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Closed Range and Relative Regularity for Products

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Let B be a closed linear transformation of the Banach space X into the Banach space Y and let A be a bounded linear transformation of Y into the Banach space Z. A simple condition is shown to be necessary and sufficient for AB to have closed range. Provided B is relatively regular there is a simple necessary and sufficient condition for AB to be relatively regular. Provided B^+ and A^+ are pseudoinverses for B and A, respectively, the condition that B^+A^+ is a pseudoinverse for AB is completely characterized.

In this note we consider the problem of when the product of two relatively regular linear transformations is relatively regular. The essence of this problem is the determination of necessary and sufficient conditions for the product of two linear transformations with closed range to have closed range. The results of this note generalize and refine the results of [1, 2], although the techniques used in this note are markedly different from the earlier methods. A good reference for the terms that we do not define is [4].

We denote the kernel or nullspace of a linear transformation T by ker T. We denote the minimum modulus of T by $\gamma(T)$ and we define it to be $\inf\{\| Tx \|/\text{dist}(x, \ker T)\colon x \text{ in domain of } T\}$, where 0/0 is defined to be positive infinity. The transformation T has closed range if and only if $\gamma(T)$ is positive (for example, see [4, p. 98]). Essential to our work is the notion of angular distance between subspaces which seems to originate in [6, p. 7]. The angular distance between two nontrivial subspaces, say M and N, is $\gamma[M, N] = \inf\{\| x - y \|\colon x \in M, y \in N, \| x \| = 1 = \| y \|\}$ and if either M or N is trivial then $\gamma[M, N]$ is defined to be one.

Throughout this note X, Y, and Z denote Banach spaces, $\mathscr{L}(Y, Z)$ denotes the bounded linear transformations of Y into Z, and $\mathscr{C}(X, Y)$ denotes the closed linear transformations of X into Y.

LEMMA. *Let $A \in \mathscr{L}(Y, Z)$, $B \in \mathscr{C}(X, Y)$ have closed ranges; that is, AY, BX are closed. Provided* $\ker A \cap BX = \{0\}$ *the range of AB is closed if and only if* $\gamma[\ker A, BX]$ *is positive.*

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Proof. Assuming $\gamma[\ker A, BX] = 0$ we take two sequences of unit vectors $\{f_n\} \subset \ker A$ and $\{Bg_n\}$ such that $\lim \| f_n - Bg_n \| = 0$. Since

$$\| ABg_n \| = \| ABg_n - Af_n \| \leqslant \| A \| \| Bg_n - f_n \|,$$

it follows that the range of $A \mid BX$ is not closed.

It is trivial that $\ker AB = \ker B$. If τ is the standard projection of Y onto $Y/\ker A$ then A induces a bounded transformation with bounded inverse between τY and Z. Thus, $ABX = A\tau BX$ is closed if and only if τBX is closed. In particular, if ABX is not closed then there are sequences $\{w_k\} \subset X$, $\{v_k\} \subset \ker A$ such that $\{Bw_k\}$ consists of unit vectors and $\lim(Bw_k - v_k) = 0$. Clearly it follows that $\lim \| v_k \| = 1$ and for $x_k = v_k / \| v_k \|$ we get $\lim \| Bw_k - x_k \| = 0$. Hence $\gamma[\ker A, BX] = 0$.

THEOREM. *Let $A \in \mathscr{L}(Y, Z)$, $B \in \mathscr{C}(X, Y)$ have closed ranges and let Y_0 denote $\ker A \cap BX$. The range of AB is closed if and only if $\gamma[\ker A/Y_0, BX/Y_0]$ is positive.*

Proof. Let τ_1, τ_2, τ_3 be the standard projections $\tau_1 \colon Y \to Y/Y_0$, $\tau_2 \colon Y \to Y/\ker A$, $\tau_3 \colon Y/Y_0 \to (Y/Y_0)/(\ker A/Y_0)$. Note that $\ker \tau_3 = (\ker A)/Y_0$ and $\tau_1 BX = BX/Y_0$. The previous lemma implies that $\tau_3\tau_1(BX)$ is closed if and only if

$$\gamma[\ker A/Y_0, BX/Y_0] > 0.$$

Since the linear transformation that A induces on $\tau_2 Y$ has a bounded inverse, it follows that $ABX = A(\tau_2 BX)$ is closed if and only if $\tau_2 BX$ is closed. Thus it suffices to prove that $\tau_2 BX$ is closed if and only if $\tau_3\tau_1(BX)$ is closed.

The proof is concluded by showing that $\tau_3\tau_1(BX)$ is isometrically isomorphic to $\tau_2(BX)$. This follows by elementary algebraic arguments.

The previous theorem gives a very general device for constructing bounded operators B such that BX is closed and B^2X is not closed. Thus, it provides a a very general answer to [8, Question (3)]. Let X_1 and X_2 be any pair of subspaces of X such that

$$X_1 \oplus X_2 \neq (X_1 + X_2)^- = X.$$

Define B to be zero on X_1 and to be the identity on X_2.

It should be remarked that the asymmetric hypothesis on A and B cannot be improved by letting $A \in \mathscr{C}(Y, Z)$. This assertion is substantiated by the following example. Let each of the spaces X, Y, Z be $l^2 \oplus l^2$; let a denote the sequence with jth entry equal to a_j and let 0 denote the sequence of zeros. The domain of A, denoted $D(A)$, is defined to be $\{(a, c) \colon \sum k^2 \mid a_k \mid^2 < \infty\}$ and $A(a, c) = (b, 0)$ where $b_j = ja_j$. Note that A is closed since its restriction to $l^2 \oplus \{0\}$ is the inverse of a bounded everywhere defined linear operator; the range of A is closed since the minimum modulus of A restricted to $l^2 \oplus \{0\}$ is one. Define B by the equation $B(a, b) = (c, d)$ where $c_j = a_j/j$ and $d_j = a_j(j^2 - 1)^{1/2}/j$ and note that the restriction of B to $l^2 \oplus \{0\}$ is isometric. Clearly ABX is closed and it is easy to see that $\ker A \cap BX = \{(0, 0)\}$. If e_j denotes the sequence with all entries equal to zero except for the jth entry which is one then $(0, e_j) \in \ker A$. Since BX contains $(e_j/j, e_j(j_2 - 1)^{1/2}/j)$ it follows that $\gamma[\ker A, BX] = 0$. Thus the theorem cannot be extended.

Our theorem implies the following useful corollary proved by Goldberg [4].

COROLLARY. *Let $A \in \mathscr{L}(Y, Z)$, $B \in \mathscr{C}(X, Y)$ have closed ranges. If* dim ker A *is finite then ABX is closed.*

Proof. Because of the compactness of the unit ball in $\ker A/Y_0$, the equation $\gamma[\ker A/Y_0, BX/Y_0] = 0$ implies $(\ker A/Y_0) \cap (BX/Y_0) \neq \{0\}$, which is impossible.

The next corollary has theoretical and computational advantages over the previous theorem since the abstract quotient spaces of that theorem are replaced by simpler considerations.

COROLLARY. *Let $A \in \mathscr{L}(Y, Z)$, $B \in \mathscr{C}(X, Y)$ have closed ranges. Assume $Y_0 = \ker A \cap BX$ has complementary subspaces in* ker A *and* BX, *say*

$$\ker A = Y_1 \oplus Y_0, \qquad BX = Y_2 \oplus Y_0 .$$

Then the following three conditions are equivalent:

(i) *The range of AB is closed,*

(ii) *$\gamma[Y_1, BX]$ is positive,*

(iii) *$\gamma[\ker A, Y_2]$ is positive.*

Proof. Assume $\gamma[Y_1, BX] > 0$ and note that this implies that $\gamma[Y_1, Y_2] > 0$. By [6, Proposition 11D, p. 8], both $Y_1 + Y_2$ and $Y_1 + BX$ are closed. Let τ be the standard map of $Y_1 + BX$ onto $Y_1 + BX/Y_0$, noting that τ is an open map by the open mapping theorem. It is routine to see that τ restricted to $Y_1 + Y_2$ is one-to-one and onto. Consequently there is a positive constant δ such that

$$\delta \| h \| \leqslant \| \tau h \| \tag{*}$$

for any $h \in Y_1 + Y_2$. Clearly we have $\tau Y_1 = \tau \ker A$ and $\tau Y_2 = \tau BX$. Take unit vectors v and w from $\tau \ker A$ and τBX, respectively, and choose f and g from Y_1 and BX, respectively, such that $\tau f = v$ and $\tau g = w$. By (*) above and [6, 11A, p. 7], we get

$$(\delta/2)\, \gamma[f, g] \max\{\|f\|, \|g\|\} \leqslant \delta \|f - g\| \leqslant \|v - w\| .$$

It follows that $\gamma[\ker A/Y_0, BX/Y_0]$ must be positive and so ABX is closed.

Assume that ABX is closed and note that $\gamma[\ker A/Y_0, BX/Y_0]$ is positive. Again let τ be the standard map of $Y_1 + BX$ onto $Y_1 + BX/Y_0$ noting that $\tau Y_1 = \tau \ker A$ and $\tau Y_2 = \tau BX$. Take unit vectors f and g from Y_1 and BX, respectively, and note that [6, 11A, p. 7] implies

$$\tfrac{1}{2}\gamma[\tau f, \tau g] \max\{\|\tau f\|, \|\tau g\|\} \leqslant \|\tau(f-g)\| \leqslant \|f-g\| .$$

This shows that $\gamma[Y_1, BX]$ is positive.

The equivalence of (i) and (iii) is proved similarly.

The preceding corollary gives a strengthened form of the main theorem of [1]. Assuming that Y is a Hilbert space we can replace Y_1 above with $\ker A \cap Y_0{}^{\perp}$ and it follows that $\gamma[\ker A \cap Y_0{}^{\perp}, BX]$ is positive if and only if $\gamma[Y_1, BX]$ is positive whenever Y_1 is a subspace which is complementary to Y_0 in $\ker A$. It is not difficult to show that $\gamma[Y_1, Y_2]$ is positive if and only if the angle between Y_1 and Y_2 is positive (see [1] for definition of angle between subspaces).

A linear transformation T is said to be relatively regular provided it has a bounded pseudoinverse, i.e., there is a bounded operator T^+ such that $TT^+T = T$ (see [3]). See [7] for generalized pseudoinverses.

COROLLARY. *Let $A \in \mathscr{L}(Y, Z)$, $B \in \mathscr{L}(X, Y)$ be relatively regular. The product AB is relatively regular if and only if the following hold:*

(i) *Either* $\gamma[Y_1, BX] > 0$ *where* $Y_1 \oplus \ker A \cap BX = \ker A$, *or* $\gamma[\ker A, Y_2] > 0$ *where* $Y_2 \oplus \ker A \cap BX = BX$,

(ii) *both* $\ker AB$ *and* ABX *have complementary subspaces.*

Proof. This is immediate from the preceding corollary and a trivial generalization of Theorem 1 of [3, p. 9].

Our final corollary gives a strengthened form of a main result in [5]. Part (iii) below for $B \in \mathscr{L}(X, Y)$ and A relatively regular is Koliha's theorem. For a further discussion and some contrasting results see [3, pp. 31–37].

COROLLARY. *Let $A \in \mathscr{L}(Y, Z)$, $B \in \mathscr{L}(X, Y)$ have closed ranges, let P be a projection onto BX, and let Q be a projection with kernel equal to* $\ker A$. *Then*

(i) *ABX is closed if and only if QPY is closed, and*

(ii) *ABX is complemented if and only if QPY is complemented, and*

(iii) *provided B is relatively regular, AB is relatively regular if and only if QP is relatively regular.*

Proof. Clearly we have

$$\ker QP = \ker Q \cap PY = \ker A \cap BX.$$

For $Y_0 = \ker A \cap BX$ we see that $\gamma[\ker Q/Y_0, PY/Y_0]$ equals $\gamma[\ker A/Y_0, BX/Y_0]$ and (i) follows.

The restriction of A to QY is one-to-one and $AQY = AY$. Since A is a linear isomorphism $QPY = QBY$ is complemented in QY if and only if $AQBX = ABX$ is complemented in AY. It follows that QPY is complemented if and only if ABX is complemented.

Let B' denote the restriction of B to X_1 where X_1 is complementary to $\ker B$. Since B' is a linear isomorphism, $\ker A \cap BX$ is complemented in $B'X_1 = BX$ if and only if $B'^{-1}(\ker A \cap BX)$ is complemented in X_1. Because $B'^{-1}(\ker A \cap BX) \oplus \ker B = \ker AB$ this proves that $\ker QP = \ker A \cap BX$ is complemented if and only if $\ker AB$ is complemented.

The next theorem gives a condition which is sufficient for B^+A^+ to be a pseudoinverse of AB; this condition is much weaker than any of the previously known sufficient conditions (see [3, 5]). The sufficient conditions given below cannot be improved since each is also necessary.

Recall that BB^+ is a projection, say P, onto BX along $\ker B^+$ and A^+A is a projection, say Q, onto A^+X along $\ker A$ (for example, see [3, pp. 9–10]). Note that B^+ and BB^+ are bounded and everywhere defined even if B is only a closed operator.

THEOREM. *Let $A \in \mathscr{L}(Y, Z)$, $B \in \mathscr{C}(X, Y)$ have closed range and let $P = BB^+$, $Q = A^+A$. Conditions* (1), (2), (3), *below, are equivalent; each is implied by condition* (4):

(1) *B^+A^+ is a pseudoinverse for AB,*

(2) *$A(PQ - QP)B = 0$,*

(3) *QP is a projection,*

(4) *PX is invariant under Q and QX is invariant under P.*

Proof. Since $A = AA^+A = AQ$ and $B = BB^+B = PB$, condition (1) above is equivalent to

$$ABB^+A^+AB = AB = AQPB$$

or

$$APQB = AQPB \text{ on the domain of } B.$$

The last equation is obviously equivalent to (2).

Equation (2) above is equivalent to

$$A(PQ - QP)BX = \{0\}$$

or

$$(PQ - QP)BX \subset \ker A = \ker Q$$

or

$$Q(PQ - QP)\,BX = \{0\}.$$

Since $BX = PX$ the last equation is equivalent to

$$Q(PQ - QP)\,PX = \{0\}$$

or

$$QPQP - Q^2P^2 = 0.$$

This is equivalent to condition (3).

Condition (4) is equivalent to $PQP = QP$ and $QPQ = PQ$. From the first of these, it follows that

$$QPQP = Q^2P,$$

which proves condition (3).

The best previously known condition for B^+A^+ to be a pseudoinverse of AB was the commutativity of P and Q.

If X is a Hilbert space then condition (4) of the above theorem is necessary for B^+A^+ to be the Moore–Penrose inverse of AB (see [2, Remark 3.2]). However, condition (4) is not necessary for B^+A^+ to be a pseudoinverse of AB. For example, if

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

then A, B and AB are projections and it follows from the preceding theorem that B^+A^+ is a pseudoinverse of AB (note that $A = A^+$, $B = B^+$). Nevertheless, AX is not invariant under $BB^+ = B$.

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