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The ring of quasimodular forms for a cocompact group

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Abstract

We describe the additive structure of the graded ring $\widetilde{M}_*$ of quasimodular forms over any discrete and cocompact group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$. We show that this ring is never finitely generated. We calculate the exact number of new generators in each weight k. This number is constant for k sufficiently large and equals $\dim_{\mathbb{C}}(I/I \cap \widetilde{I}^2)$ where I and $\widetilde{I}$ are the ideals of modular forms and quasimodular forms, respectively, of positive weight. We show that $\widetilde{M}_*$ is contained in some finitely generated ring $\widetilde{R}_*$ of meromorphic quasimodular forms with $\dim \widetilde{R}_k = O(k^2)$, i.e., the same order of growth as $\widetilde{M}_*$.

Keywords: Quasimodular forms; Modular groups

1. Introduction

Kaneko and Zagier introduced the notion of quasimodular forms in [2]. The structure of $\widetilde{M}_*(\Gamma_1)$ (where $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$ is the classical modular group) was given in [2], in which it is proved that $\widetilde{M}_*(\Gamma_1) = \mathbb{C}[E_2, E_4, E_6]$, with E_2, E_4 and E_6 being the Eisenstein series of weights 2, 4 and 6, respectively.

We study the ring of quasimodular forms over discrete and cocompact subgroups of $\mathrm{PSL}(2,\mathbb{R})$. In the second and third sections, we derive some general properties of quasimodular forms over discrete and cofinite subgroups of $\mathrm{PSL}(2,\mathbb{R})$, following [2] and [7]. In the end of the third sec-

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tion, we give an additive structure theorem for rings of quasimodular forms (Theorem 2) and an $\mathrm{sl}_2(\mathbb{C})$-module structure theorem for the ring of quasimodular forms (Theorem 3). In the fourth section, we give a cocompact/non-cocompact dichotomy (Theorem 4) which characterizes cocompact modular groups in terms of their spaces of quasimodular forms of weight 2. In the fifth section, we describe the multiplicative structure of $\widetilde{M}_*(\Gamma)$ in the cocompact case and show that this ring is never finitely generated (Theorem 6 and its corollaries). We refer to [4] for a short description of these structures.

In the sixth section, we give an algebraic characterization of cocompact groups in terms of their rings of modular forms (Theorem 7). This characterization is equivalent to another one, given in terms of canonical Rankin–Cohen rings of modular forms (Theorem 8).

In the seventh section, we prove the existence of quasimodular forms of weight 2 with prescribed poles (Theorem 9). In the eighth section, we give two constructions (Theorems 10 and 11) of finitely generated rings of meromorphic quasimodular forms over a cocompact group Γ which contain, and have the same order of growth as, their infinitely generated subring $\widetilde{M}_*(\Gamma)$. In the last section we illustrate these constructions for a specific quaternionic group Γ_6. We show that the second construction, which uses a combinatorial lemma about finitely generated semigroups of $\mathbb{R}^2$ (Lemma 6), is more precise than the first construction, which uses algebraic geometry of curves and the interpretation of certain meromorphic modular forms as sections of a line bundle over a modular curve.

2. General properties of quasimodular forms

In this section, we recall definitions and general properties of quasimodular forms, as given by Kaneko and Zagier in [2] and by Zagier in a course at the Collège de France [7].

We consider a discrete and cofinite subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$. We give the definition of modular forms, quasimodular forms, almost holomorphic modular forms and modular vectors, over the group Γ. We denote by $\mathcal{H}$ the upper half plane and by y the imaginary part of $z \in \mathcal{H}$.

Definition 1. A modular form of weight k over Γ is a holomorphic map f in $\mathcal{H}$ with moderate growth,[1] such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } z \in \mathcal{H}. \tag{1}$$

Definition 2. A quasimodular form f of weight k and depth $\leqslant p$ over Γ, is a holomorphic function f in $\mathcal{H}$ with moderate growth, such that for any $z \in \mathcal{H}$, the map

$$\begin{aligned} \Gamma &\to \mathbb{C} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \end{aligned}$$

[1] I.e., $|f(z)| \ll ((|z|^2+1)/\Im(z))^n$ for some n.

is a polynomial of degree $\leqslant p$ in $\frac{c}{cz+d}$ with functions defined on $\mathcal{H}$ as coefficients. We can write

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{p} f_j(z)\left(\frac{c}{cz+d}\right)^j, \quad \forall z \in \mathcal{H}, \tag{2}$$

with map $f_j : \mathcal{H} \to \mathbb{C}$ $(j = 0, \ldots, p)$. We say that f is depth equals p, if $f_p(z) \not\equiv 0$.

Remark 1. This definition, which is different from the one given in [2], was proposed by Werner Nahm and presented in [7]. The equivalence between this definition and the one given in [2] is a consequence of Theorem 1.

Definition 3. An almost holomorphic modular form F of weight k and depth $\leqslant p$ over Γ is a polynomial in $1/y$ of degree $\leqslant p$ whose coefficients are holomorphic maps on $\mathcal{H}$ with moderate growth, such that (1) holds (with f replaced by F) for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathcal{H}$. Since $y = (z - \bar{z})/2i$, we can write

$$F(z) = f_0(z) + \frac{f_1(z)}{z - \bar{z}} + \cdots + \frac{f_p(z)}{(z - \bar{z})^p},$$

with holomorphic maps f_i.

This way of writing F as a polynomial in $\frac{1}{z-\bar{z}}$ is more useful for making the next calculations. In [6] one finds the definition of a nearly holomorphic automorphic form.

Definition 4. A modular vector of weight k is a holomorphic map

$$E : \mathcal{H} \to \bigoplus_{j=0}^{\infty} \mathbb{C}$$
$$z \mapsto \big(f_0(z), f_1(z), \ldots\big)$$

such that the maps f_j have moderate growth and satisfy $f_j = 0$ for $j \gg 0$ and the functional equation

$$(cz+d)^{-k+2j} f_j\left(\frac{az+b}{cz+d}\right) = \sum_{l \geqslant j} \binom{l}{j} f_l(z)\left(\frac{c}{cz+d}\right)^{l-j}. \tag{3}$$

If $f_j = 0$ for $j > p$, we say that E has depth $\leqslant p$.

Notation 1. We denote by $M_* = \bigoplus_{k \geqslant 0} M_k$ (respectively $\widetilde{M}_* = \bigoplus_{k \geqslant 0} \widetilde{M}_k$, $\widehat{M}_* = \bigoplus_{k \geqslant 0} \widehat{M}_k$, or $\overrightarrow{M}_* = \bigoplus_{k \geqslant 0} \overrightarrow{M}_k$) the graded rings of modular forms (respectively quasimodular forms, almost holomorphic modular forms or modular vectors). We denote by $\widetilde{M}_*^{(\leqslant p)}$, $\widehat{M}_*^{(\leqslant p)}$, $\overrightarrow{M}_*^{(\leqslant p)}$ the subspaces of quasimodular forms (respectively almost holomorphic modular forms or modular vectors) of depth $\leqslant p$ over a given group Γ.

Theorem 1. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete cofinite subgroup and p a positive integer. We have the isomorphisms*

$$\widetilde{M}_*^{(\leqslant p)} \simeq \overrightarrow{M}_*^{(\leqslant p)} \simeq \widehat{M}_*^{(\leqslant p)}$$
$$f \mapsto (f_0, \ldots, f_p) \mapsto \sum_{j=0}^{p} \frac{f_j(z)}{(z-\bar{z})^j},$$

where the sequence of coefficients (f_j) is associated to f according to (2). *The inverse map from $\overrightarrow{M}_*^{(\leqslant p)}$ to $\widetilde{M}_*^{(\leqslant p)}$ is given by*

$$(f_0, \ldots, f_p) \mapsto f_0 = f.$$

Proof. We omit the details, given in [7]. The quickest way is to associate to a quasimodular form f the function $P_f(z,t) = \sum_{j=0}^{p} f_j(z)t^j$ with f_j defined by (2). The function P_f satisfies the transformation law

$$(cz+d)^{-k} P_f\left(\frac{az+b}{cz+d}, t\right) = P_f\left(z, \frac{t}{(cz+d)^2} + \frac{c}{cz+d}\right), \tag{4}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Writing this equation for each coordinate f_j gives the transformation law (3) for the vector $E = (f_0, f_1, \ldots)$, and applying (4) to $t = \frac{(cz+d)^2}{z-\bar{z}}$ says that the function $F(z) = P_f(z, \frac{1}{z-\bar{z}})$ satisfies (1). $\square$

This theorem implies that an almost holomorphic modular form and a modular vector are determined by their first coefficient f_0, or first coordinate f_0 respectively, and using this, one finds that the definition of quasimodular forms given in [2] (namely, as the "constant terms" f_0 of almost holomorphic modular forms) is indeed equivalent to the one used here.

3. The additive and $\mathrm{sl}_2(\mathbb{C})$-module structures of rings of quasimodular forms

In this section we show that the ring of quasimodular (or almost holomorphic modular, or vector modular) forms has a natural structure as a module over the Lie algebra $\mathrm{sl}_2(\mathbb{C})$, and describe this structure completely. The results in the first part of this section are given in [7], but our proofs are different in some cases and in any case [7] is not yet published, so we have included full details.

There exists three derivation operators on the spaces of quasimodular forms. By the isomorphisms of Theorem 1, we get the corresponding operators on the other spaces. We check that these give representations of the Lie algebra $\mathrm{sl}_2(\mathbb{C})$ on the spaces $\widetilde{M}_*$, $\widehat{M}_*$ and $\overrightarrow{M}_*$ of quasimodular forms, almost holomorphic modular forms and modular vectors.

Proposition 1. *The operator D of derivation with respect to z acts on the space of quasimodular forms. This operator increases the weight by* 2 *and the depth by* 1. *For any $k \geqslant 0$ and $p \geqslant 0$ we have*

$$D : \widetilde{M}_k^{(\leqslant p)} \to \widetilde{M}_{k+2}^{(\leqslant p+1)}.$$

Proof. Let $f \in \widetilde{M}_k^{(\leqslant p)}$. Differentiating (2) we find

$$\begin{aligned}
&(cz+d)^{-k-2} f'\left(\frac{az+b}{cz+d}\right)\\
&\quad = D\left[(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)\right] + kc(cz+d)^{-k-1} f\left(\frac{az+b}{cz+d}\right)\\
&\quad = D\left[\sum_{0\leqslant j\leqslant p} f_j(z)\left(\frac{c}{cz+d}\right)^j\right] + \frac{kc}{cz+d}\sum_{0\leqslant j\leqslant p} f_j(z)\left(\frac{c}{cz+d}\right)^j\\
&\quad = \sum_{0\leqslant j\leqslant p+1}\left[f_j'(z) + (k-j+1)f_{j-1}(z)\right]\left(\frac{c}{cz+d}\right)^j
\end{aligned}$$

with $f_{-1} \equiv f_{p+1} \equiv 0$. So f' is a quasimodular form of weight $k+2$ and depth is $\leqslant p+1$. $\square$

At the level of vector modular forms, we see that the action of D is given by

$$D(f_0, \ldots, f_j, \ldots, f_p) = \left(f_0', \ldots, f_j' + (k-j+1)f_{j-1}, \ldots\right). \tag{5}$$

Proposition 2.

(i) *If $f \in \widetilde{M}_k^{(\leqslant p)}$ is a quasimodular form with associated vector $(f_0, f_1, \ldots)$ then each f_j is quasimodular of weight $k-2j$ and depth $\leqslant p-j$. In particular, we have a map $\delta : \widetilde{M}_k \to \widetilde{M}_{k-2}$ which sends $f = f_0$ to f_1.*
(ii) *We have $f_j = \delta^j(f)/j!$ for all j.*
(iii) *The kernel of $\delta : \widetilde{M}_k \to \widetilde{M}_{k-2}$ is the space M_k.*

Proof. Part (i) is clear from Eq. (3) which shows that each f_j satisfies (2) with k replaced by $k-2j$ and p by $p-j$. In particular, applying (3) with $j=1$, we see that the modular vector in $\overrightarrow{M}_{k-2}^{(\leqslant p-1)}$ associated to $\delta(f) = f_1$ is given by

$$\delta(f_0, \ldots, f_p) = (f_1, 2f_2, \ldots, pf_p). \tag{6}$$

The same calculation for $j \geqslant 1$ proves (ii), and (ii) implies (iii), because $\delta(f) = 0 \Leftrightarrow f_j = 0$ $(\forall j \geqslant 1) \Leftrightarrow f = f_0 \in M_k$. $\square$

Corollary. *Let $k \geqslant 0$, $f \in \widetilde{M}_k^{(\leqslant p)}$ and $(f_0, \ldots, f_p)$ the associated modular vector. Then we have $f_p \in M_{k-2p}$.*

Proof. By (i) of Proposition 2, we get $f_p \in \widetilde{M}_{k-2p}^{(\leqslant 0)}$. Since a quasimodular of depth 0 is modular, we obtain $f_p \in M_{k-2p}$. $\square$

Remark 2. Since M_k vanishes for $k \leqslant 0$, we see that the depth of a quasimodular form f of weight k is at most equal to $\frac{k}{2}$.

Definition 5. Let $H : \widetilde{M}_* \to \widetilde{M}_*$ be the operator which associates kf to any quasimodular form f of weight k, i.e., $H(f) = kf$.

At the level of modular vectors the action of H is given by

$$H(f_0, \dots, f_p) = (kf_0, \dots, kf_p). \tag{7}$$

It is easy, using Eqs. (5)–(7), to check that the operators D, δ and H satisfy the commutation relations

$$\begin{aligned} &\text{(i)} && [H, D] = 2D, \\ &\text{(ii)} && [H, \delta] = -2\delta, \\ &\text{(iii)} && [\delta, D] = H. \end{aligned} \tag{8}$$

In other words, we have a representation of the Lie algebra $\mathrm{sl}(2, \mathbb{C})$ over the spaces $\widetilde{M}_*$, $\widehat{M}_*$ and $\overrightarrow{M}_*$.

Let $\mathcal{U}$ be the universal enveloping algebra of $\mathrm{sl}_2(\mathbb{C})$, which we represent as $\mathbb{C}D \oplus \mathbb{C}H \oplus \mathbb{C}\delta$ with bracket (8). We compute next the class of the operator $\delta^n D^n$ modulo $\mathcal{U}\delta$, for any $n \in \mathbb{N}$.

Lemma 1. *The class of the operator $\delta^n D^n$ modulo $\mathcal{U}\delta$ is given by*

$$\delta^n D^n \equiv n! \prod_{j=0}^{n-1} (H + j) \pmod{\mathcal{U}\delta}.$$

Proof. By induction on j, we have (in $\mathcal{U}$)

$$\delta^j D = D\delta^j + \sum_{n=0}^{j-1} \delta^n H \delta^{j-1-n} \quad (j \geqslant 0). \tag{9}$$

And by induction on n, we have

$$\delta^n H = (H + 2n)\delta^n. \tag{10}$$

We prove also by induction on the degree that for any polynomial $P \in \mathbb{Z}[X]$ we have

$$\delta P(H) = P(H + 2)\delta. \tag{11}$$

Multiplying (9) by D^{j-1} on the right and using (10) we get

$$\begin{aligned} \delta^j D^j &= D\delta^j D^{j-1} + \sum_{n=0}^{j-1} \delta^n H \delta^{j-1-n} D^{j-1} \\ &= \left(D\delta + \sum_{n=0}^{j-1} (H + 2n) \right) \delta^{j-1} D^{j-1} \\ &= \big(D\delta + j(H + j - 1) \big) \delta^{j-1} D^{j-1}. \end{aligned}$$

We denote by $\equiv$ the congruence modulo $\mathcal{U}\delta$. We have $\delta D \equiv H$. We suppose that $\delta^{j-1} D^{j-1} \equiv P_{j-1}(H)$ where P_{j-1} is a polynomial of degree $j-1$. By using (11) we get

$$\delta^j D^j \equiv D P_{j-1}(H+2)\delta + j(H+j-1)P_{j-1}(H) \equiv j(H+j-1)P_{j-1}(H).$$

Let $P_j(H)$ be the congruence class of $\delta^j D^j$ then

$$P_j(H) \equiv \delta^j D^j \equiv j\big(H+(j-1)\big)P_{j-1}(H).$$

We obtain the result $P_n \equiv n! \prod_{j=0}^{n-1}(H+j)$ by induction. □

Corollary. *Let $f \in M_k$ be a modular form of weight k and $j \geqslant 0$. We have*

$$\delta^j D^j(f) = j!^2 \binom{k+j-1}{j} f.$$

Proof. By using (ii) of Proposition 2, we get $f \in \ker(\delta)$. So the last lemma implies that

$$\delta^j D^j(f) = j! \prod_{n=0}^{j-1}(k+n) f = j!^2 \binom{k+j-1}{j} f. \qquad \square$$

Proposition 3. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cofinite subgroup. Let $k \geqslant 0$ and $p \geqslant 0$ be integers with $p < \frac{k}{2}$. Then*

$$\widetilde{M}_k^{(\leqslant p)}(\Gamma) = D^p\big(M_{k-2p}(\Gamma)\big) \oplus \widetilde{M}_k^{(\leqslant p-1)}(\Gamma).$$

Proof. By Proposition 2 and its corollary, we have $\delta^p(f) \in M_{k-2p}(\Gamma)$. By application of the corollary of Lemma 1 to $\delta^p(f)$ we get

$$\delta^p\left(D^p \delta^p f - p!^2 \binom{k-2p+p-1}{p} f\right) = 0.$$

Hence

$$p!^2 \binom{k-p-1}{p} f - D^p\big(\delta^p(f)\big) \in \widetilde{M}_k^{(\leqslant p-1)}(\Gamma).$$

In particular, if $k > 2p$ then f is the sum of the pth derivative of a modular form and of a quasimodular form of depth $\leqslant p-1$. □

We finish this section by giving an additive structure theorem and an $\mathrm{sl}_2(\mathbb{C})$-module structure theorem for rings of quasimodular forms over discrete and cofinite subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cofinite subgroup then we have an exact sequence:

$$0 \to M_2(\Gamma) \to \widetilde{M}_2(\Gamma) \xrightarrow{\delta} \mathbb{C}.$$

Since the image of δ has to have dimension 0 or 1, there are then two possibilities:

$$\text{(A)} \quad \widetilde{M}_2(\Gamma) = M_2(\Gamma),$$
$$\text{(B)} \quad \widetilde{M}_2(\Gamma) = M_2(\Gamma) + \mathbb{C}\phi, \quad \text{for some } \phi \text{ with } \delta\phi = 1.$$

We will see in the next section that case (A) occurs if and only if Γ is cocompact, but for the moment we do not need this.

Theorem 2. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and cofinite subgroup. Then*

$$\widetilde{M}_* = \mathbb{C} \oplus \bigoplus_{i=0}^{\infty} D^i M_{*>0}(\Gamma) \quad \textit{in case } \text{(A)}, \quad \textit{and}$$
$$\widetilde{M}_*(\Gamma) = \mathbb{C} \oplus \bigoplus_{i\geqslant 0} D^i M_{*>0}(\Gamma) \oplus \bigoplus_{i\geqslant 0} \mathbb{C} D^i \phi \quad \textit{in case } \text{(B)}.$$

Proof. Suppose that we are in case (A), i.e., $\delta(\widetilde{M}_2) = 0$. Then for any $f \in \widetilde{M}_k^{(\leqslant p)}$, we have $p < \frac{k}{2}$. Indeed $\delta^{\frac{k-2}{2}} f \in \widetilde{M}_2$, so $\delta^{\frac{k}{2}} f \in \delta\widetilde{M}_2 = 0$ and the depth of f is at most equal to $\frac{k-2}{2}$. By Proposition 3, f is the sum of the pth derivative of a modular form and of a quasimodular form of depth $< p$. So by induction on p, we get that $\widetilde{M}_k = \bigoplus_{i=0}^{(k-2)/2} D^i M_{k-2i}$. This implies the theorem in case (A).

We suppose now that there exists a quasimodular form ϕ of weight 2 such that $\delta(\phi) = 1$. Then ϕ has depth 1. Let $f \in \widetilde{M}_k^{(\leqslant p)}(\Gamma)$. We can suppose that $2p = k$ (since if $p < \frac{k}{2}$ then the same argument as in case (A) implies that $f \in \bigoplus_{i=0}^{k/2-1} D^i M_{k-2i}$). Then the final coefficient f_p in the expansion (2) belongs to $M_0 = \mathbb{C}$. On the other hand, $D^{p-1}\phi \in \widetilde{M}_k^{(\leqslant p)}(\Gamma)$, so $\alpha := \delta^p D^{p-1}\phi$ also belongs to $M_0 = \mathbb{C}$, and $\alpha \neq 0$ since ϕ has depth exactly 1. We have $f - \frac{f_p}{\alpha} D^{p-1}\phi \in \widetilde{M}_k^{(<p)}(\Gamma)$. By the first part of the proof and the fact that $p - 1 = \frac{k-2}{2} < \frac{k}{2}$, we get

$$f \in \mathbb{C} D^{\frac{k-2}{2}} \phi \oplus \bigoplus_{i=0}^{(k-2)/2} D^i M_{k-2i}(\Gamma).$$

This finish the proof in case (B). $\square$

We now describe the structure of $\widetilde{M}_*$ as an $\mathrm{sl}_2(\mathbb{C})$-module. For $k > 0$, let $\mathcal{A}_k$ be the $\mathrm{sl}_2(\mathbb{C})$-module defined by a basis $(x_j^{(k)})_{j\in\mathbb{N}}$ with

$$Dx_j^{(k)} = x_{j+1}^{(k)}, \quad Hx_j^{(k)} = (k + 2j)x_j^{(k)}, \quad \delta x_j^{(k)} = j(k + j - 1)x_{j-1}^{(k)}.$$

(For $j = 0$, the last equation means $\delta x_0^{(k)} = 0$.) We can think of $\mathcal{A}_k$ as $\mathbb{C}[T]$ with the $\mathrm{sl}_2(\mathbb{C})$-action given by

$$D = T, \quad H = k + 2T\frac{\partial}{\partial T}, \quad \delta = T\frac{\partial^2}{\partial T^2} + k\frac{\partial}{\partial T}.$$

We define $\mathcal{A}_0 = \mathbb{C}$ with the trivial action of $\mathrm{sl}_2(\mathbb{C})$. For $k = 2$, we define a second $\mathrm{sl}_2(\mathbb{C})$-module $\widehat{\mathcal{A}}_2$ with basis $((\hat{x}_j^{(2)}), y)$, where D, H and δ act on the $\hat{x}_j^{(2)}$ in the same way as on the $x_j^{(2)}$ except that $\delta\hat{x}_0^{(2)} = y$, and $Dy = Hy = \delta y = 0$. This is an extension of $\mathcal{A}_2$ by $\mathcal{A}_0$: there is an $\mathrm{sl}_2(\mathbb{C})$-equivariant short exact sequence $0 \to \mathcal{A}_0 \to \widehat{\mathcal{A}}_2 \to \mathcal{A}_2 \to 0$ with maps given by $1 \mapsto y$, $\hat{x}_j^{(2)} \mapsto x_j^{(2)}$, $y \mapsto 0$.

For any $k > 0$, we have a map

$$\begin{aligned} \mathcal{A}_k \otimes M_k(\Gamma) &\to \widetilde{M}_* \\ x_j^{(k)} \otimes f &\mapsto D^j f. \end{aligned} \tag{12}$$

It is obviously injective for $k > 0$ since each $D^j f$ has a different weight. In the case $k = 0$, we have a map, again injective,

$$M_0(\Gamma) \otimes \mathcal{A}_0 = \mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \subset \widetilde{M}_*(\Gamma).$$

For $k = 2$ we also have a map

$$\begin{aligned} \widehat{\mathcal{A}}_2 \otimes \widetilde{M}_2(\Gamma) &\to \widetilde{M}_*(\Gamma) \\ \hat{x}_j^{(2)} \otimes f &\mapsto D^j f, \quad y \otimes f \mapsto \delta(f), \end{aligned}$$

but it is no longer injective in general, since $y \otimes f$ maps to 0 for $f \in M_2(\Gamma)$. In case (A), when $\widetilde{M}_2(\Gamma) = M_2(\Gamma)$, this action factors through $\mathcal{A}_2$ and we have gained nothing. In case (B), when $\widetilde{M}_2(\Gamma) = M_2(\Gamma) \oplus \mathbb{C}\phi$ with $\delta(\phi) = 1$, we need the extended action only on the one-dimensional subspace $\mathbb{C}\phi$ of $\widetilde{M}_2(\Gamma)$. Putting this all together, we see that Theorem 2 is equivalent to the following theorem which gives the complete structure of $\widetilde{M}_2(\Gamma)$ as an $\mathrm{sl}_2(\mathbb{C})$-module.

Theorem 3. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and cofinite subgroup. Then we have canonical $\mathrm{sl}_2(\mathbb{C})$-equivariant isomorphisms*

$$\widetilde{M}_*(\Gamma) \cong \bigoplus_{k=0}^{\infty} \mathcal{A}_k \otimes M_k(\Gamma)$$

in case (A) *and*

$$\widetilde{M}_*(\Gamma) \cong \bigoplus_{k=0}^{\infty} \mathcal{A}_k \otimes M_k(\Gamma) \oplus \widehat{\mathcal{A}}_2 \otimes \mathbb{C}\phi,$$

in case (B). *In other words, $\widetilde{M}_*(\Gamma)$ as an $\mathrm{sl}_2(\mathbb{C})$-module is the direct sum of modules $\mathcal{A}_k$ and $\widehat{\mathcal{A}}_2$, with each $\mathcal{A}_k$ occurring with multiplicity $\dim M_k(\Gamma)$, and $\widehat{\mathcal{A}}_2$ occurring with multiplicity* 0 *or* 1 *according as Γ is of type* (A) *or* (B).

4. The cocompact/non-cocompact dichotomy

In the last section we saw that there is a dichotomy among discrete cofinite subgroups of $\mathrm{PSL}(2,\mathbb{R})$, with each group being of type (A) or (B) according as the space of quasimodular forms of weight 2 coincides with the space of modular forms of that weight or has dimension exactly one larger. We now show that these two types of groups are simply the cocompact and non-cocompact groups, respectively.

Theorem 4. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cofinite subgroup. If Γ is not cocompact, then there exists a quasimodular form ϕ of weight 2 on Γ which is not modular, and $\widetilde{M}_2 = M_2(\Gamma) \oplus \mathbb{C}\phi$. If Γ is cocompact we have $\widetilde{M}_2(\Gamma) = M_2(\Gamma)$.*

Proof. In the case of a subgroup Γ of Γ_1 we can take $\phi = E_2$. If Γ is not a subgroup, but is commensurable with Γ_1, then take a trace of E_2 and get a quasimodular form of weight 2 over Γ which is not modular. In general, if Γ is not cocompact then we can always obtain ϕ as the quasimodular form associated to the non-holomorphic modular Eisenstein series $E_{2,\Gamma}(z)$ of weight 2 (defined in the usual way as the limit as $s \to 0$ of a convergent Eisenstein series $E_{2,\Gamma}(z,s)$), which is always an almost holomorphic, but not holomorphic, modular form of weight 2.

Now suppose that Γ is cocompact and that there exists a quasimodular form f of weight 2 which is not modular. Let F be the almost holomorphic modular form associated to f. We have

$$F(z) = f(z) + \frac{c}{z-\bar{z}} \quad \text{with } c = \delta(f) \neq 0.$$

Let $\omega(z) = F(z)\,dz$. The modularity of F implies the Γ-invariance of ω. So this 1-form is defined on the quotient $X = \mathcal{H}/\Gamma$. On the other hand, we have

$$d\omega = -\frac{\partial F}{\partial \bar{z}}\,dz \wedge d\bar{z} = -\frac{c}{(z-\bar{z})^2}\,dz \wedge d\bar{z}.$$

This means that $d\omega$ is a non-zero multiple of the volume form. But then $\int_X d\omega$ is a non-zero multiple of the volume of X and hence is non-zero, which contradicts Stokes's formula since X is a variety without boundary. $\square$

Theorem 4 implies that case (A) is the cocompact case and (B) the non-cocompact case. We can therefore restate Theorem 2 as the following additive structure theorem for rings of quasimodular forms: for cocompact groups, every quasimodular form is uniquely a linear combination of derivatives of modular forms, while in the non-cocompact case every quasimodular form is uniquely a linear combination of derivatives of modular forms and of a single non-modular quasimodular form of weight 2.

5. Rings of quasimodular forms

In Sections 3 and 4 we elucidated the additive structure of the ring of quasimodular forms for a discrete and cofinite subgroup of $\mathrm{PSL}(2,\mathbb{R})$. In this section we study the multiplicative structure. We recall the multiplicative structure in the non-cocompact case. In [2] we find the corresponding structure for the particular case of Γ_1. Using Theorem 4, we easily generalize this to any non-cocompact group.

Theorem 5. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and non-cocompact subgroup, and let be ϕ be a non-modular quasimodular form of weight 2 over Γ. Then $\widetilde{M}_*(\Gamma) = M_*(\Gamma) \otimes \mathbb{C}[\phi]$.*

This theorem implies that the ring of quasimodular forms is finitely generated in the non-cocompact case, since the ring of modular forms is always finitely generated.

Proof. The existence of ϕ is guarantied by Theorem 4. The depth of ϕ is exactly 1. Let $f \in \widetilde{M}_k^{(\leqslant p)}$. Then the last coefficient f_p of (2) is modular of weight $k - 2p$. Hence $\phi^p f_p \in \widetilde{M}_k^{(\leqslant p)}$ and $f - \phi^p f_p \in \widetilde{M}_k^{(<p)}$. By induction on p, we have that $f - \phi^p f_p$ is a polynomial in modular forms and ϕ, so f also is. $\square$

For a discrete and cocompact subgroup Γ, we denote by $I = I_k$ (respectively $\widetilde{I} = \widetilde{I}_k$) the ideal of modular forms (respectively quasimodular forms) over Γ of strictly positive weight. Then $\widetilde{I}_k^2 = \sum_{0<j<k} \widetilde{M}_j \widetilde{M}_{k-j}$ is the $\mathbb{C}$-vector space of decomposable quasimodular forms of weight k. Note that the intersection $I \cap \widetilde{I}^2$ contains, but is not necessarily equal to, the vector space I^2: a modular form of positive weight can be decomposable in the space of quasimodular forms without being decomposable in the space of modular forms. We denote by P_s the quotient $I_s/(I_s \cap \widetilde{I}^2)$. Let $\epsilon_s = \dim P_s$. We denote by $\epsilon = \sum_s \epsilon_s = \dim I/(I \cap \widetilde{I}^2)$. We have

$$1 \leqslant \epsilon \leqslant \dim I/I^2 < \infty,$$

because on the one hand the modular form of smallest weight is not decomposable, and on the other hand the ring of holomorphic modular forms is finitely generated by $\dim I/I^2$ elements.

Theorem 6. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cocompact subgroup. Then for any even $k \geqslant 0$,*

$$\big(\widetilde{I}/\widetilde{I}^2\big)_k = \bigoplus_{0<s\leqslant k} D^{\frac{k-s}{2}} P_s.$$

Corollary. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cocompact subgroup. Then for all sufficiently large k, $\dim(\widetilde{I}/\widetilde{I}^2)_k = \epsilon$.*

Corollary. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cocompact subgroup. Then the ring $\widetilde{M}_*(\Gamma)$ of quasimodular forms over Γ is not finitely generated.*

Proof. The map $D^n : M_s \to \widetilde{M}_{s+2n}$ is injective for $s > 0$, since the only polynomials which are modular are constants. In particular, $\dim D^n(P_s) = \epsilon_s$. Let $\alpha_s = (k-s)/2$ $(s = 2, 4, \ldots, k)$. Then $\epsilon = \sum_{0<s\leqslant k} \dim D^{\alpha_s} P_s$. Using this, we obtain the first corollary.

Since $\dim(\widetilde{I}/\widetilde{I}^2)_k$ is equal to the number of new generators of $\widetilde{M}_k$, and this dimension is strictly positive for an infinite set of k, we deduce the second corollary.

We now prove the theorem. Let $P_s^* \subset I$ $(s = 2, 4, \ldots, k)$ be a subvector space whose image under $I \to I/(I \cap \widetilde{I}^2)$ equals P_s. We can suppose that $P_s^* \cap \widetilde{I}^2 = \{0\}$. Then by definition, we have the decomposition $I = (\bigoplus_{s>0} P_s^*) \oplus (I \cap \widetilde{I}^2)$. Let $k \geqslant 0$. From Theorem 2, we deduce that

$$\widetilde{I}_k = \widetilde{M}_k = \sum_{n\geqslant 0} D^n\big(P_{k-2n}^*\big) + \big(\widetilde{I}^2\big)_k.$$

It remains to prove that this sum is direct. This means that, if a sum

$$D^{\alpha_2}(f_2) + \cdots + D^{\alpha_k}(f_k) \in \widetilde{I}^2 \quad \left(f_s \in P_s^*\right), \tag{13}$$

then $f_2 = \cdots = f_k = 0$. If not, let l be the smallest integer such that $f_l \neq 0$. We apply the operator δ^{α_l} to (13), we get $cf_l \in \delta^{\alpha_l}(\widetilde{I}^2)$ for a certain $c \neq 0$. But δ is a derivation and $\widetilde{M}_2(\Gamma) = M_2(\Gamma)$, so $\delta(\widetilde{I}^2) \subset \widetilde{I}^2$ and by induction on n, $\delta^n(\widetilde{I}^2) \subset \widetilde{I}^2$ for any n. Hence $cf_l \in P_l^* \cap \delta^{\alpha_2}(\widetilde{I}^2) \subset P_l^* \cap \widetilde{I}^2 = \{0\}$. Then we get $f_l = 0$, a contradiction. □

6. Algebraic characterization of cocompact modular groups

We recall that a *Poisson algebra* is a commutative and associative algebra A with a Lie structure, i.e., an antisymmetric bilinear operation $[\cdot,\cdot] : A \times A \to A$ satisfying the Jacobi identity, such that for any $x \in A$, the map $[x, \cdot]$ is a derivation. If furthermore $A = \bigoplus_{n \geqslant 0} A_n$ is graded with $A_m A_n \subset A_{m+n}$, $[A_m, A_n] \subset A_{m+n+1}$, then we will call A a *graded* Poisson algebra.

Examples. (1) Let A be a graded algebra (commutative and associative) and let $d : A \to A$ be a derivation of degree 1, i.e., $d(A_n) \subset A_{n+1}$ and $d(xy) = x\,d(y) + y\,d(x)$ for every $x, y \in A$. Let $E : A \to A$ (Euler operator) be the operator of multiplication by the weight, i.e., $E(x) = nx$ for $x \in A_n$. Then the bracket defined by $[x, y] = E(x)\,d(y) - E(y)\,d(x)$, satisfies the Jacobi identity (a simple verification) and has the property that $x \to [x, y]$ is a derivation for every fixed $y \in A$ (because E and d are derivations). We will call a graded Poisson algebra *trivialisable* if it can be obtained in this way.

(2) Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and cofinite subgroup and let $A = M_*(\Gamma)$. Since all weights are even, we can choose the graduation $A_n = M_{2n}(\Gamma)$. This algebra has a Poisson structure with the usual multiplication and where the bracket $[\cdot,\cdot] = [\cdot,\cdot]_1$ is the first Rankin–Cohen bracket, defined by $[f, g] = kfg' - lgf'$ if $f \in M_k$ and $g \in M_l$. Note that E here is $\frac{1}{2}H$, where H is the operator defined in Section 3 (Definition 5), since we have changed the graduation.

Theorem 7. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete and cofinite subgroup. Then the Poisson algebra $(M_{ev}(\Gamma), [\cdot,\cdot]_1)$ is trivialisable if and only if Γ is not cocompact.*

We use the next two lemmas, of which the second is a corollary of the first.

Lemma 2. *Let M_*^{mer} be the ring of meromorphic modular forms over a discrete and cofinite subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Then any derivation*

$$\partial : M_*^{\mathrm{mer}} \to M_{*+2}^{\mathrm{mer}},$$

trivializing the first Rankin–Cohen bracket has the form $\partial = D - \phi H$, where $\phi \in \widetilde{M}_2^{\mathrm{mer}}$ with $\delta\phi = 1$.

Proof. Let ∂ be a derivation trivializing the first Rankin–Cohen bracket. Then $H(f)\partial g - H(g)\partial f = [f, g]_1 = H(f)g' - H(g)f'$ for any meromorphic modular forms f and g. Hence $\frac{g' - \partial g}{H(g)} = \frac{f' - \partial f}{H(f)}$ if f and g are of non-zero weight. This implies that the quotient $\frac{f' - \partial f}{H(f)}$ is independent of the modular form f, i.e., there exists a meromorphic quasimodular form ϕ of weight 2 such that $\partial f = f' - k\phi f$ for any $f \in M_k$. □

Lemma 3. *Let M_* be the ring of holomorphic modular forms over a discrete and cofinite subgroup of* $\mathrm{PSL}(2,\mathbb{R})$. *Then any derivation $\partial: M_* \to M_{*+2}$ trivializing the first Rankin–Cohen bracket has the form $D_\phi = D - \phi H$ with $\phi \in \widetilde{M}_2$ and $\delta\phi = 1$.*

Proof. We can extend ∂ to the ring $M_*^{\mathrm{mer}}(\Gamma)$ by the derivation formula $\partial(f/g) = \partial(f)/g - \partial(g)f/g^2$. By the last lemma, ∂ has the form D_ϕ for some $\phi \in \widetilde{M}_2^{\mathrm{mer}}$. But then $\phi f = \frac{1}{k}(f' - \partial f)$ is holomorphic for any $f \in M_k$, and this implies that ϕ itself is holomorphic, since for any finite point or cusp z_0 there exists a modular form of positive weight which is non-zero at z_0. $\square$

The converse of Lemma 3 of course also holds: if $\phi \in \widetilde{M}_2$ and $\delta\phi = 1$, then D_ϕ trivializes $[\cdot,\cdot]_1$. The proof of Theorem 7 is now immediate form Theorem 4.

Example. If $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ is the classical modular group, then there exists a derivation on $M_*(\Gamma)$ which trivializes the first Rankin–Cohen bracket. It is given by $\partial = D - \frac{1}{12}E_2 H$, where E_2 is the normalized Eisenstein series of weight 2 (Serre derivation).

We give now another consequence of the cocompact/non-cocompact dichotomy. For this we use a proposition and a definition given in [7]. We recall that a *Rankin–Cohen algebra* is a graded algebra R_* together with bilinear maps ("brackets") $[\cdot,\cdot]_n : R_k \otimes R_l \to R_{k+l+2n}$ which satisfy all algebraic identities satisfied by the usual Rankin–Cohen brackets.

Proposition 4. *Let R_* be a commutative and associative graded $\mathbb{C}$-algebra with $R_0 = \mathbb{C}.1$ together with a derivation $\partial : R_* \to R_{*+2}$ of degree 2, and let $\Phi \in R_4$. Define brackets $[\cdot,\cdot]_{\partial,\Phi,n}$ $(n \geqslant 0)$ on R_* by*

$$[f,g]_{\partial,\Phi,n} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} f_r g_s \tag{14}$$

for $f \in M_k$ and $g \in M_l$, where $f_r \in M_{k+2r}$, $g_s \in M_{l+2s}$ $(r,s \geqslant 0)$ are defined recursively by $f_0 = f$, $f_1 = \partial f$, $g_0 = g$, $g_1 = \partial g$ and

$$f_{r+1} = \partial f_r + r(r+k-1)\Phi f_{r-1}, \qquad g_{s+1} = \partial g_s + s(s+l-1)\Phi g_{s-1}$$

for $r,s \geqslant 1$. Then R_*, with these brackets, is a Rankin–Cohen algebra.

Definition 6. A Rankin–Cohen algebra R_* will be called *canonical* if its brackets are given as in Proposition 4 for some derivation $\partial : R_* \to R_*$ of degree $+2$ and some element $\Phi \in R_4$.

We can now give a third algebraic characterization of cocompact groups among all cofinite groups.

Theorem 8. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cofinite subgroup. Then the algebra $M_*(\Gamma)$ is canonical if and only if Γ is not cocompact.*

Proof. We suppose that $M_*(\Gamma)$ is a canonical Rankin–Cohen algebra. By definition this means that there exists a derivation $\partial : M_* \to M_{*+2}$ of degree $+2$ such that all Rankin–Cohen brackets $[\cdot,\cdot]_n$ are given by formula (14). In particular, for $n = 1$ this says that $[f,g]_1 = kf\partial g - lg\partial f$, i.e., ∂ trivializes the first Rankin–Cohen bracket. Using Lemma 3, we deduce that ∂ is of the form D_ϕ with ϕ a non-modular quasimodular form of weight 2. By Theorem 4, we get that Γ is non-cocompact. Conversely, if Γ is a non-cocompact group, then there exits $\phi \in \widetilde{M}_2$ such that $\delta(\phi) = 1$. Let $\partial = D - \phi H$ and $\Phi = \phi' - \phi^2$. Then, by the calculation given on pages 73–74 of [8], the Rankin–Cohen brackets over M_* are given by the brackets $[\cdot,\cdot]_{\partial,\Phi,*}$ as in Proposition 4. So M_* is canonical. $\square$

7. Existence of quasimodular forms with prescribed poles

In this section, using the Riemann–Roch theorem over algebraic curves we prove the existence, for any cocompact group Γ, of meromorphic quasimodular forms ϕ of weight 2 without poles outside the orbit of z_0, for any point $z_0 \in \mathcal{H}/\Gamma$. This result will be used in Section 8.

Theorem 9. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete and cocompact subgroup, and $z_0 \in \mathcal{H}$. Then there exists a quasimodular form ϕ of weight 2 on Γ satisfying $\delta(\phi) = 1$, with simple poles in the orbit of z_0, and without other poles. For any such form ϕ, we have $\mathrm{Res}_{z=\alpha}(\phi(z)\,dz) = \kappa$ for any α in the Γ-orbit of z_0, where $\kappa = \frac{\mathrm{Vol}(\mathcal{H}/\Gamma)}{4\pi}$.*

Remark. The form ϕ is unique up to the addition of an holomorphic modular form of weight 2. The dimension of the space of such forms is equal to the genus g of the Riemann surface $\mathcal{H}/\Gamma$.

Proof. First, we suppose that Γ acts on $\mathcal{H}$ without fixed points (this means that the action is free). Let f be a non-zero modular form of weight $k > 0$. Then $\frac{f'}{f}$ is a meromorphic quasimodular form of weight 2 with $\delta(\frac{f'}{f}) = k \neq 0$. Moreover the poles of $\frac{f'}{f}$ are simple and the set of poles is Γ-invariant. We denote by $\{P_1, \dots, P_n\}$ the poles of $\frac{f'}{f}$ in $\mathcal{H}/\Gamma$ different from the image of z_0. We want to construct a meromorphic modular form h of weight 2 such that the sum $\frac{f'}{f} + h$ has no poles outside the orbit of z_0. Let $X = \mathcal{H}/\Gamma$ be the compact Riemann surface, and g its genus. Then the hypothesis on Γ implies that X is smooth and that $g > 1$. We denote by Ω_X^1 the sheaf of holomorphic differential 1-forms over X. For any set of distinct points $\{q_1, \dots, q_m\} \subset X$ (with $m \geqslant 1$), we denote by $\Omega_X^1(q_1 + \dots + q_m)$ the sheaf of holomorphic differential 1-forms over X with simple poles at $q_1, \dots, q_m$. We will prove:

$$H^0\big(X, \Omega_X^1(q_1 + \dots + q_m)\big) \simeq \mathbb{C}^{g+m-1}.$$

Let K be the canonical divisor of X. By the Riemann–Roch theorem we have

$$l(K + q_1 + \dots + q_m) = l\big(-(q_1 + \dots + q_m)\big) + \deg(K + q_1 + \dots + q_m) - g + 1.$$

From $\deg(K) = 2g - 2$ and $l(-(q_1 + \dots + q_m)) = 0$, we deduce that:

$$l(K + q_1 + \dots + q_m) = g + m - 1.$$

If we apply the Riemann–Roch theorem to the cases $m = 1$ and $m = n + 1$, we obtain the exact sequence:

$$0 \to H^0\big(X, \Omega_X^1(z_0)\big) \to H^0\big(X, \Omega_X^1(z_0 + P_1 + \dots + P_n)\big) \xrightarrow{\mathrm{Res}} \mathbb{C}^n \to 0,$$

where Res maps a differential 1-form ω to $(\mathrm{Res}_{P_1}(\omega), \ldots, \mathrm{Res}_{P_n}(\omega))$. So we can choose h of weight 2 such that $\phi = \frac{1}{k}\frac{f'}{f} + h$ has a simple pole at z_0 and no poles outside the orbit of z_0. We also have $\delta\phi = 1$.

To compute the constant κ we apply Stokes's formula to the meromorphic differential 1-form $\omega(z) = \phi^*(z)\,dz$ over X, where $\phi^*(z) = \phi(z) + \frac{1}{z-\bar{z}}$ is the almost holomorphic modular form associated to ϕ. We know that $d\omega = -\frac{dz\wedge d\bar{z}}{(z-\bar{z})^2}$ (proof of Theorem 4 in Section 4). So $d\omega$ is $\frac{1}{2i}$ times the volume form. If we integrate $d\omega$ over $X - D_\epsilon$, where D_ϵ is a small disk around z_0, and we apply the Stokes's theorem, we get

$$\frac{1}{2i}\,\mathrm{Vol}(X - D_\epsilon) = \int\limits_{\partial(X-D_\epsilon)} \omega = 2\pi i\,\mathrm{Res}_{z=z_0}\bigl(\phi(z)\bigr) + O(\epsilon) = 2\pi i\kappa + O(\epsilon).$$

By letting ϵ to 0 we obtain $\kappa = \frac{\mathrm{Vol}(X)}{4\pi}$. This completes the proof in the case of groups acting on $\mathcal{H}$ without fixed points.

We now consider that the action of a Γ is not necessarily free. The Selberg lemma implies that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index without torsion. The first part of the proof implies that there exists a quasimodular form α over Γ' of weight 2 with at most simple poles in the orbit of the point z_0. We define

$$\beta(z) = \sum_{\gamma\in\Gamma/\Gamma'} \left((\alpha \mid \gamma)(z) - \frac{c}{cz+d}\right),$$

with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $(\alpha \mid \gamma)(z) = (cz+d)^{-2}\alpha(\frac{az+b}{cz+d})$. We will prove that β is a quasimodular form over Γ of weight 2. Let α^* be the almost holomorphic modular form associated to α. It is easy to check that

$$\beta^*(z) = \sum_{\gamma\in\Gamma/\Gamma'} (\alpha^* \mid \gamma)(z),$$

is an almost holomorphic modular form over Γ of weight 2. (Since α^* is modular, β^* corresponds to the trace of α^* over the group Γ.) On the other hand, we have $(\alpha^* \mid \gamma)(z) = [(\alpha \mid \gamma)(z) - \frac{c}{cz+d}] + \frac{1}{z-\bar{z}}$. So

$$\beta^*(z) = \sum_{\gamma\in\Gamma/\Gamma'} \left[(\alpha \mid \gamma)(z) - \frac{c}{cz+d}\right] + \sum_{\gamma\in\Gamma/\Gamma'} \frac{1}{z-\bar{z}},$$

in other words $\beta^*(z) = \beta(z) + \frac{[\Gamma:\Gamma']}{z-\bar{z}}$. This proves that β is a quasimodular form over Γ of weight 2 and $\delta(\beta) = [\Gamma : \Gamma']$. It is clear that β has at most simple poles on the orbit of z_0. Hence, $\frac{\beta}{[\Gamma:\Gamma']}$ is an appropriate form over Γ, and the value of κ is as given, since

$$\mathrm{Vol}(\mathcal{H}/\Gamma') = [\Gamma : \Gamma']\,\mathrm{Vol}(\mathcal{H}/\Gamma). \qquad \square$$

8. Embedding of quasimodular forms in finitely generated rings

We know that the ring $M_*(\Gamma)$ of modular forms is always finitely generated and that the ring $\widetilde{M}_*(\Gamma)$ of quasimodular forms never is if Γ is cocompact. In this section, we give two constructions of rings $\widetilde{\mathcal{R}}_* = \bigoplus_{k\geqslant 0}\widetilde{\mathcal{R}}_k$ ($\widetilde{\mathcal{R}}_0 = \mathbb{C}$) which contain $\widetilde{M}_*(\Gamma)$ and are finitely generated and of transcendence degree 3 (so that $\dim\widetilde{\mathcal{R}}_k = O(k^2)$, the same order of growth as $\dim\widetilde{M}_k(\Gamma)$).

We denote by $M_*(\Gamma,\{z_0\}) = \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma,\{z_0\})$ the ring of meromorphic modular forms without poles outside the Γ-orbit of z_0.

In the *first* construction, we consider the ring of "tempered" modular forms on Γ defined by

$$M_*^{\mathcal{T}}\big(\Gamma,\{z_0\}\big) = \bigoplus_{k\geqslant 0}\big\{f\in M_k\big(\Gamma,\{z_0\}\big)\ \big|\ \nu_{z_0}(f)\geqslant -k/2\big\}, \tag{15}$$

where $\nu_{z_0}(f) = \operatorname{ord}_{z_0}(f)/\operatorname{ord}_\Gamma(z_0)$ with $\operatorname{ord}_\Gamma(z_0)$ the order of the stabilizer Γ_{z_0} of z_0.

Using the interpretation of elements of $M_k^{\mathcal{T}}$ as global sections of the kth tensor power of an ample line bundle over $\mathcal{H}/\Gamma$, we show that $M_*^{\mathcal{T}}$ is a finitely generated ring. Let ϕ be a meromorphic quasimodular form as in Theorem 9. We show that $M_*^{\mathcal{T}}\otimes\mathbb{C}[\phi]$ contains the ring $\widetilde{M}_*(\Gamma)$.

In the *second* construction, we consider a meromorphic quasimodular form ϕ of weight 2 as in Construction 1. We now define a ring $\widetilde{\mathcal{R}}_*(\Gamma,\{z_0\})$ as the differential closure of $\langle M_*(\Gamma),\phi\rangle$, i.e., as the smallest ring containing $M_*(\Gamma)$ and ϕ, and closed under differentiation. This ring depends only on z_0, not on the choice of ϕ, and contains $\widetilde{M}_*(\Gamma)$ by Theorem 2. Using a combinatorial lemma about subsemigroups of $\mathbb{R}^2$, we show that $\mathcal{R}_*(\Gamma,\{z_0\})$ is finitely generated in positive weight.

Remark. Using details of these constructions, we show easily that $\widetilde{M}_*(\Gamma)\subsetneq\widetilde{\mathcal{R}}_*(\Gamma,\{z_0\})\subseteq M_*^{\mathcal{T}}(\Gamma,\{z_0\})\otimes\mathbb{C}[\phi]$, for any Γ and $z_0\in\mathcal{H}$. In the next section we show that Construction 2 is in general finer then Construction 1, by giving example of a cocompact group Γ_6 for which $\widetilde{M}_*(\Gamma_6)\subsetneq\widetilde{\mathcal{R}}(\Gamma_6,\{i\})\subsetneq M_*^{\mathcal{T}}(\Gamma_6,\{i\})\times\mathbb{C}[\phi]$.

We begin with two lemmas which are needed in both constructions.

Lemma 4. *Let $\Gamma\subset\mathrm{PSL}(2,\mathbb{R})$ be a discrete cocompact subgroup, ϕ a quasimodular form on Γ with at most simple poles in the orbit of z_0 and with $\delta(\phi)=1$, and $\omega=\phi'-\phi^2$. Then ω is a modular form of weight* 4 *with double poles in the orbit of* z_0.

Proof. We know that for any $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$ we have

$$\phi\left(\frac{az+b}{cz+d}\right) = (cz+d)^2\phi(z) + c(cz+d).$$

By differentiating, we get

$$\phi'\left(\frac{az+b}{cz+d}\right) = (cz+d)^4\phi'(z) + 2c(cz+d)^3\phi(z) + c^2(cz+d)^2.$$

On the other hand,

$$\phi^2\left(\frac{az+b}{cz+d}\right) = (cz+d)^4\phi^2(z) + 2c(cz+d)^3\phi(z) + c^2(cz+d)^2.$$

This implies

$$(\phi' - \phi^2)\left(\frac{az+b}{cz+d}\right) = (cz+d)^4(\phi' - \phi^2)(z).$$

So ω is a modular form of weight 4. Since $\phi'(z_0 + x) \sim -\kappa x^{-2}$ and $\phi^2(z_0 + x) \sim \kappa^2 x^{-2}$ for $x \to 0$ with κ as in Theorem 9, we deduce that

$$\omega(x + z_0) \sim -\kappa(\kappa + 1)x^2. \qquad \square$$

Lemma 5. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete cocompact subgroup, and ϕ a quasimodular form of weight 2 over Γ with $\delta(\phi) = 1$ which has a simple pole at z_0 and is holomorphic outside the orbit of z_0. Let*

$$D_\phi : M_k(\Gamma, \{z_0\}) \to M_{k+2}(\Gamma, \{z_0\}),$$

be the operator defined by $D_\phi(f) = f' - k\phi f$. Then for every modular form $f \in M_k(\Gamma, \{z_0\}) - \{0\}$, we have:

(i) $\mathrm{ord}_{z_0}(f) \leqslant \kappa k$.
(ii) *If* $\mathrm{ord}_{z_0}(f) < \kappa k$ *then* $D_\phi f \neq 0$ *and* $\mathrm{ord}_{z_0}(D_\phi f) = \mathrm{ord}_{z_0}(f) - 1$.
(iii) *If* $\mathrm{ord}_{z_0}(f) = \kappa k$ *and* $D_\phi(f) \neq 0$ *then*

$$\mathrm{ord}_{z_0}(f) - 1 \leqslant \mathrm{ord}_{z_0}(D_\phi f) \leqslant \mathrm{ord}_{z_0}(f) + 2\kappa.$$

Proof. The fact that D_ϕ sends modular forms to modular forms has been used several times (e.g. in Lemma 2), and it obviously also preserves the property of being holomorphic outside of $\{z_0\}$. To prove (i), we use the formula for the orders of zeros of a modular forms. The Riemann–Roch theorem shows that $M_*(\Gamma)$ contain elements f attaining this bound. If $f(z_0 + x) = cx^\alpha + O(x^{\alpha+1})$ with $\alpha \in \mathbb{Z}$ and $c \neq 0$ then $D_\phi(f)(z_0 + x) = c(\alpha - k\kappa)x^{\alpha-1} + O(x^\alpha)$, so $\mathrm{ord}_{z_0}(D_\phi f) \geqslant \alpha - 1$ and $\mathrm{ord}_{z_0}(D_\phi f) = \alpha - 1$ if $\alpha \neq k\kappa$. This implies (ii) and the first inequality in (iii). Finally, by replacing f by $D_\phi f$ in (i), we obtain the second inequality in (iii). $\square$

Now we can give the details of the two constructions.

8.1. Construction 1

Theorem 10. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete cocompact subgroup, ϕ a quasimodular form on Γ with at most simple poles in the orbit of z_0 and with $\delta(\phi) = 1$, and $M_*^{\mathcal{T}}(\Gamma, \{z_0\})$ the ring defined in* (15). *Then*:

(i) *The ring $M_*^{\mathcal{T}}(\Gamma, \{z_0\})$ is closed under D_ϕ, and is finitely generated.*
(ii) *The ring $M_*^{\mathcal{T}}(\Gamma, \{z_0\}) \otimes \mathbb{C}[\phi]$ is finitely generated, and contains the ring $\widetilde{M}_*(\Gamma)$ of holomorphic quasimodular forms on Γ.*

Proof. Let $f \in M_k^{\mathcal{T}}$. Using Lemma 5, we get that $D_\phi(f) \in M_{k+2}(\Gamma, \{z_0\})$ and $\mathrm{ord}_{z_0} D_\phi f \geqslant \alpha - 1 = \mathrm{ord}_{z_0}(f) - 1$. Then $\nu_{z_0}(D_\phi f) \geqslant \nu_{z_0}(f) - 1/\mathrm{ord}_\Gamma(z_0) \geqslant -k/2 - 1 = -(k+2)/2$. This implies that $D_\phi(f) \in M_{k+2}^{\mathcal{T}}$ and we obtain that $M_*^{\mathcal{T}}$ is closed under D_ϕ. Let $X = \mathcal{H}/\Gamma$ be the modular curve. We suppose first that it is smooth, i.e., that Γ acts on $\mathcal{H}$ without fixed points. We know also that the genus of X is strictly greater than 1 under our hypothesis on X. Let $\Omega_X(z_0)$ be the sheaf of germs of differential 1-forms with only simple poles in the orbit of $\{z_0\}$. Then we have $M_k^{\mathcal{T}} = H^0(X, \Omega_X^{\otimes k/2}(z_0))$. The line bundle $\Omega_X(z_0)$ is of positive degree $2g-1$, hence is ample over X. By a classical result in algebraic geometry of curves, it follows that $\bigoplus_k H^0(X, \Omega_X^{\otimes k/2}(z_0))$ is finitely generated. This implies that the ring $M_*^{\mathcal{T}}(\Gamma, \{z_0\})$ is finitely generated, and finishes the proof of (i) in this case.

Using (i), we get that $M_*^{\mathcal{T}} \otimes \mathbb{C}[\phi]$ is finitely generated and closed under D. On the other hand, $M_*^{\mathcal{T}} \otimes \mathbb{C}[\phi]$ contains $M_*(\Gamma)$. By Theorem 2, we finish the proof of (ii) in the case of X smooth.

We now consider the case when the action of a Γ is not necessarily free. The Selberg lemma implies that there exists a normal subgroup $\Gamma' \subseteq \Gamma$ of finite index without torsion. Let $G = \Gamma/\Gamma'$, $X' = \mathcal{H}/\Gamma'$ and $\Omega_{X'}$ the sheaf of germs of differential 1-forms over X'. Let $\{z_1, \ldots, z_n\}$ be a finite set of points in X'. We now that $\deg(\Omega_{X'}(z_1 + \cdots + z_n)) = 2g' - 2 + n > 0$ for $n \geqslant 0$. This implies that the ring $M_*^{\mathcal{T}}(\Gamma', \{z_1\} + \cdots + \{z_n\})$ of tempered modular forms on Γ' around the orbits $\{z_1\}, \ldots, \{z_n\}$ (i.e., $\bigoplus_{k \geqslant 0} H^0(X', \Omega_{X'}^{\otimes k/2}(z_1 + \cdots + z_n))$ is finitely generated. Now we take $\{z_1, \ldots, z_n\} = \{z_0\}^{\Gamma', G}$ equal to the G-orbit of the Γ'-orbit of z_0. We obtain that $M_*^{\mathcal{T}}(\Gamma', \{z_0\}^{\Gamma', G})$ is finitely generated. Since G is finite, using a classical result of algebra, we get that $M_*^{\mathcal{T}}(\Gamma, \{z_0\}) = M_*^{\mathcal{T}}(\Gamma', \{z_0\}^{\Gamma', G})^G$ (G-invariant part) is also finitely generated. We finish the proof of the theorem as before. □

8.2. Construction 2

Theorem 11. *Let Γ, z_0 and ϕ be as in Theorem 9. Let $\mathcal{R}_* = \mathcal{R}_*(\Gamma, \phi)$ be the D_ϕ closure of $M_*(\Gamma)[\omega]$ and $\widetilde{\mathcal{R}}_* = \widetilde{\mathcal{R}}_*(\Gamma, \{z_0\})$ be the D-closure of $M_*(\Gamma)[\phi]$. Then:*

(i) *we have $\widetilde{\mathcal{R}}_* = \mathcal{R}_*[\phi]$, and $\widetilde{\mathcal{R}}_*$ depends only on z_0, not on ϕ;*
(ii) *the rings $\mathcal{R}_*$ and $\widetilde{\mathcal{R}}_*$ are finitely generated;*
(iii) *the ring $\widetilde{\mathcal{R}}_*$ contains the ring of quasimodular forms $\widetilde{M}_*(\Gamma)$.*

Equivalently, let us define a sequence of rings $\mathcal{R}^j$ $(j = 0, 1, 2, \ldots)$ by

$$\mathcal{R}^0 = M_*(\Gamma)[\omega], \qquad \mathcal{R}^{j+1} = \langle \mathcal{R}^i, D_\phi(\mathcal{R}^i) \rangle. \tag{16}$$

Then the main assertion of the theorem is that $\mathcal{R}^{N+1} = \mathcal{R}^N$ for some N, so that $\mathcal{R} = \bigcup_{j \geqslant 0} \mathcal{R}^j = \mathcal{R}^N$ is finitely generated.

The fact that $\mathcal{R}_*$ is closed under the operator D_ϕ and $\omega = \phi' - \phi^2 \in \mathcal{R}_*$ implies that $\mathcal{R}_*[\phi]$ is closed by the derivation D. This implies that $\widetilde{\mathcal{R}}_*(\Gamma) = \mathcal{R}_*(\Gamma)[\phi]$. We suppose that theres exits ϕ_1 and ϕ_2 quasimodular forms of weight 2 with the same normalization $\delta(\phi_1) = \delta(\phi_2) = 1$. Let $\omega_1 = \phi_1' - \phi_1^2$ and $\omega_2 = \phi_2' - \phi_2^2$. Then $\phi_1 - \phi_2 := f$ is holomorphic modular form and $\omega_1 - \omega_2 = D_\phi(f) - f^2$. This shows that $\mathcal{R}_*(\Gamma)[\phi]$ is independent of the choice of ϕ and finishes the proof of (i).

On the other hand, $\mathcal{R}_*(\Gamma)[\phi]$ contains the ring of holomorphic modular forms $M_*(\Gamma)$. Using Theorem 2, we get that $\mathcal{R}_*(\Gamma)[\phi]$ contains $\widetilde{M}_*(\Gamma)$, which implies (iii) by using (i). To prove (ii), we need only to prove that $\mathcal{R}_*(\Gamma)$ is finitely generated. We need the following lemma about finitely generated semigroups of $\mathbb{R}^2$.

Lemma 6. *Let G be a finitely generated semigroup of $\mathbb{R}^2$. We suppose that the group generated by G is a lattice $\Lambda \subset \mathbb{R}^2$ of rank 2. Let $S = \langle G \rangle_{\mathbb{R}_+} \subset \mathbb{R}^2$ be the cone generated by G. We suppose that S is convex, with angle less than π. Then there exists $A \in S$ such that $(A + S) \cap \Lambda \subset G$.*

Proof. Let $\{P_1, \dots, P_m\}$ be a system of generators of G, chosen so that the lines (OP_{m-1}) and (OP_m) bound the cone S. By changing coordinates in $\mathbb{R}^2$, we can suppose that $P_{m-1} = (1, 0)$ and $P_m = (0, 1)$. Then $\Lambda \otimes \mathbb{Q} = \mathbb{Q}^2$ and the coordinates of each P_i are rational and non-negative. In particular, for any i there exists $a_i \in \mathbb{Z}_{>0}$ such that $a_i P_i \in \mathbb{Z}_{\geqslant 0} P_{m-1} \oplus \mathbb{Z}_{\geqslant 0} P_m$. Let $P = (x, y)$ be an arbitrary point of $S \cap \Lambda$. Then there exist $(\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$ such that $P = \alpha_1 P_1 + \dots + \alpha_m P_m$. Let $\overline{\alpha_i} \in \{0, 1, \dots, a_i - 1\}$ be the reduction of α_i modulo a_i. Then we can write P as $P = \overline{\alpha_1} P_1 + \dots + \overline{\alpha_{m-2}} P_{m-2} + \beta P_{m-1} + \gamma P_m$, with $\beta, \gamma \in \mathbb{Z}$. If the abscissa of P satisfies

$$x(P) \geqslant X_0 := \max_{\{0 \leqslant a'_1 < a_1, \dots, 0 \leqslant a'_{m-2} < a_{m-2}\}} x\big(a'_1 P_1 + \dots + a'_{m-2} P_{m-2}\big)$$

then $\beta \geqslant 0$. If the ordinate of P satisfies

$$y(P) \geqslant Y_0 := \max_{\{0 \leqslant a'_1 \leqslant a_1, \dots, 0 \leqslant a'_{m-2} \leqslant a_{m-2}\}} y\big(a'_1 P_1 + \dots + a'_{m-2} P_{m-2}\big)$$

then $\gamma \geqslant 0$. We take $A = (X_0, Y_0)$. $\square$

Back to the proof of point (ii) of Theorem 11.

Proof. We suppose that $D_\phi f \neq 0$ for every non-zero $f \in M_*(\Gamma, \{z_0\})$, this is possible, since we can always add to ϕ a modular form of weight 2, without changing properties of ϕ as given in Theorem 9.

We consider the map:

$$\lambda : M_*\big(\Gamma, \{z_0\}\big) \to \mathbb{N}^2$$
$$f \mapsto \left(\frac{k(f)}{2}, \mathrm{ord}_{z_0}(f) + \frac{k(f)}{2}\right),$$

where $k(f)$ is the weight of f and $\mathrm{ord}_{z_0}(f)$ is the vanishing order of f at z_0. We write $\lambda(f) = (\lambda_1(f), \lambda_2(f))$.

Let I be the ideal of modular forms over Γ of strictly positive weight and f_i $(i = 1, \dots, d)$ be a basis of I/I^2. Then $(f_1, \dots, f_d)$ generate $M_*(\Gamma)$ as an algebra. Let $\mathcal{R}^0 = \langle f_1, \dots, f_d, \omega \rangle = \langle M_*(\Gamma), \omega \rangle$. Let $\{\mathcal{R}^j\}$ be the sequence of subrings of $M_*(\Gamma; \{z_0\})$ defined by (16). Clearly $\mathcal{R}^j$ is finitely generated for every j. We will prove that this sequence is stationary from a certain rank N. The ring $\mathcal{R} = \mathcal{R}^N$ is then of the form given in the theorem.

Note that for every j, one has

$$f \in \mathcal{R}^j \quad \text{and} \quad \lambda_2(f) \geqslant \lambda_1(f) \quad \Rightarrow \quad f \in M_*(\Gamma), \tag{17}$$

i.e., all $f \in \mathcal{R}^j$ with $\lambda(f)$ above the line $L = \{(x, x) \mid x \in \mathbb{R}\}$ automatically belongs to $M_*(\Gamma)$. Indeed, f is holomorphic outside of $\{z_0\}$, and $\lambda_2(f) \geqslant \lambda_1(f)$ says that it is holomorphic also in $\{z_0\}$.

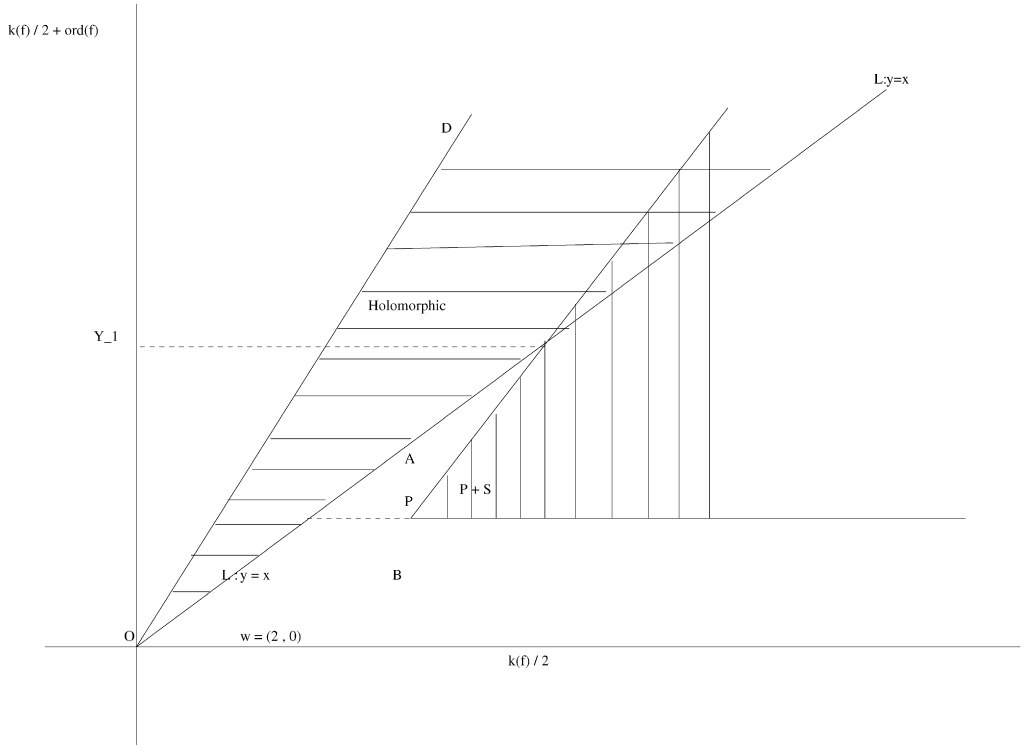

Fig. 1.

The cone $S = \langle \lambda(\mathcal{R}^0) \rangle_{\mathbb{R}_+} \subset \mathbb{R}^2$ generated by $\lambda(\mathcal{R}^0)$ is bounded above by the half-line $D = \{(x, y) \in \mathbb{R}^2_+ \mid y = (2\kappa + 1)x\}$ (consequence of Lemma 5) and below by the x-axis since $\lambda(\omega) = (2, 0)$, see Fig. 1. Lemma 5 implies that if $\lambda(f) = (x, y)$, then (for $D_\phi f \neq 0$) we have $\lambda(D_\phi f) = (\tilde{x}, \tilde{y})$ with $(\tilde{x}, \tilde{y}) = (x+1, y)$ if (x, y) lies below the line D and $(\tilde{x}, \tilde{y}) = (x+1, y+1+\theta)$ with $0 \leqslant \theta \leqslant 2\kappa$ if (x, y) lies on D. It follows that $\langle \lambda(\mathcal{R}^j) \rangle_{\mathbb{R}_+} = S$ for all j.

By the Riemann–Roch theorem, it is clear that the group generated by $\lambda(\mathcal{R}^0)$ is equal to $\mathbb{Z}^2$.

We apply the lemma to the semigroup $\lambda(\mathcal{R}^0)$, and deduce from it that there exists $P = (X_0, Y_0) \in \lambda(\mathcal{R}^0)$ such that $(P + S) \cap \mathbb{Z}^2 \subset \lambda(\mathcal{R}^0)$.

We have the essential property

$$F \in M_*\big(\Gamma; \{z_0\}\big) \quad \text{and} \quad \lambda(F) \in (P + S) \cap \mathbb{Z}^2 \quad \Rightarrow \quad F \in \mathcal{R}^0. \tag{18}$$

Indeed there exists $g \in \mathcal{R}^0$ such that $\lambda(F) = \lambda(g)$, i.e., g has the same weight and exactly the same order of zero or pole at z_0 as F. So there exists a linear combination F_1 of F and g such that $\lambda_1(F_1) = \lambda_1(F)$ and $\lambda_2(F_1) > \lambda_2(F)$. By reiterating this construction, we obtain a sequence of points $\lambda(F_i)$ which for i large are above the line L. By (17) this implies that F_i is holomorphic, so is in $M_*(\Gamma) \subset \mathcal{R}^0$. It follows that F itself belongs to $\mathcal{R}^0$.

The ring $\mathcal{R}_*$ equals $\bigcup_{j \geqslant 0} \mathcal{R}^j$. For every $f \in \mathcal{R}_*$, we set $j_0(f) = \min\{j \in \mathbb{Z}_{\geqslant 0} \mid f \in \mathcal{R}^j\}$. If $\lambda(f)$ is above the line L, then, (17) implies that $j_0(f) = 0$. If $\lambda(f) \in P + S$ then again $j_0(f) = 0$, by (18). Let Y_1 be the ordinate of the intersection point of L and the line parallel to D and containing P.

We now prove by downward induction on ν the following property:

$$\mathcal{E}(\nu):\quad \text{The set } \{j_0(f) \mid f \in \mathcal{R}_*,\ \lambda_2(f) \geqslant \nu\} \text{ is finite.}$$

We just saw that $\lambda_2(f) \geqslant Y_1$ implies $j_0(f) = 0$. Therefore $\mathcal{E}(Y_1)$ is true. Suppose that $\mathcal{E}(\nu + 1)$ is true, for a certain $\nu \leqslant Y_1 - 1$. We want to establish $\mathcal{E}(\nu)$. If $\nu \notin \lambda_2(\mathcal{R}_*)$ then there is nothing to prove. If $\nu \in \lambda_2(\mathcal{R}_*)$ then there exists a non-zero $H \in \mathcal{R}_*$ with $\lambda_2(H) = \nu$. Let μ be the minimum of values of $\lambda_1(f)$ with $f \in \mathcal{R}_* - \{0\}$ and $\lambda_2(f) = \nu$, and fix $H_\nu \in \mathcal{R}_* - \{0\}$ with $\lambda(H_\nu) = (\mu, \nu)$. Now, if $f \in \mathcal{R}_*$ and $\lambda_2(f) = \nu$ then there exists $i \geqslant 0$ such that $\lambda(f) = \lambda(\omega^i H_\nu)$ or $= \lambda(\omega^i D_\phi H_\nu)$. Hence, there exists a linear combination g of f and $\omega^i H_\nu$ (or $\omega^i D_\phi H_\nu$) with $\lambda_2(g) \geqslant \nu + 1$. By induction, we get that $j_0(g) \leqslant j_0(H_{\nu+1}) + 1 \leqslant N = \max_{n \leqslant Y_1}(j_0(H_n) + 1)$. This implies that $\mathcal{E}(\nu)$ is true and finishes the induction. Finally, we deduce that $\mathcal{E}(0)$ is true, which is equivalent to point (ii) of Theorem 11. Moreover $\mathcal{R}_* = \mathcal{R}^N$. □

8.3. Comparison

Using the same proof as point (i) of Theorem 10, we show that $R_*(\Gamma, \{z_0\}) \subseteq M_*^{\mathcal{T}}(\Gamma, \{z_0\})$. On the other hand, we prove for the example $\Gamma = \Gamma_6$ and $z_0 = i$ (see next section) that there exists an element $f \in M_4^{\mathcal{T}}(\Gamma_6, \{i\})$ which is not a polynomial on holomorphic modular forms and on the special meromorphic modular form ω of weight 4. Moreover, f is not in the image of D_ϕ, so $R_*(\Gamma_6, i) \subsetneq M_*^{\mathcal{T}}(\Gamma_6, \{i\})$. This implies that $\widetilde{\mathcal{R}}_*(\Gamma_6, \{i\}) \subsetneq M_*^{\mathcal{T}}(\Gamma_6, \{i\}) \otimes \mathbb{C}[\phi]$. This proves that in general, $M_*^{\mathcal{T}}(\Gamma, \{z_0\}) \otimes \mathbb{C}[\phi]$ does not coincide with its subring $\widetilde{\mathcal{R}}_*(\Gamma, \{z_0\})$.

9. Examples

In this section we illustrate the results of this paper with an explicit example of a cocompact group Γ_6. We construct a finitely generated ring $\widetilde{\mathcal{R}}_*(\Gamma_6)$ of meromorphic quasimodular forms of positive weight containing the ring $\widetilde{M}_*(\Gamma_6)$ of holomorphic quasimodular forms.

Let $B = (-1, 3)_{\mathbb{Q}}$ be the quaternion algebra of discriminant 6, defined over $\mathbb{Q}$ with a basis $(1, i, j, ij)$ and relations $i^2 = -1$, $j^2 = 3$, $ij + ji = 0$. Let N be the norm defined on B by $N(x + iy + jz + ijt) = x^2 + y^2 - 3z^2 - 3t^2$. Let $A_6 = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+ij}{2}$, which is a maximal order in B. We can also define A_6 as the set

$$\left\{\frac{x + yi + zj + tij}{2} \;\middle|\; x, y, z, t \in \mathbb{Z},\ x \equiv y \equiv z \equiv t \pmod 2\right\}.$$

We denote by A_6^1 the multiplicative group of units of norm 1 in A_6. We can embed A_6^1 into $\mathrm{SL}(2, \mathbb{R})$ by $i \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $j \mapsto \left(\begin{smallmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{smallmatrix}\right)$. We denote by Γ_6 the image of A_6^1 in $\mathrm{PSL}(2, \mathbb{R})$. See [1] and [3] for more details about the construction of cocompact groups and fundamental domains for Shimura curves. We show in [5] that the ring of modular forms on Γ_6 is generated by three forms A, B and C of weight 4, 4 and 10, respectively, with the unique relation

$$C^2 = A\left(B^2 - 4A^2\right)\left(B^2 + 12A^2\right). \tag{19}$$

Using the formula for the number of zeros of a modular form, we see that A vanishes only at i, the elliptic point of order 2 in $\mathcal{H}/\Gamma_6$. We define $\phi = \frac{A'}{4A}$. Then ϕ is a meromorphic quasimodular form of weight 2 with $\delta(\phi) = 1$ and has all its poles in the Γ_6-orbit of i. Let $\omega = \phi' - \phi^2$. Then

ω is a meromorphic modular form of weight 4 on Γ_6 with no poles outside the Γ_6-orbit of i. We show in [5] that $\omega = -2A - \frac{3}{2}B^2/A$ and that one has the differential system

$$\text{(S):}\quad \begin{cases} D_\phi(A) = 0, \\ D_\phi(B) = -2C/A, \\ D_\phi(C) = -4B^3 - 16A^2B, \\ D_\phi(\omega) = 6BC/A^2, \end{cases}$$

where $D_\phi(f) = f' - k\phi f$ for $f \in M_k$, as usual.

Let $\mathcal{R}^0 := \langle M_*(\Gamma_6), \omega\rangle$ be the graded ring generated by holomorphic modular forms on Γ_6 and ω. From (S) we see that $\mathcal{R}^1 := \langle \mathcal{R}^0, D_\phi\mathcal{R}^0\rangle = \langle \mathcal{R}_0, C/A, BC/A^2\rangle$. Since $D_\phi(C/A) = -16AB - 4B^3/A$ and B^2/A is a linear combination of A and ω, we get that $D_\phi(C/A) \in \mathcal{R}_0 \subseteq \mathcal{R}_1$. On the other hand, $D_\phi(BC/A^2) = -16B^2 - 4B^4/A^2 - 2C^2/A^3$. Using the relation

$$\frac{C^2}{A^3} = \left(\frac{B^2}{A} - 4A\right)\left(\frac{B^2}{A} + 12A\right)$$

which is another way of writing the relation (19), we get that

$$D_\phi\bigl(BC/A^2\bigr) \in \mathcal{R}_0 \subseteq \mathcal{R}_1.$$

In other words the ring $\mathcal{R}_* := \mathcal{R}^1$ is closed under the derivation D_ϕ. Finally, $\widetilde{\mathcal{R}}_* = \mathcal{R}_* \otimes \mathbb{C}[\phi] = \mathcal{R}_*[\phi] = \mathbb{C}[A, B, B^2/A, C/A, BC/A^2, \phi]$ is closed under the usual derivation D and contains $M_*(\Gamma_6)$. Using Theorem 2, we deduce that $\widetilde{\mathcal{R}}_*$ contains $\widetilde{M}_*(\Gamma_6)$. On the other hand, we have $B^3/A^2 \in M_4^{\mathcal{T}}(\Gamma_6, \{i\}) \subset M_*^{\mathcal{T}}(\Gamma_6, \{i\}) \otimes \mathbb{C}[\phi]$ and from the description of $\widetilde{\mathcal{R}}$ just given we see that $B^3/A^2 \notin \widetilde{\mathcal{R}}$. This implies that $M_*^{\mathcal{T}}(\Gamma_6, \{i\})$ strictly contains $\mathcal{R}_*(\Gamma_6, \{i\})$ and $M_*^{\mathcal{T}}(\Gamma_6, \{i\}) \otimes \mathbb{C}[\phi]$ strictly contains $\widetilde{R}_*(\Gamma_6, \{i\})$.

The last example shows that Theorem 11 can be true with $N = 1$. Using the same group Γ_6, we will show that if we change the CM point $z_0 = i$ to a new CM point $z_1 = \frac{\sqrt{3}+i\sqrt{6}}{3}$, then N changes from 1 to 3. We use a new basis of the ring of modular forms over Γ_6. Performing the calculations, we find that $A + B = L$ is a holomorphic modular form of weight 4 which vanishes only in z_1. Let $p = L'/4L$, a quasimodular form of weight 2. The modular form $C \in M_{10}(\Gamma_6)$ is equal to $4[L, A]_1$. Let $P = p' - p^2$. Then P is a modular form of weight 4 over Γ_6 without poles outside the Γ_6-orbit of z_1, given explicitly by

$$P = -\frac{5}{16}A^3/L^2 + \frac{17}{16}A^2/L - \frac{31}{32}A + \frac{1}{4}L.$$

The unique relation between L, A and C is

$$(*)\quad C^2 = A^5 - \frac{17}{3}A^4L + \frac{65}{6}A^3L^2 + 2AL^4.$$

We have the following differential system:

$$(\mathrm{S}'):\quad \begin{cases} D_p(L) = 0, \\ D_p(A) = C/L, \\ D_p(C) = -8PAL - \dfrac{17}{6}A^3 + \dfrac{17}{2}A^2L + 6AL^2 + L^3, \\ D_p(P) = -\dfrac{15}{16}A^2LC + \dfrac{17}{8}CA/L^2 - \dfrac{31}{32}C/L. \end{cases}$$

In this example $\mathcal{R}^0 = \langle A, L, C, P\rangle$.

Using (S′) and ($*$) repeatedly, we get that $\mathcal{R}^1 = \langle \mathcal{R}^0, C/L, CA/L^2\rangle$, $\mathcal{R}^2 = \langle \mathcal{R}^1, A/L\rangle$, $\mathcal{R}^3 = \langle \mathcal{R}^2, C/L^2\rangle$, and $\mathcal{R}^3 = \mathcal{R}^4$ and hence $\mathcal{R}^3 = \mathcal{R}_*$.

We can also take z_0 to be an arbitrary point in $\mathcal{H}/\Gamma_6 - \{i\}$. We obtain essentially the same structure. We can find a constant λ such that $L = A - \lambda B$ vanishes only at z_0 and we find $N = 3$, i.e., $\mathcal{R}^3 = \mathcal{R}^4 = \mathcal{R}_*$, as in the last example.

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