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Blow-up phenomena for a family of Burgers-like equations

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ABSTRACT

By introducing a stress multiplier we derive a family of Burgers-like equations. We investigate the blow-up phenomena of the equations both on the real line $\mathbb{R}$ and on the circle $\mathbb{S}$ to get a comparison with the Degasperis–Procesi equation. On the line $\mathbb{R}$, we first establish the local well-posedness and the blow-up scenario. Then we use conservation laws of the equations to get the estimate for the L^∞-norm of the strong solutions, by which we prove that the solutions to the equations may blow up in the form of wave breaking for certain initial profiles. Analogous results are provided in the periodic case. Especially, we find differences between the Burgers-like equations and the Degasperis–Procesi equation, see Remark 4.1.

1. Introduction

In [14], D.D. Holm and M.F. Staley studied the following 1D evolution equation that describes the balance between convection and stretching for small viscosity in the dynamics of one-dimensional nonlinear waves in fluids:

$$m_t + um_x + bu_x m = \nu m_{xx}, \tag{1.1}$$

where $u = g * m$ denotes $u(x) = \int_{\mathbb{R}} g(x-y)m(y)\,dy$.

Setting $g(x) = e^{-\frac{|x|}{\alpha}}$, $m = u - \alpha^2 u_{xx}$, Eq. (1.1) becomes the viscous b-family of equations. Expressed solely in terms of velocity $u(t,x)$, (1.1) reads as

$$\begin{aligned} u_t + (b+1)uu_x - \nu u_{xx} &= \alpha^2(u_{xxt} + uu_{xxx} + bu_x u_{xx} - \nu u_{xxxx}) \\ &= \alpha^2 \partial_x\left(u_{xt} + uu_{xx} - \nu u_{xxx} + \frac{b-1}{2}u_x^2\right) \\ &= \alpha^2 \partial_x^2\left(u_t + uu_x - \nu u_{xx} + \frac{b-3}{2}u_x^2\right). \end{aligned}$$

When $\alpha \to 0$, equation above reduces to

$$u_t + (b+1)uu_x - \nu u_{xx} = 0,$$

and then recovers formally the usual Burgers equation either by rescaling dimensions or by setting $b = 0$. The viscous b-family of equations may also be rearranged into the following convective form,

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$$\left(1-\alpha^2\partial_x^2\right)(u_t+uu_x-\nu u_{xx})=-\partial_x\left(\frac{b}{2}u^2+\frac{3-b}{2}\alpha^2u_x^2\right). \tag{1.2}$$

For purpose of comparison, the authors in [14] introduced the stress multiplier λ as a parameter. When $\lambda\neq 1$, one deforms the convective form of the b-family of Eq. (1.2) into the following family of Burgers-like equations,

$$(u_t+uu_x-\nu u_{xx})=-\lambda p_x, \quad \text{with } \left(1-\alpha^2\partial_x^2\right)p=\frac{b}{2}u^2+\frac{3-b}{2}\alpha^2u_x^2. \tag{1.3}$$

Obviously, when $\lambda=0$, Eq. (1.3) becomes the Burgers equation, and when $\lambda=1$, it is the viscous b-family of equations.

The authors' interest there was in seeking solutions of the Burgers-like equation (1.3), either on the real line and vanishing at spatial infinity or in a periodic domain, for various values of b, α, λ, and ν. Under these boundary conditions, (1.3) becomes the convective form (1.2) of the viscous b-family when $\lambda\to 1$, and reduces to the usual Burgers equation when $\lambda=0$. They were also concerned with the evolution of peakon initial data upon introducing the parameters ν and λ, under the condition $(3-b)\lambda=1$. This relation between b and λ ensures that Eq. (1.3) controls the α-weighted H^1-norm of the velocity $\|u\|^2_{H^1_\alpha}=\int_{-\infty}^{\infty}(u^2+\alpha^2u_x^2)\,dx$, $\alpha\neq 0$. Obviously, the Camassa–Holm equation ($b=2$, $\lambda=1$, in (1.3)) satisfies this condition. The Camassa–Holm equation has been thoroughly studied after it was first derived physically by Camassa and Holm in 1993, see [1]. For more results, we refer readers to [2,4–8,10,18] and references therein.

In our paper, we study Eq. (1.3) in the case $b=3$, $\nu=0$, $\alpha=1$, i.e.,

$$u_t+uu_x=-\lambda p_x, \quad \text{with } \left(1-\partial_x^2\right)p=\frac{3}{2}u^2. \tag{1.4}$$

Expressed solely in terms of velocity $u(t,x)$, it becomes

$$u_t-u_{xxt}+(3\lambda+1)uu_x=3u_xu_{xx}+uu_{xxx}. \tag{1.5}$$

Eq. (1.4) can be viewed as an extension of the Degasperis–Procesi equation. It includes formally the Burgers equation as its special case. As we will see later, such an extension helps us get a family of equations which have several properties in common with the Degasperis–Procesi equation. When $\lambda=1$ Eq. (1.4) recovers the Degasperis–Procesi equation, which has also been thoroughly studied in recent years. For example, Yin established the local well-posedness to the equation with initial data $u_0\in H^s(\mathbb{R})$, $s>\frac{3}{2}$ on the line [21] and on the circle [22] and derived the blow-up results. Lenells classified all the traveling wave solutions in [19]. Henry [15] and Mustafa [20] proved that the smooth solutions to Degasperis–Procesi equation have infinite propagation speed. Besides, Coclite and Karlsen [9] derived the global existence results for the entropy weak solutions belonging to various classes.

Motivated by [11], where the authors proved that $\int_{\mathbb{R}} v(u-u_{xx})\,dx$ is a conserved quantity with $u=4v-\partial_x^2v$, we find that $\int_{\mathbb{R}} v(u-u_{xx})\,dx$ is also a conserved quantity when $u=(3\lambda+1)v-\partial_x^2v$, $\lambda>-\frac{1}{3}$. On the real line, we use this conservation law to establish the a priori estimate for the L^∞-norm of strong solutions. Combining the a priori estimate with the approaches used in analyzing the blow-up phenomena of the Degasperis–Procesi equation in [17], we can establish blow-up results for Eq. (1.4). We will see that the larger λ is, the livelier steepening it produces and the shorter the lifespan of the blow-up solution is. In the periodic case, we provide similar results. Besides, we find that there are differences in the blow-up phenomena between (1.4) and the Degasperis–Procesi equation. We know that the solution of the Degasperis–Procesi equation exists globally provided that $y_0=u_0-u_{0xx}$ is of one sign, where $u_0\in H^s(\mathbb{S})$, $s>\frac{3}{2}$. However, it is not true any more for Eq. (1.3), see Theorem 4.2.

The remainder of our paper is organized as follows. In Section 2, we establish the local well-posedness of Eq. (1.4) with initial data $u_0\in H^s$, $s>\frac{3}{2}$ and derive the blow-up scenario of the strong solutions. In Section 3, we obtain the a priori estimate for the L^∞-norm of the strong solutions, by which we provide two blow-up results. Section 4, the last section, is mainly devoted to establish the analogous results of Eq. (1.4) in the periodic case.

2. Local well-posedness and blow-up scenario on the line

In this section, by applying Kato's theory we first establish the local well-posedness of the following equation

$$\begin{cases} u_t+uu_x=-\lambda\partial_x\left(1-\partial_x^2\right)^{-1}\left(\frac{3}{2}u^2\right), & t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R}. \end{cases} \tag{2.1}$$

Setting $G(x):=\frac{1}{2}e^{-|x|}$, $x\in\mathbb{R}$, then $(1-\partial_x^2)^{-1}f=G*f$ for all $f\in L^2(\mathbb{R})$. Therefore we can rewrite Eq. (2.1) in the following equivalent form

$$\begin{cases} u_t+uu_x=-\lambda\partial_xG*\left(\frac{3}{2}u^2\right), & t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R}. \end{cases} \tag{2.2}$$

We now give our local well-posedness result, and after that, we will present a precise blow-up scenario of the strong solutions to Eq. (2.1).

Theorem 2.1. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. There exists a maximum $T = T(\lambda, u_0) > 0$, and a unique solution u to Eq.* (2.1)*, which depends continuously on initial data u_0 such that*

$$u = u(\cdot, u_0) \in C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big).$$

Moreover, T may be chosen to be independent of s in the sense that if $u_0 \in H^r(\mathbb{R})$ for some $r \neq s$, $r > \frac{3}{2}$ too, then for the same T,

$$u \in C\big([0,T); H^r(\mathbb{R})\big) \cap C^1\big([0,T); H^{r-1}(\mathbb{R})\big).$$

Furthermore, given $u_0 \in H^\infty(\mathbb{R}) = \bigcap_{r \geqslant 0} H^r(\mathbb{R})$, then $u \in C([0,T), H^\infty(\mathbb{R}))$.

Proof. Define the operator $A(u) := u\partial_x$. Set $f(u) = -\lambda\partial_x(1-\partial_x^2)^{-1}(\frac{3}{2}u^2)$, $Y = H^s(\mathbb{R})$, $X = H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$, and $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. Similar to the proof of Theorem 2.2 and the proof of Theorem 2.3 in [21], we can prove Theorem 2.1 by applying Kato's theory, and we omit the details for brevity. □

The maximum value of T in Theorem 2.1 is called the lifespan of the solution. In general, if $T < \infty$ in the sense that $\lim_{t\to T} \|u(\cdot,t)\|_{H^s(\mathbb{R})} = \infty$, we say that the solution blows up in finite time.

Next, we recall a useful lemma.

Lemma 2.1. *(See [16].) If $s > 0$, then*

$$\big\|[\Lambda^s, f]g\big\|_{L^2(\mathbb{R})} \leqslant C\big(\|\partial_x f\|_{L^\infty(\mathbb{R})}\big\|\Lambda^{s-1}g\big\|_{L^2(\mathbb{R})} + \big\|\Lambda^s f\big\|_{L^2(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}\big),$$

where c is a constant depending only on s.

Before providing the precise blow-up scenario of strong solutions to Eq. (2.1), we prove the following useful result.

Theorem 2.2. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and T be the lifespan of the corresponding solution with initial data u_0. If there exists $M > 0$ such that $\|u_x(t,x)\|_{L^\infty(\mathbb{R})} \leqslant M$, $t \in [0,T)$, then the $H^s(\mathbb{R})$-norm of $u(t,\cdot)$ does not blow up on $[0,T)$.*

Proof. Let u be the solution to Eq. (2.1) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and let T be the lifespan of the solution u. Applying the operator Λ^s to Eq. (2.2), multiplying by $\Lambda^s u$, and integrating over $\mathbb{R}$, we have

$$\frac{d}{dt}\|u\|_s^2 = -2(uu_x, u)_s + 2\big(u, f(u)\big)_s. \tag{2.3}$$

Here, $\|\cdot\|_s$ and $(\cdot,\cdot)_s$ denote the norm and the inner product in $H^s(\mathbb{R})$, and

$$f(u) = -\lambda\partial_x\big(1-\partial_x^2\big)^{-1}\left(\frac{3}{2}u^2\right) = -3\lambda\big(1-\partial_x^2\big)^{-1}(uu_x).$$

Using Lemma 2.1, we have

$$\begin{aligned}
\big|(uu_x, u)_s\big| &= \big|\big(\Lambda^s(uu_x), \Lambda^s u\big)_0\big| \\
&= \big|\big([\Lambda^s, u]u_x, \Lambda^s u\big)_0 + \big(u\Lambda^s u_x, \Lambda^s u\big)_0\big| \\
&\leqslant \big\|[\Lambda^s, u]u_x\big\|_{L^2}\big\|\Lambda^s u\big\|_{L^2} + \frac{1}{2}\big|\big(u_x\Lambda^s u, \Lambda^s u\big)_0\big| \\
&\leqslant \left(C\|u_x\|_{L^\infty} + \frac{1}{2}\|u_x\|_{L^\infty}\right)\|u\|_s^2 \\
&\leqslant C\|u_x\|_{L^\infty}\|u\|_s^2.
\end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned}
\big|\big(f(u), u\big)_s\big| &= 3|\lambda|\big|\big(\Lambda^s\big(1-\partial_x^2\big)^{-1}(uu_x), \Lambda^s u\big)_0\big| \\
&\leqslant 3|\lambda|\big|\big(\Lambda^{s-1}(uu_x), \Lambda^{s-1}u\big)_0\big| \\
&\leqslant 3|\lambda|\big|\big([\Lambda^{s-1}, u]u_x, \Lambda^{s-1}u\big)_0 + \big(u\Lambda^{s-1}u_x, \Lambda^{s-1}u\big)_0\big| \\
&\leqslant C\left(\big\|[\Lambda^{s-1}, u]u_x\big\|_{L^2}\big\|\Lambda^{s-1}u\big\|_{L^2} + \frac{1}{2}\big|\big(u_x\Lambda^{s-1}u, \Lambda^{s-1}u\big)_0\big|\right) \\
&\leqslant C\|u_x\|_{L^\infty}\|u\|_s^2.
\end{aligned} \tag{2.5}$$

Here C may denote different positive constant from line to line. Combining (2.3) with (2.4), (2.5), we obtain

$$\frac{d}{dt}\|u\|_s^2 \leqslant CM\|u\|_s^2.$$

From Gronwall's inequality, we have

$$\|u\|_s^2 \leqslant e^{CMt}\|u_0\|_s^2,$$

and this completes the proof. □

Now we give the precise blow-up scenario for the problem (2.1).

Theorem 2.3. *Assume that $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, $\lambda \geqslant \frac{1}{6}$, then the solution blows up in finite time if and only if the slope of the solution becomes unbounded from below in finite time.*

Proof. By Theorem 2.1 and the density argument, we only need to prove the theorem for $s = 3$. Let $T > 0$ be the lifespan of the solution u with initial data $u_0 \in H^3(\mathbb{R})$. Setting $y = u - u_{xx}$, we rewrite Eq. (2.1) in the following equivalent form

$$\begin{cases} y_t + uy_x + 3u_x y = (3-3\lambda)uu_x, & t > 0,\ x \in \mathbb{R}, \\ y(0,x) = u_0(x) - u_{0xx}(x), & x \in \mathbb{R}. \end{cases} \tag{2.6}$$

Multiplying Eq. (2.6) by y and integrating by parts, we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} y^2\,dx &= -2\int_{\mathbb{R}} uyy_x\,dx - 6\int_{\mathbb{R}} u_x y^2\,dx + (6-6\lambda)\int_{\mathbb{R}} uu_x y\,dx \\ &= -5\int_{\mathbb{R}} u_x y^2\,dx - (3\lambda-3)\int_{\mathbb{R}} u_x^3\,dx. \end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} y^2\,dx &= -5\int_{\mathbb{R}} u_x y^2\,dx + (6-6\lambda)\int_{\mathbb{R}} u_x y^2\,dx + (6-6\lambda)\int_{\mathbb{R}} u_x u_{xx} y\,dx \\ &= (1-6\lambda)\int_{\mathbb{R}} u_x y^2\,dx + (3\lambda-3)\int_{\mathbb{R}} u_x^3\,dx + (6\lambda-6)\int_{\mathbb{R}} u_x u_{xx}^2\,dx. \end{aligned} \tag{2.8}$$

Assume that $u_x > -M$, $M > 0$. When $\lambda \geqslant 1$, from (2.7) we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} y^2\,dx &\leqslant 5M\int_{\mathbb{R}} y^2\,dx + (3\lambda-3)M\int_{\mathbb{R}} u_x^2\,dx \\ &\leqslant 5M\int_{\mathbb{R}} y^2\,dx + (3\lambda-3)M\int_{\mathbb{R}} y^2\,dx. \end{aligned} \tag{2.9}$$

Hence, we deduce that

$$\|y(t)\|_{L^2}^2 \leqslant e^{(3\lambda+2)Mt}\|y_0\|_{L^2}^2. \tag{2.10}$$

When $\frac{1}{6} \leqslant \lambda < 1$, using (2.8), we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} y^2\,dx &\leqslant (6\lambda-1)M\int_{\mathbb{R}} y^2\,dx + (3-3\lambda)M\int_{\mathbb{R}} u_x^2\,dx + (6-6\lambda)M\int_{\mathbb{R}} u_{xx}^2\,dx \\ &\leqslant (6\lambda-1)M\int_{\mathbb{R}} y^2\,dx + (9-9\lambda)M\int_{\mathbb{R}} y^2\,dx \\ &\leqslant (8-3\lambda)M\int_{\mathbb{R}} y^2\,dx. \end{aligned} \tag{2.11}$$

Then we deduce that

$$\|y(t)\|_{L^2}^2 \leqslant e^{(8-3\lambda)Mt}\|y_0\|_{L^2}^2. \tag{2.12}$$

Note that $\|u_x\|_{L^\infty} \leqslant \|u\|_2 \leqslant c\|y\|_{L^2}$. Combining (2.10), (2.12) with Theorem 2.2, we know that the solution will not blow up in finite time.

The sufficiency is obvious. Thus, we complete the proof. □

Remark 2.1. Theorem 2.3 covers the corresponding result for the Degasperis–Procesi equation in [21].

3. A priori estimates and blow-up results on the line

In this section, firstly, we shall prove a conservation law for strong solutions to Eq. (2.6). Then we use this conservation law to deduce the a priori estimate for the L^∞-norm of the strong solutions. At last, we provide two blow-up results and investigate the blow-up rate of the strong solutions.

Lemma 3.1. *Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, $\lambda > -\frac{1}{3}$, and $u(t,x)$ is the corresponding solution to problem* (2.1) *given by Theorem* 2.1, *then we have*

$$\int_{\mathbb{R}} yv\,dx = \int_{\mathbb{R}} y_0 v_0\,dx,$$

where $y(t,x) = u(t,x) - u_{xx}(t,x)$ and $v(t,x) = ((3\lambda+1) - \partial_x^2)^{-1}u$. Furthermore, $\|u(t)\|_{L^2}^2 \leqslant K\|u_0\|_{L^2}^2$, where $K = \max\{(3\lambda+1), (3\lambda+1)^{-1}\}$.

Proof. Again, we prove the above theorem for $s = 3$. By Theorem 2.1 and the density argument, the theorem will hold for $s > \frac{3}{2}$. Let $T > 0$, be the lifespan of the strong solution to Eq. (2.1) with initial data $u_0 \in H^3(\mathbb{R})$, such that $u \in C([0,T); H^3(\mathbb{R})) \cap C^1([0,T); H^2(\mathbb{R}))$. By (2.6) we deduce that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} yv\,dx &= \int_{\mathbb{R}} y_t v\,dx + \int_{\mathbb{R}} yv_t\,dx \\ &= 2\int_{\mathbb{R}} y_t v\,dx \\ &= -2\int_{\mathbb{R}} y_x uv\,dx - 6\int_{\mathbb{R}} u_x yv\,dx + (6-6\lambda)\int_{\mathbb{R}} uu_x v\,dx \\ &= -2\int_{\mathbb{R}} (yu)_x v\,dx - 4\int_{\mathbb{R}} u_x yv\,dx + (6-6\lambda)\int_{\mathbb{R}} uu_x v\,dx. \end{aligned} \tag{3.1}$$

Since $y = u - u_{xx}$, $(3\lambda+1)v - \partial_x^2 v = u$, we have

$$\begin{aligned} -2\int_{\mathbb{R}} (yu)_x v\,dx - 4\int_{\mathbb{R}} u_x yv\,dx &= 2\int_{\mathbb{R}} v_x u^2\,dx - 2\int_{\mathbb{R}} v_x uu_{xx}\,dx - 4\int_{\mathbb{R}} vuu_x\,dx + 4\int_{\mathbb{R}} vu_x u_{xx}\,dx \\ &= 2\int_{\mathbb{R}} v_x u^2\,dx + 2\int_{\mathbb{R}} (v_x u)_x u_x\,dx + 2\int_{\mathbb{R}} v_x u^2\,dx - 2\int_{\mathbb{R}} v_x u_x^2\,dx \\ &= 4\int_{\mathbb{R}} v_x u^2\,dx + 2\int_{\mathbb{R}} v_{xx} uu_x\,dx \\ &= 4\int_{\mathbb{R}} v_x u^2\,dx - \int_{\mathbb{R}} v_{xxx} u^2\,dx \\ &= 4\int_{\mathbb{R}} v_x u^2\,dx - \int_{\mathbb{R}} \big[(3\lambda+1)v_x - u_x\big]u^2\,dx \\ &= -(6-6\lambda)\int_{\mathbb{R}} vuu_x\,dx. \end{aligned} \tag{3.2}$$

Combining (3.1) with (3.2), we get

$$\int_{\mathbb{R}} yv\,dx = \int_{\mathbb{R}} y_0 v_0\,dx.$$

When $\lambda \geqslant 0$, we have

$$\begin{aligned}
\|u(t)\|_{L^2}^2 = \|\hat{u}(t)\|_{L^2}^2 &\leqslant (3\lambda+1)\int_{\mathbb{R}} \frac{1+\xi^2}{(3\lambda+1)+\xi^2}\big|\hat{u}(\xi)\big|^2\,d\xi \\
&= (3\lambda+1)\big(\hat{y}(t),\hat{v}(t)\big) = (3\lambda+1)\big(y(t),v(t)\big) \\
&= (3\lambda+1)(y_0,v_0) = (3\lambda+1)(\hat{y}_0,\hat{v}_0) \\
&\leqslant (3\lambda+1)\int_{\mathbb{R}} \frac{1+\xi^2}{3\lambda+1+\xi^2}\big|\hat{u}_0(\xi)\big|^2\,d\xi \\
&\leqslant (3\lambda+1)\|\hat{u}_0\|_{L^2}^2 \\
&= (3\lambda+1)\|u_0\|_{L^2}^2. \qquad (3.3)
\end{aligned}$$

Similarly, when $-\frac{1}{3} < \lambda < 0$, we obtain

$$\|u(t)\|_{L^2}^2 = \|\hat{u}(t)\|_{L^2}^2 \leqslant \int_{\mathbb{R}} \frac{1+\xi^2}{(3\lambda+1)+\xi^2}\big|\hat{u}(\xi)\big|^2\,d\xi \leqslant (3\lambda+1)^{-1}\|u_0\|_{L^2}^2. \qquad (3.4)$$

This completes the proof of the lemma. □

Remark 3.1. When $\lambda = 1$, we have the corresponding result for the Degasperis–Procesi equation, see Lemma 3.1 in [12].

Remark 3.2. The lower bound $-\frac{1}{3}$ for the parameter is natural. Since for $\lambda \leqslant -\frac{1}{3}$, the value $3\lambda+1$ belongs to the spectrum of the operator ∂_x^2.

Before we establish the blow-up results for Eq. (2.1), we recall a lemma which will be used later. Consider the following differential equation

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \qquad (3.5)$$

One can obtain the following lemma.

Lemma 3.2. *Let $u_0 \in H^s(\mathbb{R})$, $s \geqslant 3$ and $T > 0$ be the lifespan of the corresponding solution u to Eq.* (2.1)*. Then problem* (3.5) *has a unique solution $q \in C^1([0,T)\times\mathbb{R},\mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x\big(s,q(s,x)\big)\,ds\right) > 0,$$

for $(t,x) \in [0,T)\times\mathbb{R}$.

Using Lemmas 3.1, 3.2, we now estimate the L^∞-norm of the strong solutions to Eq. (2.1).

Lemma 3.3. *Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and $\lambda > -\frac{1}{3}$. Let $T > 0$ be the lifespan of the corresponding strong solution u to Eq.* (2.1) *given by Theorem* 2.1*. Then we have*

$$\|u(t,x)\|_{L^\infty} \leqslant \frac{3K|\lambda|}{4}\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}, \quad \forall t \in [0,T],$$

where $K = \max\{(3\lambda+1),(3\lambda+1)^{-1}\}$.

Proof. Again, we only need to prove the above theorem for $s = 3$. Let $T > 0$, be the lifespan of the strong solution to Eq. (2.1) (or (2.2)) with initial data $u_0 \in H^3(\mathbb{R})$. We have

$$\begin{aligned}
-3\lambda G*(uu_x) &= \frac{-3\lambda}{2}\int_{\mathbb{R}} e^{-|x-y|}uu_y\,dy \\
&= \frac{3\lambda}{4}\int_{-\infty}^{x} e^{-|x-y|}u^2\,dy - \frac{3\lambda}{4}\int_{x}^{\infty} e^{-|x-y|}u^2\,dy. \qquad (3.6)
\end{aligned}$$

By (3.5), we obtain

$$\frac{d}{dt}u\big(t,q(t,x)\big)=u_t\big(t,q(t,x)\big)+u_x\big(t,q(t,x)\big)q_t(t,x)=(u_t+uu_x)\big(t,q(t,x)\big). \tag{3.7}$$

Combining (3.6), (3.7) with Eq. (2.2), we have

$$\left|\frac{d}{dt}u\big(t,q(t,x)\big)\right|\leqslant\frac{3|\lambda|}{4}\int_{\mathbb{R}}e^{-|x-y|}u^2\,dy\leqslant\frac{3|\lambda|}{4}\|u\|_{L^2}^2\leqslant\frac{3K|\lambda|}{4}\|u_0\|_{L^2}^2. \tag{3.8}$$

Thus

$$\big|u\big(t,q(t,x)\big)\big|\leqslant\big\|u\big(t,q(t,x)\big)\big\|_{L^\infty}\leqslant\frac{3K|\lambda|}{4}\|u_0\|_{L^2}^2t+\|u_0\|_{L^\infty}.$$

Since $u_x(t,y)$ is uniformly bounded for $(t,y)\in[0,t]\times\mathbb{R}$ with $t\in[0,T)$, from Lemma 3.2 we have for every $t\in[0,T)$ a constant $c(t)>0$ such that

$$e^{-c(t)}\leqslant q_x(t,x)\leqslant e^{c(t)},\quad x\in\mathbb{R}.$$

We then deduce that $q(t,\cdot)$ is strictly increasing on $\mathbb{R}$ with $\lim_{x\to\pm\infty}q(t,x)=\pm\infty$ as long as $t\in[0,T)$.

Thus,

$$\big\|u(t,x)\big\|_{L^\infty}=\big\|u\big(t,q(t,x)\big)\big\|_{L^\infty}\leqslant\frac{3K|\lambda|}{4}\|u_0\|_{L^2}^2t+\|u_0\|_{L^\infty}. \tag{3.9}$$

This completes the proof of the lemma. □

Remark 3.3. Theorems 2.2, 2.3 and Lemma 3.3 ensure that, when $\lambda>-\frac{1}{3}$, the first blow-up of a strong solution to Eq. (2.1) must occur as wave breaking, i.e., the slope of the solution becomes unbounded in finite time whereas its amplitude remains bounded in finite time. This is the same as the Degasperis–Procesi equation.

Now we are in the position to provide our blow-up results.

Theorem 3.1. *Assume that $u_0\in H^s(\mathbb{R})$, $s>\frac{3}{2}$, is odd. Then*

(i) *when $\lambda>0$, $u'(0)\leqslant0$, the corresponding solution to problem* (2.1) *blows up, and the lifespan of the solution is strictly less than $-\frac{1}{u'(0)}$;*
(ii) *when $-\frac{1}{3}<\lambda\leqslant0$, if $u'(0)<-\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}$, the corresponding solution to problem* (2.1) *blows up, and the lifespan of the solution is strictly less than $\frac{1}{2\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}\ln\frac{u'(0)-\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}{u'(0)+\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}$.*

Proof. Similarly, we only prove theorem above for $s=3$. Since u_0 is odd, $u(t,x)$ is odd for any $t\in[0,T)$. Hence,

$$u(t,0)=u_{xx}(t,0)=0.$$

From (2.2) we have

$$u_{tx}=-u_x^2-uu_{xx}+\frac{3\lambda}{2}u^2-\lambda G*\left(\frac{3}{2}u^2\right).$$

Let $f(t)=u_x(t,0)$ for $t\in[0,T)$. We obtain

$$f'(t)=-f(t)^2-\lambda G*\left(\frac{3}{2}u^2\right)(t,0).$$

If $\lambda>0$, since $G*(\frac{3}{2}u^2)>0$, we get

$$f'(t)<-f(t)^2,$$

and then

$$0>\frac{1}{f(t)}\geqslant\frac{1}{f(0)}+t.$$

So, we obtain $T<-\frac{1}{f(0)}$.

If $\lambda = 0$, we have

$$f'(t) = -f(t)^2,$$

and then

$$0 > \frac{1}{f(t)} = \frac{1}{f(0)} + t.$$

Thus, we have $T < -\frac{1}{f(0)}$.

If $-\frac{1}{3} < \lambda < 0$, we obtain

$$f'(t) = -f(t)^2 - \lambda G * \left(\frac{3}{2}u^2\right)(t,0) \leqslant -f(t)^2 - \lambda\frac{3}{2}\left\|u(t)\right\|_{L^2}^2.$$

From (3.4), we deduce that

$$\begin{aligned} f'(t) &\leqslant -f(t)^2 - \frac{3\lambda}{6\lambda+2}\|u_0\|_{L^2}^2 \\ &= \left(\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} - f(t)\right)\left(\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} + f(t)\right). \end{aligned} \tag{3.10}$$

Since $f(0) < -\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}$, we know $f'(0) < 0$. From the continuity of $f(t)$, we have

$$f(t) < -\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}, \quad t \in [0,T). \tag{3.11}$$

From (3.10), we get

$$\frac{f'(t)}{(\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} - f(t))^2} \leqslant \frac{\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} + f(t)}{\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} - f(t)}.$$

That is

$$\frac{1}{2\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}\left(\frac{\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} + f(t)}{\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} - f(t)}\right)' \leqslant \frac{\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} + f(t)}{\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} - f(t)}. \tag{3.12}$$

Combining (3.11) with (3.12), we obtain

$$0 > \frac{f(t) + \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}{f(t) - \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}} - 1 \geqslant \frac{f(0) + \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}{f(0) - \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}} e^{2\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2} t} - 1. \tag{3.13}$$

Since

$$0 < \frac{f(0) + \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}{f(0) - \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}} < 1,$$

we have

$$t \leqslant T = \frac{1}{2\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}} \ln\frac{f(0) - \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}{f(0) + \sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}},$$

and $\lim_{t \nearrow T} f(t) = -\infty$. This completes the proof. □

We now present the second blow-up result.

Theorem 3.2. *Let $\delta > 0$ and $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then*

(i) *when $\lambda = 0$, if there exists $x_0 \in \mathbb{R}$ such that $u'(x_0) < 0$, the corresponding solution to (2.1) blows up in finite time and the lifespan of the solution is strictly less than $-\frac{1}{u'(x_0)}$;*

(ii) *when* $-\frac{1}{3} < \lambda < 0$, *if there exists* $x_0 \in \mathbb{R}$ *such that* $u'(x_0) < -\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}$, *the corresponding solution to* (2.1) *blows up in finite time and the lifespan of the solution is strictly less than* $\frac{1}{2\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}} \ln \frac{u'(x_0)-\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}{u'(x_0)+\sqrt{\frac{-3\lambda}{6\lambda+2}}\|u_0\|_{L^2}}$;

(iii) *when* $\lambda > 0$, *if there exists* $x_0 \in \mathbb{R}$ *such that*

$$u'(x_0) < -\frac{\sqrt{6\lambda}(1+\delta)}{4}\left(\left(\|u_0\|_{L^\infty}^2 + \frac{\sqrt{6\lambda}}{2}(3\lambda+1)\ln\left(1+\frac{2}{\delta}\right)\|u_0\|_{L^2}^2\right)^{\frac{1}{2}} + \|u_0\|_{L^\infty}\right),$$

the corresponding solution blows up in finite time and the lifespan of the solution is strictly less than $T_0 = \frac{2(\|u_0\|_{L^\infty}^2 + \frac{\sqrt{6\lambda}}{2}(3\lambda+1)\ln(1+\frac{2}{\delta})\|u_0\|_{L^2}^2)^{\frac{1}{2}} - 2\|u_0\|_{L^\infty}}{3\lambda(3\lambda+1)\|u_0\|_{L^2}^2}$.

Proof. Again, we only need to prove the theorem for $s = 3$. As in the proof of Theorem 3.1, let T be the lifespan of the solution corresponding to the initial data $u_0 \in H^3(\mathbb{R})$. From (2.2), we obtain

$$u_{tx} = -u_x^2 - uu_{xx} + \frac{3\lambda}{2}u^2 - \lambda G * \left(\frac{3}{2}u^2\right).$$

As we know

$$\frac{du_x(t,q(t,x))}{dt} = u_{xt}\big(t,q(t,x)\big) + u_{xx}\big(t,q(t,x)\big)q_t(t,x) = (u_{xt}+uu_{xx})\big(t,q(t,x)\big).$$

When $\lambda = 0$, the proof is trivial. So we only consider the assertions (ii) and (iii).

When $-\frac{1}{3} < \lambda < 0$, we have

$$\begin{aligned}\frac{du_x(t,q(t,x))}{dt} &= -u_x^2\big(t,q(t,x)\big) + \frac{3\lambda}{2}u^2\big(t,q(t,x)\big) - \lambda G * \left(\frac{3}{2}u^2\big(t,q(t,x)\big)\right)\\ &\leqslant -u_x^2\big(t,q(t,x)\big) - \lambda G * \left(\frac{3}{2}u^2\big(t,q(t,x)\big)\right)\\ &\leqslant -u_x^2\big(t,q(t,x)\big) - \frac{3\lambda}{2}\|u\|_{L^2}^2\\ &\leqslant -u_x^2\big(t,q(t,x)\big) - \frac{3\lambda}{2(3\lambda+1)}\|u_0\|_{L^2}^2. \end{aligned} \tag{3.14}$$

Let $f(t) = u_x(t,q(t,x))$, then $f(0) = u_x(0,x_0) = u'(x_0)$. From (3.14), we have

$$f'(t) \leqslant -f(t)^2 - \frac{3\lambda}{2(3\lambda+1)}\|u_0\|_{L^2}^2.$$

Similar to Theorem 3.1, we can prove the assertion (ii) and we omit the details for concision.

When $\lambda > 0$, we have

$$\begin{aligned}\frac{du_x(t,q(t,x))}{dt} &\leqslant -u_x^2\big(t,q(t,x)\big) + \frac{3\lambda}{2}u^2\big(t,q(t,x)\big)\\ &\leqslant -u_x^2\big(t,q(t,x)\big) + \frac{3\lambda}{2}\left[\frac{3\lambda(3\lambda+1)}{4}\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}\right]^2. \end{aligned} \tag{3.15}$$

Our aim next is to find a T_0 such that, under the assumptions of the theorem, by solving inequality

$$\frac{du_x(t,q(t,x))}{dt} \leqslant -u_x^2\big(t,q(t,x)\big) + \frac{3\lambda}{2}\left[\frac{3\lambda(3\lambda+1)}{4}\|u_0\|_{L^2}^2 T_0 + \|u_0\|_{L^\infty}\right]^2,$$

we can get $t < T_0$.

Let

$$T_0 = \frac{2(\|u_0\|_{L^\infty}^2 + \frac{\sqrt{6\lambda}}{2}(3\lambda+1)\ln(1+\frac{2}{\delta})\|u_0\|_{L^2}^2)^{\frac{1}{2}} - 2\|u_0\|_{L^\infty}}{3\lambda(3\lambda+1)\|u_0\|_{L^2}^2},$$

$$C(T_0) = \frac{\sqrt{6\lambda}}{2}\left(\frac{3\lambda(3\lambda+1)}{4}\|u_0\|_{L^2}^2 T_0 + \|u_0\|_{L^\infty}\right).$$

Setting $u'(t, q(t, x)) = g(t)$, then $u'(x_0) = g(0)$. By the assumptions of the theorem, we have

$$g(0) < -(1+\delta)C(T_0),$$

and

$$0 < \frac{g(0) - C(T_0)}{g(0) + C(T_0)} = 1 - \frac{2C(T_0)}{g(0) + C(T_0)} \leqslant 1 + \frac{2}{\delta}. \tag{3.16}$$

It is not difficult to verify that

$$\frac{1}{2C(T_0)} \ln\left(1 + \frac{2}{\delta}\right) \leqslant T_0. \tag{3.17}$$

Combining (3.16), (3.17), we have

$$\frac{1}{2C(T_0)} \ln \frac{g(0) - C(T_0)}{g(0) + C(T_0)} \leqslant \frac{1}{2C(T_0)} \ln\left(1 + \frac{2}{\delta}\right) \leqslant T_0. \tag{3.18}$$

Thus, we deduce by (3.15) that

$$g'(t) \leqslant g(t)^2 + C^2(T_0), \quad t \in [0, T_0] \cap [0, T). \tag{3.19}$$

By (3.18) and the assumptions of the theorem, we obtain

$$g(t) < -C(T_0), \quad t \in [0, T_0] \cap [0, T).$$

Arguments similar to those in the second part of Theorem 3.1 help us obtain

$$0 < T < \frac{1}{2C(T_0)} \ln \frac{g(0) - C(T_0)}{g(0) + C(T_0)} < T_0,$$

and $\lim_{t \nearrow T} g(t) = -\infty$. The proof is completed. □

Remark 3.4. In Theorem 3.2, for given $\lambda > 0$, when $\delta \to +\infty$, the lifespan of the solution will tend to zero. This means that the steeper the slope at some point is, the quicker the solution blows up. On the other hand, for fixed $\delta > 0$, when $\lambda > 0$, $\lambda \to +\infty$, the lifespan T tends to zero too. This is due to the fact that the lager λ is, the larger the coefficient increases in the steepening term $(3\lambda + 1)uu_x$ in (1.5), and the livelier steepening it produces.

Remark 3.5. Note that singularities formed in Theorems 3.1, 3.2 are in the form of wave breaking. When $\lambda \to 1$, the two theorems recover the blow-up results for the Degasperis–Procesi equation. Hence, by introducing a stress multiplier, we obtain a family of equations which accommodate the same wave-breaking mechanism as the Degasperis–Procesi equation.

Theorem 3.3. *Let $\lambda > -\frac{1}{3}$, and $T < \infty$ be the blow-up time of the corresponding solution to problem* (2.1) *with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then we have*

$$\lim_{t \to T} \left(\inf_{x \in \mathbb{R}} \{u_x(t, x)\} (T - t) \right) = -1,$$

while the solution remains bounded.

The proof of this theorem is similar to that of Theorem 3.1 in [12], so we will not provide it.

Remark 3.6. Note that the blow-up rate of breaking waves to Eq. (2.1) is -1 for any $\lambda > -\frac{1}{3}$. For $\lambda = 1$, it is the corresponding result for the Degasperis–Procesi equation, see Theorem 3.1 in [12]. Thus λ does not affect the blow-up rate of the solutions.

4. Blow-up phenomena in the periodic case

In this section, we will consider the following periodic Burgers-like equation

$$\begin{cases} u_t + uu_x = -\lambda \partial_x G * \left(\frac{3}{2}u^2\right), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ u(t, x) = u(t, x+1), & t \geqslant 0,\ x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $G(x)=\frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, $[x]$ stands for the integer part of $x\in\mathbb{R}$, and $(1-\partial_x^2)^{-1}f=G*f$ for $f\in L^2(\mathbb{S}),\mathbb{S}=\mathbb{R}/\mathbb{Z}$. Note that Eq. (4.1) is equivalent to the following equation

$$\begin{cases} u_t+uu_x=-\lambda\partial_x\big(1-\partial_x^2\big)^{-1}\left(\dfrac{3}{2}u^2\right), & t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R},\\ u(t,x)=u(t,x+1), & t\geqslant 0,\ x\in\mathbb{R}. \end{cases} \tag{4.2}$$

Setting $y=u-u_{xx}$, we have

$$\begin{cases} y_t+uy_x+3u_xy=(3-3\lambda)uu_x, & t>0,\ x\in\mathbb{R},\\ y(0,x)=u_0(x)-u_{0xx}(x), & x\in\mathbb{R},\\ y(t,x)=y(t,x+1), & x\in\mathbb{R},\ t\geqslant 0. \end{cases} \tag{4.3}$$

We first provide the local well-posedness and the blow-up scenario of strong solutions. Then we establish the blow-up results. Since the proof of these results is essentially similar to that we presented in the last section, we omit the details.

Lemma 4.1. *Let $u_0\in H^s(\mathbb{S})$, $s>\frac{3}{2}$. There exists a maximum $T=T(\lambda,u_0)>0$, and a unique solution u to Eq.* (4.1)*, which depends continuously on initial data u_0 such that*

$$u=u(\cdot,u_0)\in C\big([0,T);H^s(\mathbb{S})\big)\cap C^1\big([0,T);H^{s-1}(\mathbb{S})\big).$$

Moreover, T is independent of s in the sense similar to Theorem 2.1.

The proof is similar to that of Theorem 3.1 above, or Theorem 2.1 in [22].

Lemma 4.2. *Let $u_0\in H^s(\mathbb{S})$, $s>\frac{3}{2}$, and T be the lifespan of the corresponding solution with initial data u_0. If there exists $M>0$, such that $\|u_x(t,x)\|_{L^\infty(\mathbb{S})}\leqslant M$, $t\in[0,T)$, then the $H^s(\mathbb{S})$-norm of $u(t,\cdot)$ does not blow up on* $[0,T)$.

The proof is similar to that of Theorem 2.2.

Lemma 4.3. *Assume that $u_0\in H^s(\mathbb{S})$, $s>\frac{3}{2}$, $\lambda\geqslant\frac{1}{6}$, then the solution blows up if and only if the slope of the solution becomes unbounded from below in finite time.*

The proof is similar to that of Theorem 2.3.

Lemma 4.4. *Assume that $u_0\in H^s(\mathbb{S})$, $s>\frac{3}{2}$, $\lambda>-\frac{1}{3}$. Let T be the lifespan of the corresponding strong solution u to Eq.* (2.1) *guaranteed by Theorem* 2.1*. Then we have*

$$\int_{\mathbb{S}} yv\,dx=\int_{\mathbb{S}} y_0v_0\,dx,$$

where $y(t,x)=u(t,x)-u_{xx}(t,x)$ and $v(t,x)=((3\lambda+1)-\partial_x^2)^{-1}u$.

Besides,

$$\big\|u(t)\big\|_{L^2}^2\leqslant K\|u_0\|_{L^2}^2,$$
$$\big\|u(t,x)\big\|_{L^\infty}\leqslant\frac{3cK|\lambda|}{4}\|u_0\|_{L^2}^2t+\|u_0\|_{L^\infty},\quad \forall t\in[0,T],$$

where $K=\max\{(3\lambda+1),(3\lambda+1)^{-1}\}$, $c=\coth(\frac{1}{2})=\frac{\cosh(1/2)}{\sinh(1/2)}$.

Combining Lemmas 3.1, 3.3 in last section and Lemmas 3.1, 3.2 in [13], we can prove this lemma. In the following, for convenience, we use c to denote $\coth(\frac{1}{2})$.

Our first blow-up result is as follows.

Theorem 4.1. *Assume $u_0\in H^s(\mathbb{S})$, $s>\frac{3}{2}$, and let $\delta>0$. Then*

(i) *when $\lambda=0$, if there exists $x_0\in\mathbb{S}$ such that $u'(x_0)<0$, the corresponding solution to* (4.1) *blows up in finite time and the lifespan of the solution is strictly less than $-\frac{1}{u'(x_0)}$;*

(ii) *when $-\frac{1}{3}<\lambda<0$, if there exists $x_0\in\mathbb{S}$ such that $u'(x_0)<-\sqrt{\frac{-3\lambda c}{6\lambda+2}}\|u_0\|_{L^2}$, the corresponding solution to* (4.1) *blows up in finite time and the lifespan of the solution is strictly less than $\frac{1}{2\sqrt{\frac{-3\lambda c}{6\lambda+2}}\|u_0\|_{L^2}}\ln\frac{u'(x_0)-\sqrt{\frac{-3\lambda c}{6\lambda+2}}\|u_0\|_{L^2}}{u'(x_0)+\sqrt{\frac{-3\lambda c}{6\lambda+2}}\|u_0\|_{L^2}}$;*

(iii) *when $\lambda > 0$, if there exists $x_0 \in \mathbb{S}$, such that*

$$u'(x_0) < -\frac{\sqrt{6}(1+\delta)}{4}\left(\left(\|u_0\|_{L^\infty}^2 + \frac{\sqrt{6\lambda}}{2}c(3\lambda+1)\ln\left(1+\frac{2}{\delta}\right)\|u_0\|_{L^2}^2\right)^{\frac{1}{2}} + \|u_0\|_{L^\infty}\right),$$

the corresponding solution to (4.1) *blows up in finite time and the lifespan of the solution is strictly less than* $T_0 = \frac{2(\|u_0\|_{L^\infty}^2 + \frac{\sqrt{6\lambda}}{2}c(3\lambda+1)\ln(1+\frac{2}{\delta})\|u_0\|_{L^2}^2)^{\frac{1}{2}} - \|u_0\|_{L^\infty}}{3c\lambda(3\lambda+1)\|u_0\|_{L^2}^2}$.

The proof is similar to that of Theorem 3.2, and we will not provide it again.
Following this theorem we have corollaries as follows.

Corollary 4.1. *Assume $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, is even and not a constant. Then for sufficiently large n, the corresponding solution to* (4.1) *with initial data $v_0(x) = u_0(nx)$ blows up in finite time when $\lambda > -\frac{1}{3}$.*

Corollary 4.2. *Assume $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, is not a constant. If*

$$\left|\min_{x\in\mathbb{S}} u_0'(x)\right| > \left|\max_{x\in\mathbb{S}} u_0'(x)\right|,$$

then for sufficiently large n, the corresponding solution to (4.1) *with initial data $v_0(x) = u_0(nx)$ blows up in finite time when $\lambda > -\frac{1}{3}$.*

The above two corollaries are similar to those for the Degasperis–Procesi equation, see [13].

Theorem 4.2. *Assume $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Then*

(i) *when $-\frac{1}{3} < \lambda < 0$, if $\int_{\mathbb{S}} u_{0x}^3\,dx < -(\frac{-9\lambda c}{12\lambda+4}\|u_0\|_{L^2}^2)^{3/2}$, the corresponding solution to* (4.1) *blows up in finite time*;
(ii) *when $\lambda = 0$, if $\int_{\mathbb{S}} u_{0x}^3\,dx < 0$, the corresponding solution to* (4.1) *blows up in finite time*;
(iii) *when $0 < \lambda < \frac{16}{9}$, if u_0 is not a constant and the corresponding solution to* (4.1) *has a zero for any $t > 0$, the solution to* (4.1) *blows up in finite time.*

Proof. As before, we prove the theorem for the case $s = 3$. Let $u(t,x)$ be the strong solution with initial data $u_0 \in H^3(\mathbb{S})$, and T be its lifespan. By (4.1) we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= 3\int_{\mathbb{S}} u_x^2 u_{tx}\,dx = 3\int_{\mathbb{S}} u_x^2\left(-u_x^2 - uu_{xx} + \frac{3\lambda}{2}u^2 - G*\left(\frac{3\lambda}{2}u^2\right)\right)dx \\ &= -3\int_{\mathbb{S}} u_x^4\,dx - 3\int_{\mathbb{S}} uu_x^2u_{xx}\,dx + \frac{9\lambda}{2}\int_{\mathbb{S}} u_x^2u^2\,dx - \frac{9\lambda}{2}\int_{\mathbb{S}} u_x^2 G*u^2\,dx \\ &= -2\int_{\mathbb{S}} u_x^4\,dx + \frac{9\lambda}{2}\int_{\mathbb{S}} u_x^2u^2\,dx - \frac{9\lambda}{2}\int_{\mathbb{S}} u_x^2 G*u^2\,dx. \end{aligned} \tag{4.4}$$

For $\lambda = 0$, we have

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx = -2\int_{\mathbb{S}} u_x^4\,dx \leqslant -2\left(\int_{\mathbb{S}} u_x^3\,dx\right)^{\frac{4}{3}}. \tag{4.5}$$

Setting $g(t) = \int_{\mathbb{S}} u_x^3\,dx$, then $g(0) = \int_{\mathbb{S}} u_{0x}^3\,dx$. By assumptions of the theorem, we obtain by solving (4.5) that

$$0 > \frac{1}{g(t)} \geqslant \left(\frac{2}{3}t + \frac{1}{g(0)^{1/3}}\right)^3.$$

Hence, we have proved the assertion (i).
When $-\frac{1}{3} < \lambda < 0$, from (4.4) we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &\leqslant -2\int_{\mathbb{S}} u_x^4\,dx - \frac{9\lambda c}{2}\int_{\mathbb{S}} u_x^2\,dx\int_{\mathbb{S}} u^2\,dx \\ &\leqslant 2\left(\int_{\mathbb{S}} u_x^3\,dx\right)^{\frac{4}{3}} - \frac{9\lambda c}{2(3\lambda+1)}\|u_0\|_{L^2}^2\int_{\mathbb{S}} u_x^2\,dx. \end{aligned} \tag{4.6}$$

Setting $g(t)=\int_{\mathbb{S}} u_x^3\,dx$, $-\frac{9\lambda c}{2(3\lambda+1)}\|u_0\|_{L^2}^2=M$, by Hölder inequality and (4.6) we deduce that

$$g'(t)\leqslant -2g(t)^{4/3}+Mg(t)^{2/3}. \tag{4.7}$$

Thus,

$$3\frac{d}{dt}\left(g(t)^{1/3}\right)\leqslant -2g(t)^{2/3}+M. \tag{4.8}$$

By assumption of the theorem, $(g(0))^{1/3}=(\int_{\mathbb{S}} u_{0x}^3\,dx)^{1/3}<-\sqrt{\frac{M}{2}}$. By the same arguments as we used in Theorem 3.1, we prove the assertion (ii).

When $0<\lambda<\frac{16}{9}$, the proof is the same as that of Theorem 3.8 in [13] (or Theorem 4.1 in [3]). We omit the details. □

Remark 4.1. Note that assertions (i), (ii) imply that the strong solution to (4.1) with initial data u_0 may blow up even $y_0=u_0-u_{0xx}$ does not change sign. (We only need $\int_{\mathbb{S}} u_{0x}^3\,dx$ to be small enough.) This is different from the Degasperis–Procesi equation.

Remark 4.2. When $0<\lambda<\frac{16}{9}$, assertion (iii) is parallel to that of the Degasperis–Procesi equation, see [13].

By Theorem 4.2, we have the following corollaries.

Corollary 4.3. *Assume $u_0\in H^3(\mathbb{S})$, is not a constant and $\int_{\mathbb{S}} u_0^3\,dx=0$. Then when $\frac{16}{9}>\lambda>0$, the corresponding solution to* (4.1) *blows up.*

Corollary 4.4. *Assume $u_0\in H^3(\mathbb{S})$, is not a constant and u_0 is odd. Then when $\frac{16}{9}>\lambda>0$, the corresponding solution to* (4.1) *blows up.*

Corollary 4.5. *Assume $u_0\in H^3(\mathbb{S})$, is not a constant and $\int_{\mathbb{S}} u_0\,dx=0$ or $\int_{\mathbb{S}} y_0\,dx=0$. Then when $\frac{16}{9}>\lambda>0$, the corresponding solution to* (4.1) *blows up.*

For the blow-up rate, we have the following theorem.

Theorem 4.3. *Let $T>0$ be the blow-up time of the corresponding solution u to problem* (4.1) *with initial data $u_0\in H^s(\mathbb{S})$, $s>\frac{3}{2}$. Then we have*

$$\lim_{t\to T}\left(\min_{x\in\mathbb{S}}\left\{u_x(t,x)\right\}(T-t)\right)=-1,$$

while the solution remains bounded.

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