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On an Inequality of Bynum and Drew

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An extension of the Bynum–Drew inequality in the abstract L_p space, $1<p\leqslant 2$, is presented. The extended inequality is applied to show the strong uniqueness of best approximations and existence of fixed points for uniformly Lipschitzian mappings and to prove an estimate for the self-Jung constants. © 1990 Academic Press, Inc

1. The Extended Inequality

Let X be an *abstract* L_p *space* [3] with $1<p\leqslant 2$, i.e., let X be a Banach lattice for which $\|x+y\|^p=\|x\|^p+\|y\|^p$, whenever $x, y\in X$ and $x\wedge y=0$. Then the Bynum–Drew inequality

$$\left\|\frac{x+y}{2}\right\|^2\leqslant\frac{1}{2}(\|x\|^2+\|y\|^2)-(p-1)\left\|\frac{x-y}{2}\right\|^2 \tag{1}$$

holds for all x, y in X [2]. Moreover, $p-1$ is the best constant in inequality (1) provided that there exist at least two nonzero elements $u, v\in X$ such that $u\wedge v=0$. This inequality can be extended as follows.

THEOREM 1. *If X is an abstract L_p space with $1<p\leqslant 2$, then*

$$\|(1-t)x+ty\|^2\leqslant(1-t)\|x\|^2+t\|y\|^2-(p-1)t(1-t)\|x-y\|^2 \tag{2}$$

for all $x, y\in X$ and $0<t<1$.

Proof. Let T be the set of all $t\in(0,1)$ such that inequality (2) holds for all $x, y\in X$. By (1) and (2) we conclude that $\frac{1}{2}\in T$ and

(i) $1-t\in T$ whenever $t\in T$.

Moreover, we have

(ii) $ts\in T$ whenever $t, s\in T$.

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Indeed, by applying inequality (2) for $t, s \in T$ twice, we get

$$\begin{aligned} t(\|x\|^2 - \|y\|^2) &\leqslant [\|x\|^2 - \|x + t(y-x)\|^2] - (p-1)\, t(1-t)\, \|x-y\|^2 \\ &\leqslant [(\|x\|^2 - \|x + st(y-x)\|^2)/s - (p-1)(1-s)\, \|t(x-y)\|^2] \\ &\quad - (p-1)\, t(1-t)\, \|x-y\|^2 \\ &= [\|x\|^2 - \|(1-ts)\, x + tsy\|^2]/s - (p-1)\, t(1-ts)\, \|x-y\|^2, \end{aligned}$$

which yields (ii). Now, one can apply (i) and (ii) to show, by an easy induction on n, that T includes all dyadic rationals $t = k/2^n$ $(1 \leqslant k < 2^n$, $n = 1, 2, \ldots)$. Hence the closure of T is equal to $[0, 1]$, which completes the proof. ∎

If the abstract L_p space X contains two nonzero elements u, v such that $u \wedge v = 0$, then the largest possible constant in inequality (2) is equal to $p-1$. This means that the real-valued function f, defined by the formula

$$f(x, y, t) = (1-t)\|x\|^2 + t\, \|y\|^2 - \|(1-t)\, x + ty\|^2$$

for all x, y in the space X and $0 < t < 1$, satisfies the identity

$$\inf\left\{\frac{f(x, y, t)}{\|x-y\|^2} : x, y \in X, x \neq y\right\} = (p-1)\, t(1-t) \tag{3}$$

which seems to be independently interesting. For the proof, we choose the elements $u_s, v_s \in X$ such that $u_s \wedge v_s = 0$, $\|u_s\| = 1 + s$, and $\|v_s\| = 1 - s$ $(0 \leqslant s < 1)$, and define $x_s = u_s + v_s$ $(0 < s < 1)$ and $y = u_0 + v_0$. Then we have $\|x_s\|^p = (1+s)^p + (1-s)^p$, $\|y\|^p = 2$, $\|x_s - y\|^p = 2s^p$, and $\|(1-t)\, x_s + ty\|^p = [(1-t)(1+s) + t]^p + [(1-t)(1-s) + t]^p$. Hence one can use L'Hôpital's theorem twice to obtain that

$$f(x_s, y, t)/\|x_s - y\|^2 \to (p-1)\, t(1-t)$$

as $s \to 0$, which in view of Theorem 1 finishes the proof of identity (3).

2. Applications

If M is a nonempty subset of an abstract L_p space X, then a mapping $T\colon M \to M$ is said to be uniformly Lipschitzian with a uniform Lipschitz constant $k > 1$ if

$$\|T^n x - T^n y\| \leqslant k\, \|x - y\|$$

for all x, y in M and all integers $n \geqslant 1$. By Theorem 1 we can apply

Theorem 4.2 from [4] to get the following improvement of the corresponding part of Theorem 4.3 from [5].

COROLLARY 1. *If M is a nonempty closed convex bounded subset of an abstract L_p space, $1 < p \leqslant 2$, then a uniformly Lipschitzian mapping $T: M \to M$ with a uniform Lipschitz constant $k < \sqrt{p}$ has a fixed point in M.*

For the other applications of Theorem 1, consider the class $\mathfrak{C}$ of all bounded nonempty subsets C of X and assume that a countable covering $[C_n] = \{C_1, C_2, ...,\}$ of each subset $C \in \mathfrak{C}$ is prescribed, i.e., $C = \bigcup C_n$. In the following we shall simply write $C = [C_n] \in \mathfrak{C}$ to denote that $[C_n]$ is a covering of $C \in \mathfrak{C}$. Then following [7] an element $x_M \in M$ is called an *asymptotic Chebyshev center of* $C = [C_n]$ with respect to a nonempty closed convex subset M of X if

$$f_C(x_M) = \inf_{x \in M} f_C(x),$$

where the convex functional $f_C: X \to [0, \infty)$ is defined by

$$f_C(x) = \limsup_{n \to \infty} \sup_{z \in C_n} \|x - z\|.$$

Since an abstract L_p space X is uniformly convex, it follows that such an element x_M is uniquely determined [7] for any $1 < p < \infty$.

COROLLARY 2. *If X is an abstract L_p space, $1 < p \leqslant 2$, and x_M is an asymptotic Chebyshev center of $C = [C_n] \in \mathfrak{C}$ with respect to a nonempty closed convex subset M of X, then*

$$[f_C(x_M)]^2 \leqslant [f_C(x)]^2 - (p-1)\|x_M - x\|^2$$

for all x in M.

Proof. Using the definition of x_M combined with Theorem 1 and the convexity of M, we get

$$\begin{aligned}
&[f_C(x_M)]^2 \\
&\quad \leqslant [f_C((1-t)\,x_M + tx)]^2 \leqslant \limsup_{n \to \infty} \sup_{z \in C_n} [(1-t)\|x_M - z\|^2 \\
&\quad + t\,\|x - z\|^2 - (p-1)\,t(1-t)\|x_M - x\|^2] \leqslant (1-t)[f_C(x_M)]^2 \\
&\quad + t[f_C(x)]^2 - (p-1)\,t(1-t)\|x_M - x\|^2.
\end{aligned}$$

Hence one can divide both sides of the derived inequality by t and pass to the limit $t \to 0$ in order to get the desired inequality. ∎

Note that, in the particular case when $C_n = C = z \in X$, the element $m = x_M$ is the best approximation in M to z. Hence Corollary 2 yields the following improvement of the corresponding parts of Theorem 3.2 from [5] and Corollary 4.1 from [6] which concern the strong uniqueness of best L_p-approximations.

COROLLARY 3. *If $m \in M$ is a best approximation in M to an element $z \in X$, then we have*

$$\|z - m\|^2 \leqslant \|z - x\|^2 - (p-1)\|m - x\|^2$$

for all x in M.

Finally, let us define the constant

$$J_s(X) = 2 \sup_{C} \inf_{x \in C} f_C(x),$$

where the supremum is taken over all closed convex subsets $C = [C_n]$ of X with the diameter diam $(C) = 1$. In particular, if $C_n = C$ for all n then the constant $J_s(X)$ is the self-Jung constant which has been introduced and studied recently by Amir in [1].

COROLLARY 4. *If X is an abstract L_p space, $1 < p \leqslant 2$, then we have $J_s(X) \leqslant 2/\sqrt{p}$.*

Proof. Let $C = [C_n]$ be a closed convex subset with diam $(C) = 1$. By Corollary 2 we have

$$[f_C(x_C)]^2 + (p-1)\|x_C - x\|^2 \leqslant [f_C(x)]^2$$

for all x in C. Taking the supremum over $x \in C$ on both sides of this inequality, and then using twice the obvious inequality

$$f_C(x) \leqslant \sup_{z \in C} \|x - z\|,$$

we get

$$p[f_C(x_C)]^2 \leqslant \text{diam}\,(C) = 1.$$

Now one can take the supremum over C to complete the proof. ∎

It should be noticed that, in view of Proposition 2.2 and Corollary 2.11 from [1], the estimate of self-Jung constants can not be improved for an abstract infinite-dimensional L_p space, whenever either $p = 2$ or $p \to 1$.

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