathcal{H}_\sigma) \cap \Omega \right) = 1.
\]
Then, for any given \( t \geq 0 \), the mapping \( \omega \to \omega(t) \) has a \( P \)-modification on \( B_t \) which is \( B_t \)-measurable with values in \( (\mathcal{H}, B(\mathcal{H})) \), where \( B(\mathcal{H}) \) is the Borel \( \sigma \)-field of \( \mathcal{H} \).

For proving Lemma 6.2 we need the following lemma.

**Lemma 6.4.** Let \( X \) and \( Y \) be two Banach spaces, such that \( X \subset Y \) with a continuous injection. If a function \( \phi(t) \) belongs to \( L^\infty([0, T+\epsilon], X) \), \( \epsilon > 0 \) and is weakly right continuous and has weakly left limits with values in \( Y \) for \( t \in [0, T+\epsilon] \), then \( \phi \) is weakly right continuous and has weakly left limits with values in \( X \), when \( t \in [0, T] \).

**Proof.** If we replace \( Y \) by the closure of \( X \) in \( Y \), we may suppose that \( X \) is dense in \( Y \). Hence the dense continuous imbedding of \( X \) into \( Y \) gives by duality a dense continuous imbedding of \( Y' \) (dual of \( Y \)), into \( X' \) (dual of \( X \)):
\[
Y' \subset X'.
\]
By assumption, for each \( \eta \in Y' \),
\[
\langle \phi(t), \eta \rangle_{Y', Y'} \to \langle \phi(t₀), \eta \rangle_{Y', Y'}, \quad \text{as } t \downarrow t₀, \quad \forall t₀ \in [0, T+\epsilon), \quad (6.1)
\]
and \( \exists \, y_{t₀} \in Y \) such that
\[
\langle \phi(s), \eta \rangle_{Y', Y'} \to \langle y_{t₀}, \eta \rangle_{Y', Y'} \quad \text{as } s \uparrow t₀, \quad \forall t₀ \in (0, T+\epsilon]. \quad (6.2)
\]
We next to prove that (6.1) and (6.2) are true for \( \eta \in X', \phi(t), y_t \in X, t \in (0, T] \).
We first prove that for each \( t \in (0, T] \), \( \phi(t), y_t \in X \) and

\[
\| \phi(t) \|_X \leq 2\| \phi \|_{L^\infty([0, T+\epsilon], X)}, \tag{6.3}
\]
\[
\| y_t \|_X \leq 2\| \phi \|_{L^\infty([0, T+\epsilon], X)}. \tag{6.4}
\]

For every \( t_0 \in [0, T] \), define

\[
\tilde{\Phi}_{t_0}(t) = \begin{cases} 
0, & 0 \leq t < t_0, \text{ or } t > T + \epsilon, \\
\phi(t), & t_0 \leq t \leq T + \epsilon.
\end{cases}
\]

We can find a sequence of smooth functions \( \phi_m \) from \([0, T + \epsilon]\) into \( X \), such that

\[
\| \phi_m(t) \|_X \leq \| \tilde{\Phi}_{t_0} \|_{L^\infty([0, T+\epsilon]; X)} \leq \| \phi \|_{L^\infty([0, T+\epsilon]; X)}
\]

and

\[
\left\langle \phi_m(t), \eta \right\rangle_{Y, Y'} \to \frac{\lim_{s \uparrow t} \left\langle \tilde{\Phi}_{t_0}(s), \eta \right\rangle_{Y, Y'} + \lim_{s \downarrow t} \left\langle \tilde{\Phi}_{t_0}(s), \eta \right\rangle_{Y, Y'}}{2}, \quad \text{for } m \to \infty, \forall \eta \in Y'.
\]

Since

\[
\left\langle \phi_m(t_0), \eta \right\rangle_{Y, Y'} \to \left\langle \frac{\phi(t_0)}{2}, \eta \right\rangle_{Y, Y'}, \quad \text{for } m \to \infty, \forall \eta \in Y',
\]

and

\[
\left\| \phi_m(t), \eta \right\|_{Y, Y'} \leq \| \phi \|_{L^\infty([0, T+\epsilon]; X)} \| \eta \|_{X'}, \quad \forall m, \forall t \in [0, T + \epsilon], \forall \eta \in Y',
\]

we have the limit

\[
\left\| \phi(t_0), \eta \right\|_{Y, Y'} \leq \| \phi \|_{L^\infty([0, T+\epsilon]; X)} \| \eta \|_{X'}, \quad \forall \eta \in Y'.
\]

This inequality shows that \( \phi(t) \in X \), for every \( t \in [0, T + \epsilon] \) and that (6.3) holds. For any \( t \in (0, T] \), since

\[
\left\| y_t, \eta \right\|_{Y, Y'} = \lim_{s \uparrow t} \left\| \frac{\phi(s)}{2}, \eta \right\|_{Y, Y'} \leq \| \phi \|_{L^\infty([0, T+\epsilon]; X)} \| \eta \|_{X'}, \quad \forall \eta \in Y',
\]

\( y_t \in X \), for \( t \in (0, T + \epsilon] \), and thus (6.4) holds.

Finally let us prove (6.1), (6.2) hold for \( \eta \) in \( X' \). Since \( Y' \) is dense in \( X' \), there exists, for each \( \delta > 0 \), some \( \eta_\delta \in Y' \) such that

\[
\| \eta - \eta_\delta \|_{X'} \leq \delta,
\]

then for \( t > t_0 \),

\[
\left\langle \phi(t), \phi(t_0), \eta \right\rangle_{X, X'} = \left\langle \phi(t) - \phi(t_0), \eta - \eta_\delta \right\rangle_{X, X'} + \left\langle \phi(t) - \phi(t_0), \eta_\delta \right\rangle_{X, X'}.
\]
we have
\[ \langle \phi(t), \phi(t_0), \eta \rangle_{X,X'} \leq 4\delta \| \phi \|_{L^\infty([0,T+\epsilon];X)} + \| \phi(t) - \phi(t_0), \eta \delta \rangle_{X,X'} \].

Since \( \eta \delta \in Y' \), the right continuity assumption implies that
\[ \langle \phi(t) - \phi(t_0), \eta \delta \rangle_{X,X'} \to 0 \quad \text{as} \quad t \downarrow t_0 \]

and hence
\[
\lim_{t \downarrow t_0} \| \phi(t) - \phi(t_0), \eta \delta \rangle_{X,X'} \leq 4\delta \| \phi \|_{L^\infty([0,T+\epsilon];X)}.
\]

Since \( \delta > 0 \) is arbitrarily small, (6.1) is proved. (6.2) can be proved similarly.

\[ \square \]

Proof of Lemma 6.2. Firstly, we prove the equality. By the resonance theorem and the covering theorem, \( D([0,\infty); \mathcal{H}_\sigma) \cap \Omega \) is in \( L^\infty_{loc}([0,\infty); \mathcal{H}) \cap \Omega \). The other inclusion follows from Lemma 6.4.

Secondly, we prove measurability. Notice that the map
\[ f^n_t(\omega) = \| \omega(t) \|_{\mathcal{H}_\sigma} \quad \Omega \to [0,\infty) \]
is continuous for each \( n \). Let \( \{h_i\}_{i \in \mathbb{N}} \) be a complete orthonormal system of \( \mathcal{H} \), \( \mathcal{H}_n = \text{span}\{h_i, i = 1, \ldots, n\} \). For each \( t \geq 0 \), by Lemma 6.1
\[
\{ \omega \in \Omega: \| \omega(t) \|_{\mathcal{H}} \leq R \} = \bigcap_{n \geq 1} \{ f^n_t(\omega)(t) \leq R \} \in \mathcal{B}. \tag{6.5}
\]

Let \( D \subset [0,\infty) \) be a continuous dense set. It’s easy to prove that
\[
L^\infty_{loc}([0,\infty); \mathcal{H}) \cap \Omega = \bigcap_{T=1}^\infty \bigcup_{R=1}^\infty \bigcap_{t \in D\cap[0,T]} \{ \omega \in \Omega: \| \omega(t) \|_{\mathcal{H}} \leq R \}.
\]

By (6.5), \( L^\infty_{loc}([0,\infty); \mathcal{H}) \cap \Omega \) is a Borel subset in \( \mathcal{B} \).

Similarly, for each \( n, T \geq 0 \),
\[
g^n_T(\omega) = \int_0^T |\omega(s)|^2_{\mathcal{V}_n} ds
\]
is continuous in \( \mathcal{V} \), and
\[
\left\{ \omega \in \Omega \mid \int_0^T |\omega(s)|^2_{\mathcal{V}} ds \leq R \right\} = \bigcup_{n \geq 1} \left\{ \omega \in \Omega \mid g^n_T(\omega) \leq R \right\} \in \mathcal{B},
\]

hence,
\[
L^2_{loc}([0,\infty); \mathcal{V}) \cap \Omega = \bigcap_{T=1}^\infty \bigcup_{R=1}^\infty \left\{ \omega \in \Omega \mid \int_0^T |\omega(s)|^2_{\mathcal{V}} ds \leq R \right\}.
\]

which implies that \( L^2_{loc}([0,\infty); \mathcal{V}) \) is a Borel subset in \( \mathcal{B} \). \[ \square \]
Proof. The uniqueness is obvious. Let $\delta_\eta$ be the Dirac measure on $\Omega_s$ at $\eta$, i.e. $\delta_\eta((\alpha \in \Omega_s): \alpha(t) = \eta(t), 0 \leq t \leq s)) = 1$ and $\Phi : \Omega_s \to \Omega^2$ be the map defined by $\Phi(\Omega_s(t)) = \omega(t)$, $t \geq s$. By Lemma 6.1, $\Phi$ is measurable on $(\Omega, B^2)$, and therefore $P \circ \Phi^{-1}$ is well defined. Define $\tilde{P} = \delta_\eta \times (P \circ \Phi^{-1})$ on $\tilde{X} \equiv \Omega_s \times \Omega_s$. Set $X = \{(\alpha, \beta) \in \tilde{X}: \alpha(s) = \beta(s)\}$, $X$ is a Borel subset of $\tilde{X}$ (denote $f(\alpha, \beta) = \alpha(s) - \beta(s): \Omega_s \times \Omega_s \to \mathbb{V}'$, it's easy to see that $X$ is a Borel subset of $\tilde{X}$). Note that $\tilde{P}(X) \geq \delta_\eta((\alpha \in \Omega_s): \alpha(s) = \eta(s))) = 1$. Thus $\tilde{P}$ can be restricted to $X$. Define $\psi : X \to \Omega$ as

$$
\psi((\alpha, \beta))(t) = \begin{cases}
\alpha(t), & 0 \leq t < s, \\
\beta(t), & t \geq s.
\end{cases}
$$

It is a continuous map from $X$ to $\Omega$, and the restriction of $\tilde{P}$ to $X$ determines, via $\psi$, a probability measure on $(\Omega, B)$. This is the desired measure $\delta_\eta \otimes_s P$. \qed

Remark 6.1. Note that if $x, y \in \Omega_s$, $d(x, y) = 0$ if and only if $x(t) = y(t)$ for every $t \geq 0$; if $x, y \in \Omega_s$, $d(x, y) = 0$, if and only if $x(t) = y(t)$ for every $0 \leq t \leq s$.

Lemma 6.6. Fixed $t \geq 0$, suppose that $\omega \to Q_\omega$ is a mapping of $\Omega$ into probability measures on $$(\Omega, B)$$ satisfies
Given a probability measure $P$ on $(\Omega, \mathcal{B})$, there exists a unique probability measure $P \otimes_t Q_\cdot$ on $(\Omega, \mathcal{B})$ such that $P \otimes_t Q_\cdot$ equals $P$ on $(\Omega, \mathcal{B}_t)$ and $[\delta_\omega \otimes_t Q_\omega]$ is a RCPD of $P \otimes_t Q_\cdot |\mathcal{B}_t$.

**Proof.** The uniqueness is obvious. We prove the existence of $P \otimes_t Q_\cdot$.

Let $N = \{x \in \Omega: \xi(s_1) \in A_1, \ldots, \xi(s_n) \in A_n\}$, where $n \geq 1$, $0 \leq s_1 < \cdots < s_n$, and $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{V}')$, then

\[
\delta_\omega \otimes_t Q_\omega(N) = \chi_0,t(\cdot)Q_\omega(N) + \sum_{k=1}^{n-1} \chi_{s_k,s_{k+1}}(\xi(s_1)) \cdots \chi_{s_k,s_{k+1}}(\xi(s_k)) \times Q_\omega(\xi(s_{k+1}) \in \Gamma_{k+1}, \ldots, \xi(s_n) \in A_n) + \chi_{s_n,\infty}(\xi(s_1)) \cdots \chi_{s_n,\infty}(\xi(s_n)).
\]

It is clear that $\omega \mapsto \delta_\omega \otimes_t Q_\omega(N)$ is $\mathcal{B}_t$-measurable. By Lemma 6.1 and the monotone class theorem, the map $\omega \mapsto \delta_\omega \otimes_t Q_\omega(N)$ is $\mathcal{B}_t$-measurable for all $N \in \mathcal{B}$.

Set

\[G(N) = E^P[\delta_\omega \otimes_t Q_\cdot(N)], \quad N \in \mathcal{B}.
\]

It is easy to prove that $G$ has the desired properties of $P \otimes_t Q_\cdot$. \qed

6.2. The Markov property and existence of Markov selections

We first extended some concepts and Theorems in [7] to the space $D([0, \infty); \mathcal{H}_\sigma)$. Since the proving is almost the same as in [7], we only state them without proving.

Given a family $(P_x)_{x \in \mathcal{H}}$ of probability measures, the Markov property can be stated as

\[P_x^{(\omega)}|\mathcal{B}_t = P_{\omega(t)}, \quad \text{for } P_x-\text{a.s. } \omega \in \Omega
\]

for each $x \in \mathcal{H}$ and $t \geq 0$.

**Definition 6.2** (Almost sure Markov property). Let $x \mapsto P_x$ be a measurable map from $\mathcal{H}$ to $Pr(\Omega)$ such that

\[P_x[D([0, \infty); \mathcal{H}_\sigma) \cap \Omega] = 1 \quad \text{for all } x \in \mathcal{H}.
\]

The family $(P_x)_{x \in \mathcal{H}}$ has the almost sure Markov property if for each $x \in \mathcal{H}$ there is a set $T \subset [0, \infty)$ with null Lebesgue measure, such that

\[P_x^{(\omega)}|\mathcal{B}_t = P_{\omega(t)}, \quad \text{for } t \notin T, \omega \in \Omega, \text{ } P_x-\text{a.s.}
\]

Denote by $\text{Comp}(Pr(\Omega))$ the family of all compact subsets of $Pr(\Omega)$.

**Definition 6.3** (Almost sure pre-Markov family). Consider a measurable map $C: \mathcal{H} \rightarrow \text{Comp}(Pr(\Omega))$ such that

\[P_x[D([0, \infty); \mathcal{H}_\sigma) \cap \Omega] = 1 \quad \text{for all } x \in \mathcal{H} \text{ and } P \in C(x).
\]

The family $(C(x))_{x \in \mathcal{H}}$ is almost surely pre-Markov if for each $x \in \mathcal{H}$ and $P \in C(x)$, there is a set $T \subset [0, \infty)$ with null Lebesgue measure, such that for all $t \notin T$, the following properties hold:
1. (Disintegration) there exists $N \in B_\mathcal{H}$ with $P(N) = 0$ such that for all $\omega \not\in N$,

$$\omega \in \mathcal{H} \quad \text{and} \quad P\vert_{\mathcal{H}_\omega}^{(\omega)} \in \Phi_\mathcal{T}(\omega(t));$$

2. (Reconstruction) for each $B_\mathcal{H}$-measurable map $\omega \to Q_\omega : \Omega \to \text{Pr}(\Omega^1)$ such that there is $N \in B_\mathcal{T}$ with $P(N) = 0$ and for all $\omega \not\in N$,

$$\omega(t) \in \mathcal{H} \quad \text{and} \quad Q_\omega \in \Phi_\mathcal{T}(\omega(t));$$

then $P \otimes_\mathcal{T} Q \in \text{C}(\omega)$.

**Remark 6.2.** If every $C(x)$ is a singleton, the a.s. pre-Markov family is indeed an a.s. Markov family of probability measures, as stated in Definition 6.2.

For each $f \in C_b(V', R)$, $\lambda > 0$, $P \in \text{Pr}(\Omega)$, $x \in \mathcal{H}$, let

$$J_{\lambda,f}(P) = E^P \left[ \int_0^{\infty} e^{-\lambda t} f(\xi(t)) \, dt \right],$$

$$R^+_\lambda f(x) = \sup_{P \in \text{C}(x)} J_{\lambda,f}(P),$$

$$C_{\lambda,f}(x) = \left\{ P \in \text{C}(x) \mid J_{\lambda,f}(P) = R^+_\lambda f(x) \right\}.$$

**Lemma 6.7.** (See [7].) Let $(C(x))_{x \in \mathcal{H}}$ be an a.s. pre-Markov family with non-empty convex values, $\lambda > 0$ and $f \in C_b(V', R)$. Then $R^+_\lambda f(x)$ is well defined and $(C_{\lambda,f}(x))_{x \in \mathcal{H}}$ is again an a.s. pre-Markov family with non-empty convex values.

**Theorem 6.1.** (See [7].) Let $(C(x))_{x \in \mathcal{H}}$ be an a.s. pre-Markov family with non-empty convex values. Then there is a measurable map $x \to P_x$ on $\mathcal{H}$ with values in $\text{Pr}(\Omega)$ such that $P_x \in C(x)$ for all $x \in \mathcal{H}$ and $(P_x)_{x \in \mathcal{H}}$ has the a.s. Markov property.

If in Definition 6.3 each set $T$ of exceptional times is empty, call it a pre-Markov family.

**Theorem 6.2.** (See [7].) Let $(C(x))_{x \in \mathcal{H}}$ be a pre-Markov family with non-empty convex values. Then there is a measurable map $x \to P_x$ from $\mathcal{H}$ to $\text{Pr}(\Omega)$ such that $P_x \in C(x)$ for all $x \in \mathcal{H}$ and $(P_x)_{x \in \mathcal{H}}$ has the Markov property.

### 7. Markov selection for the Navier–Stokes equations

The Markov selection for the 3D Navier–Stokes equations with Wiener process has been considered in [5,7,8] recently. Using their argument, we extended the results to the Lévy Noise.

Let $T = [0, 1]^3$ be the 3D torus and let $\mathcal{D}^\infty$ be the space of infinitely differentiable divergence-free periodic vector fields on $R^3$ with zero mean, $H$ be the closure of $\mathcal{D}^\infty$ in the norm of $L^2(T, R^3)$, and $D(A) = \{ u \in H \mid \Delta u \in H \}$. Let $A : D(A) \to H$ be the Stokes operator

$$Au = -\Delta u, \quad u \in D(A).$$

It is a positive linear self-adjoint operator on $H$ and we can define the powers $A^\alpha$, $\alpha \in R$, with domain $D(A^\alpha)$. By proper identifications of dual spaces, $V \subset H \subset V' \subset D(A)'$. The bi-linear operator $B : V \times V \to V'$ is defined as
\[ B(u, v) = \mathbb{P}_{dix}(u \cdot \Delta)v \]

where \( \mathbb{P}_{dix} \) is the projection onto divergence-free vector fields.

Let \((e_i)_{i \in \mathbb{N}}\) be a complete orthonormal system of eigenvectors and denote \( \sigma^2 = \sum_{i=1}^{\infty} \sigma_i^2, \sigma_i \in \mathbb{R} \).

We shall consider the 3D Navier–Stokes equations in its abstract form

\[
du + \left( vAu + B(u, u) \right) dt = \sum_{i} \sigma_i e_i dL_i(t) \quad (7.1)
\]

where \( \{L_i(t) = \int_0^{t+} \int_{|x| \leq 1} x \tilde{N}_i(ds, dx)\}_{i \in \mathbb{N}} \) are independent Lévy processes on a complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\), \( \tilde{N}_i(dt, dx) = N_i(dt, dx) - dt \lambda(dx) \), \( \{N_i(dt, dx)\}_{i \in \mathbb{N}} \) are independent Poisson random measure with the characteristic measure \( \lambda(dx) \) satisfying: \( \int_{|x| \leq 1} x^2 \lambda(dx) < \infty \).

7.1. The solutions to the martingale problem

In view of the results of previous, we consider the particular case where \( \mathcal{V} = V, \mathcal{H} = H \) and \( \mathcal{V}' = D(A)' \). We set

\[ \Omega_{NS} = D([0, \infty); D(A)' \). \]

This space will play the role of state space for the solutions to (7.1). By Lemma 6.1, we denote by \( B_{NS} \) the \( \sigma \)-field of Borel sets of \( \Omega_{NS} \), and, for each \( t \geq 0 \), by \( B_{t, NS} = \sigma(\omega_{|[0,t]}: \omega \in \Omega_{NS}) \) and \( B_{\infty, NS} = \sigma(\omega_{|[t,\infty]}: \omega \in \Omega_{NS}) \) the \( \sigma \)-fields of past and future, with respect to time \( t \), events. For Markov selection, we define the solutions to the martingale problem (7.1) as follows

**Definition 7.1.** Given \( x \in H \), a probability \( P_x \) on \((\Omega_{NS}, B_{NS})\) is a solution starting at \( x \) to the martingale problem associated to the Navier–Stokes equation (7.1) if

**[MP1]** \( P_x([L_{loc}^\infty([0, \infty); H] \cap L_{loc}^2([0, \infty); V]) = 1. \)

**[MP2]** For each \( \{e_i\}_{i \in \mathbb{N}} \) the process \( M^{e_i}_t \), defined \( P_x \)-a.s. on \((\Omega_{NS}, B_{NS})\) as

\[
M^{e_i}_t = (\xi_t - \xi_0, e_i) + \int_0^t \langle \xi_s, Ae_i \rangle ds - \int_0^t \langle B(\xi_s, e_i), \xi_s \rangle ds
\]

is square integrable. Furthermore, \( \{M^{e_i}_t\}_{i \in \mathbb{N}} \) are independent Lévy process on \((\Omega_{NS}, B_{NS}, (B_{t, NS})_{t \geq 0}, P_x)\), and the characteristic function is

\[
E^{P_x}\left[ e^{iu(\cdot) - M^{e_i}_t} \right] = \exp \left( \int_{|u| \leq 1} (e^{iu} - 1 - iu\lambda(dx)) \right)
\]

for \( t_2 \geq t_1 \geq 0. \)

**[MP3]** \( \delta_s \) is the marginal of \( P_x \) at time \( t = 0. \)

**[MP4]** There exists a constant \( C > 0 \) and a Lebesgue null set \( T_{P_x} \subset (0, \infty) \) such that for all \( 0 \leq s \notin T_{P_x} \) and all \( t \geq s, \)

\[
E^{P_x}\left( \sup_{r \in [s,t]} |\xi(r)|_H^2 + \int_s^t \|\xi(r)\|_V^2 dr \right) \leq C \left[ |\xi(s)|_H^2 + \sigma^2 \int_{|x| \leq 1} x^2 \lambda(dx)(t-s) \right], \quad P_x \text{-a.s.}
\]
Theorem 7.1. Assume $\sigma^2 < \infty$. For each $x \in H$, there exists at least one martingale solution $P_x \in \mathcal{P}(\Omega_{NS})$ to Eq. (7.1) in the sense of Definition 7.1; and there exists a Markov selection $(P_x)_{x \in H}$.

To prove Theorem 7.1, we need the following three lemmas about regular probabilities. The continuous version refer to [8] and the proof is similar to [8]. The symbol used in the following three lemmas refer to Section 6.

Lemma 7.1. (See [8].) Let $P$ be a probability measure on $(\Omega, \mathcal{B})$, and for $s \geq 0$, denote $Q_s := P(. \mid \mathcal{B}_s)$ be an RCPD of $P$ with respect to $\mathcal{B}_s$. Then for any $\xi \in L^1(\Omega, \mathcal{B}, P)$ and $t \geq s$ there exists a $P$-null set $\Lambda_{t, s}$, satisfying for all $x \in \Lambda_{t, s}$,

$$
E^P(\xi \mid \mathcal{B}_t) = E^{Q_s}(\xi \mid \mathcal{B}_t) = E^{Q_s}(\xi \mid \mathcal{B}_s), \quad Q_s-a.s.
$$

Lemma 7.2. (See [8].) Let $D := \{(t, s) : 0 \leq s \leq t < \infty\}$, let $\xi, \eta : D \to \mathbb{R}_+$ be two measurable processes on $(\Omega, \mathcal{B})$. Given $P \in \mathcal{P}(\Omega)$ and $r \geq 0$, suppose that

(i) for each $s \geq 0$, the map $t \to \xi(t, s)$ is a.s. increasing, and $t \to \eta(t, s)$ is a.s. right continuous, $\eta(t, s)$ is $\mathcal{B}_s$-measurable for any $t \geq s$;

(ii) for each $(t, s) \in D$,

$$
\xi(t, s), \eta(t, s) \in L^1(\Omega, \mathcal{B}, P)
$$

and

$$
\xi(t, .), \eta(t, .) \in L^1(0, t; L^1(\Omega, \mathcal{B}, P));
$$

(iii) for any $x \in \Omega$, and $t \geq s \geq r$,

$$
\xi(t, s, \Phi_r x) = \xi(t - r, s - r, x)
$$

and

$$
\eta(t, s, \Phi_r x) = \eta(t - r, s - r, x).
$$

Then the following three statements are equivalent:

(1) There is a Lebesgue null set $T_r \subset (r, \infty)$ such that for any $r \leq s \notin T_r$ and $t \geq s$,

$$
E^P(\xi(t, s) \mid \mathcal{B}_s) \leq \eta(t, s), \quad P-a.s.
$$

(2) For some $P$-null set $N \in \mathcal{B}_r$ and each $x \in N^c$, there is a Lebesgue null set $T_{r, x} \subset (r, \infty)$ such that for any $r \leq s \notin T_{r, x}$ and any $t \geq s$,

$$
E^{Q_s}(\xi(t, s) \mid \mathcal{B}_s) \leq \eta(t, s), \quad Q_s-a.s.
$$

(3) For some $P$-null set $N \in \mathcal{B}_r$ and each $x \in N^c$, there is a Lebesgue null set $T_{r, x} \subset (0, \infty)$ such that for any $0 \leq s \notin T_{r, x}$ and any $t \geq s$,

$$
E^{Q_s \circ \Phi_r}(\xi(t, s) \mid \mathcal{B}_s) \leq \eta(t, s), \quad Q_s \circ \Phi_r-a.s.
$$

Moreover $T_r = \emptyset \iff T_{r, x} = \emptyset.$
Lemma 7.3. (See [8]) Let \((M(t))_{t \geq 0}\) and \((K(t))_{t \geq 0}\) be \(\mathcal{B}_t\)-adapted real valued process on \((\Omega, \mathcal{B})\) which satisfy for \(x \in \Omega, t \geq r \geq 0\),
\[
M(t, \Phi_r x) = M(t - r, x), \quad K(t, \Phi_r x) = K(t - r, x).
\]
(7.2)

Given \(P \in \text{Pr}(\Omega)\) and \(r \geq 0\), assume that for each \(t \geq 0\), \(E^P(K(t)) < \infty\). Then the following statements are equivalent:

1. \((M_t, \mathcal{B}_t, P)_{t \geq r}\) is a càdlàg martingale with square variation process \((K(t))_{t \geq r}\).
2. There exists a \(P\)-null set \(N \in \mathcal{B}_r\) such that for all \(x \notin N\), \((M_t, \mathcal{B}_t, Q^r_x)_{t \geq 0}\) is a càdlàg martingale with square variation process \((K(t))_{t \geq r}\) and \(E^P[E^Q_r[K(t)]] < \infty\).
3. There exists a \(P\)-null set \(N \in \mathcal{B}_r\) such that for all \(x \notin N\), \((M_t, \mathcal{B}_t, Q^r_x \circ \Phi_r)_{t \geq 0}\) is a càdlàg martingale with square variation process \((K(t))_{t \geq r}\).

Proof. (1) \(\Rightarrow\) (2) First, we prove that if \(r \leq t_1 \leq t_2\), then there is a \(P\)-null set \(N_{t_1,t_2} \in \mathcal{B}_r\), such that for all \(x \notin N_{t_1,t_2}\),
\[
E^Q_r[M_{t_2} \mid \mathcal{B}_{t_1}] = M_{t_1}, \quad Q^r_x\text{-a.s.}
\]
(7.3)

Indeed, let \(A \in \mathcal{B}_{t_1}\), then for each \(B \in \mathcal{B}_r\), we have that \(A \cap B \in \mathcal{B}_{t_1}\) and
\[
E^P[I_B \cdot E^Q_r[M_{t_2} I_A]] = E^P[M_{t_2} I_{A \cap B}]
\]
\[
= E^P[M_{t_1} I_{A \cap B}]
\]
\[
= E^P[I_B \cdot E^Q_r[M_{t_2} I_A]]
\]
so that \(E^Q_r[M_{t_2} I_A] = E^Q_r[M_{t_1} I_A]\) out of a \(P\)-null set in \(\mathcal{B}_r\). Since \(\mathcal{B}_r\) is countably generated, the \(P\)-null set can be chosen independently of \(A\).

Next, let \(D\) be a dense set in \([r, \infty)\), then by the previous argument we can find a \(P\)-null set \(N \in \mathcal{B}_r\) such that \((7.3)\) is true for \(x \notin N\) and \(t_1, t_2 \in D\). By Lemma 1.29 in [15], \((7.3)\) is true for all \(t \geq r\).

One can proceed similarly to prove that \(M_t\) is \(Q^r_x\)-square integrable with quadratic variation \((K_t)_{t \geq r}\), since \(M^2_t\) is a sub-martingale and \(M^2_t - K_t\) is a martingale. Finally, \(E^P[E^Q_r[K_t]] = E^P[K_t]\).

(2) \(\Rightarrow\) (1) Since \(x \rightarrow Q^r_x [K_t]\) is \(P\)-integrable, \(M_t\) is \(P\)-square integrable and it is easy to see that \(M_t\) is a martingale with quadratic variation \(K_t\).

(2) \(\Leftrightarrow\) (3) is direct from (7.2). Indeed, for any \(A \in \mathcal{B}_{s-r}\),
\[
E^Q_r \circ \Phi_r (I_A \cdot M_{s-r}) = E^Q_r (I_{\Phi_r A} \cdot M(s - r, \Phi_r^{-1}(.)])
\]
\[
= E^Q_r (I_{\Phi_r A} \cdot M(s))
\]
\[
= E^Q_r (I_{\Phi_r A} \cdot M(t))
\]
\[
= E^Q_r \circ \Phi_r (I_A \cdot M_{t-r}),
\]
this completes the proof. \(\Box\)

As a consequence, we have the following BDG’s inequality.

Corollary 7.1. Let \((M_t, \mathcal{B}_t, P)_{t \geq r}\) be a càdlàg square integrable martingale with \(M_r = 0\), then \(P\)-a.s.
\[
E^P \left( \sup_{s \in [r,t]} |M_s| \mid \mathcal{B}_r \right) \leq CE^P([M^2_t]^{1/2} \mid \mathcal{B}_r).
\]
Proof of Theorem 7.1. The proof will be developed in the following lemmas.

Lemma 7.4. Assume $\sigma^2 < \infty$. For each $x \in H$, there exists at least one martingale solution $P_x \in Pr(\Omega_{NS})$ to Eq. (7.1) in the sense of Definition 7.1.

Proof. Refer to Section 3, we only prove [MP4]. From Theorem 3.3 we have

Proof of Theorem 7.1. For each $x \in H$, there exists at least one martingale solution $P_x \in Pr(\Omega_{NS})$ to Eq. (7.1) in the sense of Definition 7.1.
By Theorem 3.3, Remark 3.1 and Skorokhod embedding theorem, there exists a stochastic basis $(\Omega^t, \mathcal{F}^t, P^t)$ and $\Omega_{NS} \cap L^2_{loc}(\mathbb{R})$-valued random variables $\tilde{x}_n, \tilde{x}$ have the law of $P_n, P$ on $\Omega_{NS} \cap L^2_{loc}(\mathbb{R})$ respectively, and $\tilde{x}_n \to \tilde{x}$ in $\Omega_{NS} \cap L^2_{loc}(\mathbb{R})$, $P^t$-a.s. (by choosing a sub-sequence if necessary). So, for any $B \in \mathcal{F}^0$, thus there exists a Lebesgue null set $\Omega \subseteq (0, \infty)$ such that for all $s \notin T$, $\lim_{n \to \infty} E P^t[|\tilde{x}_n(s) - \tilde{x}(s)|^2_{H}] = 0$. Thus there exists a Lebesgue null set $T \subset (0, \infty)$ such that for all $s \notin T$, $\lim_{n \to \infty} E P^t[|\tilde{x}_n(s) - \tilde{x}(s)|^2_{H}] = 0$.

For any $r \notin T$ and $t \geq r$, we want to prove $P$-a.s.

$$E P^t[y(t, r, \xi) | B^T_r] \leq C \left( \langle \xi(r) \rangle_H^2 + (t - r)^2 \int_{|x| \leq 1} \lambda(x) dx \right)$$

which is equivalent to prove that for any $B_r^T$-measurable and bounded continuous function $g$ on $\Omega_{NS}$,

$$E P^t[y(t, r, \xi) g(\xi)] \leq CE P^t \left[ \left( \langle \xi(r) \rangle_H^2 + (t - r)^2 \int_{|x| \leq 1} \lambda(x) dx \right) g(\xi) \right].$$

By Fatou’s lemma

$$E P^t[y_n(t, r, \xi) g(\xi)] = E P^t[y_n(t, r, \tilde{x}) g(\tilde{x})]$$

$$\leq \liminf_{m \to \infty} E P^t[y_n(t, r, \tilde{x}_m) g(\tilde{x}_m)]$$

$$= \liminf_{m \to \infty} E P^t[y_n(t, r, \xi) g(\xi)]$$

$$\leq C \liminf_{m \to \infty} E P^t \left[ \left( \langle \xi(r) \rangle_H^2 + (t - r)^2 \int_{|x| \leq 1} \lambda(x) dx \right) g(\xi) \right]$$

$$= C \liminf_{m \to \infty} E P^t \left[ \left( \langle \tilde{x}_m(r) \rangle_H^2 + (t - r)^2 \int_{|x| \leq 1} \lambda(x) dx \right) g(\tilde{x}_m) \right]$$

$$= CE P^t \left[ \left( \langle \tilde{x}(r) \rangle_H^2 + (t - r)^2 \int_{|x| \leq 1} \lambda(x) dx \right) g(\tilde{x}) \right]$$

$$= CE P^t \left[ \left( \langle \xi(r) \rangle_H^2 + (t - r)^2 \int_{|x| \leq 1} \lambda(x) dx \right) g(\xi) \right]$$

which means that [MP4] in Definition 7.1 holds for $P$ by taking limitation in $n$. 

Define for each $x \in H$ the subset of $Pr(\Omega_{NS})$ as

$$C_{NS}(x) = \{ P \in Pr(\Omega_{NS}) \mid P \text{ solves the martingale problem (7.1) starting at } \delta_{x} \}. \quad (7.4)$$

**Lemma 7.5.** Given $x \in H$, the set $C_{NS}(x)$ is non-empty, convex and compact, and satisfies

1. for every $P \in C_{NS}(x)$, $P[D((0, \infty); H_\sigma)] = 1$.
2. the map $C_{NS} : H \to \text{Comp}(Pr(\Omega_{NS}))$ is Borel measurable.
Proof. Due to Lemma 7.4, the set $C_{NS}(x)$ is non-empty. And, from [MP1] of Definition 7.1 and Lemma 6.2, it follows that $P[D([0, \infty); H_{\sigma})] = 1$ for every $P \in C_{NS}(x)$. And it is easy to check each $C_{NS}(x)$ is convex.

Refer to [7], compactness and measurability follow from the following claim:

For each sequence $\{x_n\}_{n \in N} \subset H$ and $P_n \in C_{NS}(x_n)$, if $x_n \to x$ in $H$, then there exists $n_k \uparrow \infty$ and $P \in C_{NS}(x)$, such that $P_{n_k} \to P$ with respect to weak convergence in $Pr(\Omega_{NS})$.

In order to prove the claim, let $x_n \to x$ in $H$ and $P_n \in C_{NS}(x_n)$, we first show that $(P_n)_{n \in N}$ is tight on $\Omega_{NS} \cap L^2_{loc}([0, \infty); H)$.

By [MP4] of Definition 7.1, we have that for all $T > 0$,

$$E^P_n \left[ \sup_{t \in [0, T]} |\xi_t|^2_H + \int_0^T \|\xi_s\|^2_Y \, ds \right] \leq C \left( \sigma^2, T, |x_n|^2_H, \int_{|x| \leq 1} x^2 \lambda(dx) \right).$$

Next, let $(\Sigma, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space, $\{u(t)\}_{t \geq 0}$ be a process on $\Sigma$ whose law is $P_n$ and such that $\{u(t)\}_{t \geq 0}$ is a weak martingale solution to (3.1). In particular,

$$u(t) = x_n - \int_0^t (Au(s) + B(u(s), u(s))) \, ds + \sum_{i=0}^{\infty} \sigma_i L_i(t) e_i, \quad P\text{-a.s.}$$

in $D(A')$.

Using the same method in Theorem 3.3 and Remark 3.1, we have that $(P_n)_{n \in N}$ is tight on $\Omega_{NS} \cap L^2_{loc}([0, \infty); H)$, denote $P = \lim_{n \to \infty} P_n$, and we also have $P$ satisfy [MP1], [MP2], [MP3]. By using the method as in Lemma 7.4, [MP4] follows. □

Lemma 7.6. The disintegration property of Definition 6.3 holds for the family $(C_{NS}(x))_{x \in H}$.

Proof. Fix $x_0 \in H$ and $P \in C_{NS}(x_0)$. Let $Q_x^r := P(\cdot \mid B_{NS}^r)(x)$ be an RCPD of $P$ with respect to $\mathcal{B}_r$. We want to show that there is a $P$-null set $N \in \mathcal{B}_r$ such that for all $x \notin N$,

$$Q_x^r \circ \Phi_r \in C_{NS}(x(r)).$$

That is, we need to check $Q_x^r \circ \Phi_r$ satisfies [MP1]-[MP4].

[MP1] Set

$$A_t = \{ \xi \in \Omega_{NS} : \xi|_{[0,t]} \in L^\infty(0, t; H) \cap L^2(0, t; V) \} \in B_{NS}^r,$$

$$A^t = \{ \xi \in \Omega_{NS} : \xi|_{(t, \infty)} \in L^\infty([t, \infty); H) \cap L^2_{loc}([t, \infty); V) \} \in B_{NS}^r.$$

Notice that $P(A_t \cap A^t = 0)$ by property [MP1]. Hence,

$$1 = P[A_t \cap A^t] = \int_{A_t} Q_x^r[A^t] P(dx),$$

and thus there is a $P$-null set $N_1 \in B_{NS}^r$ such that $Q_x^r[S^t] = 1$ for all $x \notin N_1$.

[MP2] By (3) of Lemma 7.3 there exists a $P$-null set $N_2 \in B_{NS}^r$ such that for all $x \notin N_2$, $Q_x^r \circ \Phi_r$ satisfies [MP2].
[MP3] We choose ξ and η in Lemma 7.2 as follows

\[ \xi(t, s) := \sup_{s' \in [s, t]} |x(s')|^2_H + \int_s^t \|x(s')\|^2_{C^2} \, ds', \quad \eta(t, s) = C \left[ |x(s)|^2_H + \sigma^2 \int_{|x| \leq 1} x^2 \lambda(dx)(t - s) \right]. \]

It’s clear that for each \( s \in [0, t] \), \( \eta(t, s) \) is \( B^\text{NS}_s \)-measurable, \( t \to \eta(t, s) \) is continuous, \( t \to \xi(t, s) \) is increasing, and (iii) in Lemma 7.2 holds. The integrability conditions on \( \xi \) and \( \eta \) in Lemma 7.2 follow from [MP4], i.e.

\[ E^P (\xi(t, 0)) \leq C \left[ |x_0|^2_H + \sigma^2 \int_{|x| \leq 1} x^2 \lambda(dx)t \right]. \]

Thus, by (2) of Lemma 7.2, there exists a \( P \)-null set \( N_3 \subset B^\text{NS}_t \) such that for all \( x \notin N_3 \), \( Q^F_{x} \circ \Phi_t \) satisfies [MP4].

Finally, letting \( N := N_1 \cup N_2 \cup N_3 \), we obtain the desired result. \( \square \)

**Lemma 7.7.** The reconstruction property of Definition 6.3 holds for the family \( (C^\text{NS}(x))_{x \in H} \).

**Proof.** Fix \( x_0 \in H \), and \( P \in C^\text{NS}(x_0) \), let \( Q^F_{x} \in Pr(\Omega^F) \) satisfying the assumptions in Definition 6.3. Our aim is to show \( P \otimes_r Q^F \in C^\text{NS}(x_0) \).

[MP1] \( P \otimes_r Q^F [A_t] \subset C^\text{NS}(x_0) \), let \( Q^F_{x} = \int_{\Omega^F} Q^F_{x} [A_t] P(dx) = P(A_t) = 1 \), since \( Q^F_{x} [A_t] = 1 \) holds due to [MP1] for \( Q^F_{x} \in \Phi_{r} C^\text{NS}(x(t)) \).

[MP3] Since \( P \) agrees with \( P \otimes_r Q^F \) on \( B^\text{NS}_t \), \( P \otimes_r Q^F (y : y(0) = x_0) = 1 \).

[MP2] and [MP4] can be obtained directly from Lemmas 7.2 and 7.3 and the fact that \( P \) agrees with \( P \otimes_r Q^F \) on \( B^\text{NS}_t \). \( \square \)

So Theorem 7.1 is proved. \( \square \)

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**References**