Theorem B of Hall–Higman Revisited

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Applying Green’s theory of vertices and sources, Thompson [5] described an elegant approach to the celebrated Theorem B of Hall and Higman [2]. We extend and complete here Thompson’s work from the point of block theory. Our discussion will show, at least, that the crucial quantity appearing in Theorem B is nothing but the defect of some block involved.¹

THEOREM B. Let G be a finite p-solvable group with a cyclic Sylow p-subgroup \( P = \langle x \rangle \) of order \( p^n \) and with \( O_p(G) = 1 \). Suppose G has a block B of defect \( a \), say, containing a faithful \( kG \)-module V over some field \( k \) of characteristic \( p \). Then the degree of the minimal polynomial of \( x \) on \( V \) either is \( d = p^n \) or \( d = p^{n-a}(p^a - 1) \). The latter happens only if the following holds:

(a) In case \( p = 2 \) the integer \( q = 2^n - 1 \) is a Mersenne prime, and the commutator group \( [O_2(G), x^{2^n-1}] \) is a nonabelian special \( q \)-group (which is normal in G).

(b) If \( p \) is odd, either \( B \) is of defect \( a = 1 \) or \( p = 3 \) and \( a = 2 \). Moreover, \( p \) is a Fermat prime and \( [O_p(G), x^{p^n-1}] \) is a nonabelian special 2-group of order larger than \( (p^a - 1)^2 \).

Of course, this applies to the original situation of Theorem B: Suppose that \( G_0 \) is \( p \)-solvable with \( O_p(G_0) = 1 \) and that \( V_0 \) is a faithful \( kG_0 \)-module. Let \( x \) be an element in \( G_0 \) of order \( p^n \) and let \( d_0 \) be the degree of the minimal polynomial of \( x \) acting on \( V_0 \). Assume \( d_0 \neq p^n \). Set \( H = O_p(G_0) \) and \( G = \langle x \rangle H \). There exists an indecomposable summand \( V \) of the restriction \( (V_0)_G \) with \( H/C_H(V) \) being not centralized by \( x^{p^n-1} \). Then \( O_p(G/C_H(V)) = 1 \) and \( V \) is a faithful module for \( G/C_H(V) \). If \( a \) denotes the defect of the block of \( G \) containing \( V \), then \( p^{n-a}(p^a - 1) \) is the degree of the minimal polynomial of \( x \) on \( V \) and so \( d_0 \geq p^{n-a}(p^a - 1) \).

¹ This has been recognized also by W. Feit.

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Actually there may be various such summands of \((V_\alpha)_{\alpha}\) belonging to blocks with distinct defects \(a_i\), notably when \(p = 2\). Then \([H, x^{p^a - 1}]\) is a direct product of nonabelian special \(q_i\)-groups where \(q_i = 2^{a_i} - 1\) varies over the resulting Mersenne primes; when \(p\) is odd, \([H, x^{p^a - 1}]\) is a nonabelian special 2-group. (Use that \(V_\alpha\) is faithful and argue as in (2) and (3) of the proof of the proposition below.) Note that this commutator group is in the Fitting subgroup of \(G_0\).

We emphasize that Theorem B imposes strong conditions on the defect \(a\) in case \(d \neq p^a\). Clearly \(d = p^a\) if and only if \(V_\alpha\) is projective-free, i.e., \(V_\alpha\) has no projective summand \(\neq 0\). We shall prove that then \(V_\alpha\) is homogeneous; every indecomposable summand \((\neq 0)\) of \(V_\alpha\) is of the same dimension \(d = p^a - a(p^a - 1)\).

Any of the specified situations can be realized. We outline this for the exceptional case under (b): Let \(Q\) be the central product of three quaternion groups of order 8. The group of outer automorphisms of \(Q\) is isomorphic to \(O^{-}(6,2)\). We deduce that \(Q\) has an automorphism of order 9. Let \(n \geq 2\) and let \(H = Q_1 \times \cdots \times Q_{3^n-2}\) with \(Q_i \cong Q\). Then \(H\) admits an automorphism \(x\) of order \(3^n\) permuting the direct factors \(Q_i\) transitively. Let \(G = \langle x \rangle H\) be the semidirect product. Let \(W\) be the unique (up to isomorphism) irreducible \(kH\)-module with \(C_k(W) = Q_1 \times \cdots \times Q_{3^n-2}, k \cong \mathbb{F}_q\) being a field of characteristic \(p = 3\) (dim \(W = 8\)). There exists an (irreducible) \(k(\langle x^{3^n-2} \rangle H)\)-module \(U\) with \(U_H = W\), and the induced module \(V = U^G\) is a faithful irreducible \(kG\)-module in a 3-block with defect \(a = 2\).

For the proof of Theorem B to the usual Hall-Higman reduction does not work. The principal objective will be to produce a suitable normal \(q\)-subgroup. In an earlier version of the paper we even needed the Feit-Thompson theorem on solvability of groups of odd order to handle the case \(p = 2\). However, it is possible to avoid this deep result.

### 1. Preliminaries

For convenience we collect some known lemmas. First an elementary fact concerning Mersenne and Fermat primes; a proof can be found in Suzuki [4].

**Lemmas 1.** Let \(a \geq 1\) be an integer.

(a) If \(2^a - 1\) is a (positive) power of some prime \(q\), then \(2^a - 1 = q\).

(b) If \(p\) is an odd prime such that \(p^a - 1\) is a power of 2, then either \(a = 1\) or \(p = 3\) and \(a = 2\).

For the remainder of this section we fix an algebraically closed field \(k\) of characteristic \(p > 0\) and a finite group \(G\).
LEMMA 2. Let $k_1 \subseteq k_2$ be subfields of $k$ and let $b_1, b_2$ be block idempotents of $k_1 G$ and $k_2 G$, respectively. If $b_1 b_2 = b_2$, then $b_1$ and $b_2$ have the same defect groups.

This follows directly from definition of defect groups. Use that taking relative traces in group algebras is compatible with scalar extensions.

LEMMA 3. Let $H$ be a normal subgroup of $G$ such that $G/H$ is a $p$-group.

(a) If $V$ is an irreducible $kG$-module, $V_H$ is the direct sum of pairwise non-isomorphic irreducible $kH$-modules, i.e., the ramification index of $V$ with respect to $H$ is 1.

(b) If $W$ is an irreducible $kH$-module which is invariant in $G$ (under conjugation), there is a unique (up to isomorphism) $kG$-module $V$ such that $V_H \cong W$.

Assertion (a) follows by inducing up to $G$ the projective cover of any irreducible summand of $V_H$ and applying Green's theorem [1] and Frobenius reciprocity. Of course, $V_H$ is completely reducible (Clifford). As for (b), let $V$ be any irreducible submodule of the induced module $W^G$ and use (a). Uniqueness follows since now every composition factor of $W^G \cong V \otimes k[G/H]$ is isomorphic to $V$. (If $G/H$ is cyclic, $W^G$ is also uniserial.)

LEMMA 4. Suppose $G$ has a normal $p$-complement, say $H$. Let $B$ be a $p$-block of $G$ with defect group $D$. Then

(i) $B$ contains a unique irreducible $kG$-module $V$;
(ii) $V \cong U^G$ for some irreducible $k\langle DH\rangle$-module $U$;
(iii) $W = U_H$ is irreducible and $DH$ is the inertial group of $W$ in $G$.

Choose an irreducible $kG$-module $V \in B$ and let $W$ be an irreducible constituent of $V_H$. Let $I$ be the inertial group of $W$. By Lemma 3 there is a unique (irreducible) $kI$-module $U$ with $U_H \cong W$. Clifford's theorem yields that $V \cong U^G$. Now every composition factor of $W^G = (W^I)^G$ is isomorphic to $V$, and $W^G$ is projective as $H$ is a $p'$-group (and indecomposable by Green's theorem; if $I/H$ is cyclic, $W^G$ is also uniserial). This gives (i). We deduce that $D$ is a vertex of $V$. It remains to show that any Sylow $p$-subgroup $\bar{D}$ of $I$ is a vertex of $V$ too.

It is evident that $V \cong U^G$ is relatively $\bar{D}$-projective. Furthermore, as $\dim U = \dim W$ is a $p'$-number, $|G:I|$ is the largest power of $p$ dividing $\dim V$. Hence the result.
2. The Special Case

We refine Thompson’s [5] arguments. As before $k$ denotes an algebraically closed field of characteristic $p > 0$.

**Proposition.** Let $G = PH$ where $P = \langle x \rangle$ is a cyclic $p$-group of order $p^n > 1$ and $H \neq 1$ is a normal $q$-subgroup of $G$ for some prime $q \neq p$. Assume the commutator group $[H, P_0] = H$, where $P_0 = \langle x^{p^n - 1} \rangle$ is the subgroup of order $p$ in $P$. Suppose that $V$ is a faithful indecomposable $kG$-module belonging to a block with defect $a$. If $V_p$ is projective-free, then

(i) $p^a - 1$ is a power of $q$;
(ii) $H$ is a nonabelian special $q$-group of order $|H| > (p^a - 1)^2$;
(iii) $V$ is irreducible with $\dim V = p^n - a(p^a - 1)$, and $V_p$ is indecomposable.

In particular, $p^a - a(p^a - 1)$ is the degree of the minimal polynomial of $x$ acting on $V$.

**Proof.** Let $D \subseteq P$ be a defect group of the block containing $V$. We have $P_0 \subseteq D$ as $V$ is not projective. Note that $DH$ is a normal subgroup of $G$ and that $V$ is relatively $D$-projective. By Green’s theorem [1] there exists an indecomposable $k[DH]$-module $U$ such that $V \cong U^G$. Let $\bar{U}$ be an irreducible submodule of $U$. By Lemma 4 all composition factors of $U$ are isomorphic to $\bar{U}$ (a block with defect 0, i.e., $U$ is even uniserial). We also know that $\bar{U}_H$ is an irreducible constituent of $V_H$ and that $\bar{U}_H^G$ is the unique irreducible constituent of $V$. For the sake of clarity we break the proof into several steps.

1. $P_0$ acts fixed-point-freely on $H/H'$:

   Since $H$ is a $p'$-group, we have
   $$H/H' = [H/H', P_0] \times C_{H/H'}(P_0).$$

   Apply the hypothesis $[H, P_0] = H$.

2. $P_0$ centralizes every $P$-invariant abelian subgroup of $H$:

   Suppose $A$ is a $P$-invariant abelian subgroup of $H$ which is not centralized by $P_0$. Replacing $A$ by $[A, P_0]$ we may assume that $[A, P_0] = A \neq 1$. Since $V$ is faithful, there exists an irreducible constituent $W$ of $V_{pA}$ on which $A$ acts nontrivially. It follows that $PA/C_A(W)$ is a Frobenius group. From Lemma 4 we infer that $W$ belongs to a block of defect 0, i.e., $W$ is projective. This is impossible as $V_p$ is projective-free by hypothesis.

3. $H$ is a nonabelian special $q$-group:

   This follows from (2) and the hypothesis $[H, P_0] = H \neq 1$ by a result of Thompson (cf. [3, Satz III.13.6]).
(4) \( Q = H/C_H(U) \) is extra-special and \( D \) centralizes \( Q' \):

Since \( V \) is faithful and \( H \) is a \( p' \)-group, \( C = C_H(U) = C_H(\bar{U}) \) is a proper \( D \)-invariant normal subgroup of \( H \). From \([H, P_0] = H\) it follows that \( P_0 \) does not centralize \( Q = H/C \). Consequently \( O_p(DH/C) = 1 \) and \( \bar{U} \) is a faithful and irreducible \( k[DH/C] \)-module. Since \( \bar{U}_Q \) is irreducible, application of Schur's lemma gives that \( Z(Q) \) is a cyclic central subgroup of \( DH/C \). As \( P_0 \subset D \) acts fixed-point-freely on \( Q/Q' \cong H/CH' \) by (1), \( Q \) cannot be abelian. The assertion follows.

(5) Let \( |Q| = q^{2m+1} \). Then \( \dim \bar{U} = q^m \):

Noting that \( \bar{U}_q \) is irreducible and faithful, this follows from the ordinary character theory of extra-special \( q \)-groups.

(6) \( U_D \) is indecomposable:

From \( V \cong U^G \) (and the structure of \( G \)) we deduce that

\[
V_p \cong (U_D)^p.
\]

In particular, \( U_D \) is projective-free as \( V_p \) is projective-free. In view of (1) \( DH/CH' \) (\( C = C_H(U) \)) is a Frobenius group. Hence Mackey decomposition yields that \( \bar{U} \) remains indecomposable when restricted to \( D(CH') \).

(Use that \( U \) is relatively \( DH' \)-projective.) Since by (4) \( D(CH')/C = Q' \times DC/C \), we can conclude that even \( U_D \) is indecomposable.

(7) \( U = \bar{U} \) is irreducible and \( \dim U = p^a - 1 \):

\( D \) acts on the \( \mathbb{F}_q \)-vector space \( Q/Q' \) preserving the obvious symplectic (commutator) form. By Maschke \( Q/Q' \) is a completely reducible \( \mathbb{F}_q D \)-module. As \( P_0 \subset D \) acts fixed-point-freely on \( Q/Q' \), every irreducible \( \mathbb{F}_q D \)-submodule is faithful of the same dimension. From (5) and (6) it follows that \( q^m < p^a = |D| \). We can conclude that \( D \) acts irreducibly on \( Q/Q' \) and that \( p^a \) divides \( q^m + 1 \). Hence the result.

(8) Conclusion:

Because of Lemma 4, \( V \cong U^G \) is irreducible as well and has the stated dimension. By (6) \( V_p \cong (U_D)^p \) is indecomposable. This implies that \( \dim V \) is the degree of the minimal polynomial of \( x \) on \( V \), completing the proof.

3. The General Case

Theorem \( B' \) is an immediate consequence of Lemma 1 and the following result.

**Theorem.** Assume the hypotheses of Theorem \( B' \). Suppose \( V_p \) is
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faithful module for $\mathcal{G} = \text{PH}/C_H(\mathcal{V})$ and that $O_p(\mathcal{G}) = 1$. Clearly $\mathcal{V}_p$ is projective-free.

Let $\mathcal{D} \subseteq P$ be a defect group of the block of $\text{PH}$ containing $\mathcal{V}$. Then $\mathcal{D}$ maps isomorphically onto a defect group of the block of $\mathcal{G}$ containing $\mathcal{V}$. The ramification index of the unique irreducible $kG$-module $\mathcal{V}$ in $B$ with respect to $\text{PH}$ is 1 because $G/\text{PH}$ is cyclic. As $G/\text{PH}$ is a $p'$-group also, we get $\mathcal{D} = D$.

Now $\mathcal{G}$ is $p$-solvable with $O_p(\mathcal{G}) = 1$ and with a cyclic Sylow $p$-subgroup of order $p^n$, and $\mathcal{V}$ is a faithful indecomposable $kG$-module belonging to a block with defect $\alpha$ such that $\mathcal{V}_p$ is projective-free. A corresponding statement holds for the conjugates of $\mathcal{V}$. Thus minimality of $|G|$ forces $G = \text{PH}$.

(4) $D = P$ (and so $a = n$):

Of course, $V$ is relatively $D$-projective. In view of Green's theorem [1], there is an indecomposable $k[DH]$-module $U$ with $V \cong U^G$. Note that $DH$ is a normal subgroup of $G = \text{PH}$ containing $P$. It follows that $V_{DH}$ is a direct sum of $P$-conjugates of $U$ and that $P$ cannot centralize $H/C_H(U)$. Thus $O_p(DH/C_H(U)) = 1$, and $U$ is a faithful module for $DH/C_H(U)$ belonging to a block with defect $\alpha$. We have $V_p \cong (U_p)^P$. Hence $U_p$ is projective-free. If every indecomposable summand of $U_p$ is of dimension $d_0$, then every indecomposable summand of $V_p$ is of dimension $p^n - a d_0$. Thus we may conclude that $D = P$ by the choice of $G$.

(5) $V$ is irreducible:

Every composition factor of $V$ is isomorphic to $\mathcal{V}$ and $\mathcal{V}$ is faithful. Assume $V$ is not irreducible. Then any indecomposable summand of $\mathcal{V}_p$ is of dimension $p^n - 1$, because of the choice of $V$ and of (4). The theory of modules over principal ideal domains shows that then also every indecomposable summand of $V_p$ is of the maximal possible dimension $d = p^n - 1$ (Stickelberger). The other statements of the theorem would follow too.

(6) $V_\alpha$ is irreducible:

This follows from (3), (4), (5) and Lemma 4.

(7) Suppose there is an odd prime $q$ dividing $|H : C_H(P_0)|$. Then $p = 2$ and $q = 2^{a_q} - 1$ for some $a_q \leq n$, and $V_p$ has an indecomposable summand of dimension $q \cdot 2^{n - a_q}$.

By Sylow's theorem $P$ normalizes some Sylow $q$-subgroup $S_q$ of $H$. We have $T_q = [S_q, P_0] \neq 1$, and $T_q = [T_q, P_0]$ is $P$-invariant as well. Let $U$ be an indecomposable summand of $V_{PT_q}$ on which $T_q$ acts nontrivially. (The letter $U$ will be used with variable meaning.) Then $P_0$ does not centralize $T_q/C_{T_q}(U)$. It follows that $U$ is a faithful module for $PT_q/C_{T_q}(U)$. Clearly $U_p$ is projective-free. Hence the proposition applies to $PT_q/C_{T_q}(U)$ and $U$. 
Hence, if \( a_q \) denotes the defect of the block containing \( U \), then \( p^{a_q} - 1 \) is a power of \( q \). This forces \( p = 2 \) as \( q \) is odd. From Lemma 1 we infer that even \( q = 2^{a_q} - 1 \). We also know from the proposition that \( U_p \) is indecomposable of dimension \( q \cdot 2^{n-a_q} \), as desired.

(8) If \( p \) is odd, then \( [H, P_0] \) is a 2-group:

By (7) the index \( |H : C_H(P_0)| \) is a power of 2 here. Let \( S_2 \) be a \( P \)-invariant Sylow 2-subgroup of \( H \). Then \( H = C_H(P_0) S_2 \) and

\[
|H, P_0| = |S_2, P_0| \subseteq S_2,
\]
as asserted.

(9) \( P_0 \) centralizes every proper \( P \)-invariant normal subgroup of \( H \):

Assume the contrary. Let \( N \) be a proper \( P \)-invariant normal subgroup of \( H \) which is not centralized by \( P_0 \). Replacing \( N \) by \([N, P_0]\) we may assume that \( N = [N, P_0] \neq 1 \). Note that \( N \) is normal in \( G \). By (4) there exists an indecomposable summand \( U \) of \( V_{P_N} \) with vertex \( P \). This \( U \) belongs to a block of \( P_N \) with defect \( a = n \). Let \( \bar{U} \) be the unique irreducible constituent of \( U \). Then \( W = \bar{U}_N \) is irreducible (Lemma 4). From Clifford’s theorem it follows that every irreducible summand of \( V_N \) is \( G \)-conjugate to \( W \).

Now \( V \) is faithful. We deduce that \( C_N(U) = C_N(W) \) is a proper \( P \)-invariant normal subgroup of \( N \). Since \([N, P_0] = N, P_0 \) does not centralize \( N/C_N(U) \) and so \( O_p(PN/C_N(U)) = 1 \). Thus \( U \) is a faithful (and indecomposable) module for \( PN/C_N(U) \) belonging to a block with defect \( a = n \). Also \( U_P \) is projective-free. By minimality of \( |G| \) we obtain that \( p^n - 1 \) is a power of some prime \( q \) and that \( N/C_N(U) \) is a nonabelian special \( q \)-group of order larger than \( (p^n - 1)^2 \). From the proposition it follows that \( U = \bar{U} \) is irreducible and that \( U_P \) is indecomposable. Moreover,

\[
dim U = \dim W = p^n - 1.
\]

Since \( N/C_N(W) \) is a \( q \)-group and \( \bigcap_{g \in G} C_N(W)^g = 1 \), \( N \) itself is a \( q \)-group. As in the proof of the proposition, by showing that \( P_0 \) centralizes every \( P \)-invariant abelian subgroup of \( N \), we obtain that \( N \) is even a special \( q \)-group.

Suppose \( \bar{U} (\neq 0) \) is another indecomposable summand of \( V_{P_N} \). Then \( \bar{U}_N \) is a direct sum of \( G \)-conjugates of \( W \). Let \( \bar{a} \) be the defect of the block containing \( \bar{U} \). From the proposition it follows that

\[
dim \bar{U} = p^{\bar{a}} - (p^{\bar{a}} - 1)
\]

with \( p^{\bar{a}} - 1 \) being a power of \( q \). But \( \dim \bar{U} \) must be a multiple of \( \dim W = p^n - 1 \). This forces \( \bar{a} = n \). Since also \( \bar{U}_P \) is indecomposable, we see that every indecomposable summand of \( V_P \) is of dimension \( d = p^n - 1 \).

As \( G \) is a counterexample we now must have \([H, P_0] = H\), because
otherwise we could specialize \( N = [H, P_0] \). We claim that nevertheless \( H \) is a \( q \)-group. If \( p \) is odd, then \( q = 2 \) and \( H \) is a 2-group by (8). In case \( p = 2 \) we infer from (7) that \( [H : C_H(P_0)] \) is a power of \( q \), because all indecomposable summands of \( V_p \) have the same dimension \( 2^n - 1 \). Hence \( H = [H, P_0] \) is a \( q \)-group as claimed. We are in the situation of the proposition, in view of (3), which is impossible by the choice of \( G \).

\[
(H, P_0) = H \text{ and } p = 2;
\]

From (2) and (9) it follows that \( [H, P_0] = H \). If \( p \) is odd, then \( H \) is a 2-group by (8), and the proposition applies.

It remains to handle the case \( p = 2 \). Using solvability of \( H = O_{2n}(G) \) it is fairly easy to complete the proof also in this case. We are able, however, to avoid the Feit–Thompson theorem. Let \( \pi \) denote the set of all primes dividing \( [H : C_G(P_0)] \). For every \( q \in \pi \) fix a \( P \)-invariant Sylow \( q \)-subgroup \( S_q \) of \( H \) and write \( T_q = [S_q, P_0] \). Recall that then \( q = 2^{a_q} - 1 \) is a Mersenne prime by (7).

\[
(11) \quad |\pi| > 1:
\]

Assume \( \pi = \{q\} \) consists of one element. Then \( [H : C_H(P_0)] \) is a power of \( q \) and \( [H, P_0] = [S_q, P_0] \subseteq S_q \). But then, by (10), \( H = [H, P_0] \) is a \( q \)-group and the proposition applies.

\[
(12) \quad H = \langle S_q ; q \in \pi \rangle:\n\]

Let \( \bar{H} = \langle S_q ; q \in \pi \rangle \). By definition of the set \( \pi \) of primes, the indices \( [H : \bar{H}] \) and \( [H : C_H(P_0)] \) are relatively prime. Hence \( H = C_H(P_0)\bar{H} \) and so \( [H, P_0] = [\bar{H}, P_0] \subseteq \bar{H} \), because \( \bar{H} \) is \( P \)-invariant. Consequently \( H = \bar{H} \) by (10).

\[
(13) \quad \text{If } R \text{ is a proper } P \text{-invariant subgroup of } H, \text{ then } [R, P_0] \text{ is a nilpotent } \pi \text{-group:}
\]

We may assume that \( R = [R, P_0] \neq 1 \). Let \( U \) be an indecomposable summand of \( V_{PR} \) on which \( R \) acts nontrivially. Then \( P_0 \) does not centralize \( R/C_R(U) \) and therefore \( O_2(PR/C_R(U)) = 1 \). By minimality of \( |G| \) we deduce that \( R/C_R(U) \) is a \( q \)-group for some \( q \in \pi \). Since \( V \) is faithful, the assertion follows.

\[
(14) \quad T = \langle T_q ; q \in \pi \rangle \text{ is nilpotent:}
\]

It is evident that \( T \) is \( P \)-invariant and that \( [T, P_0] = T \). In view of (13) it therefore suffices to show that \( T \) is a proper subgroup of \( H \). Let \( U \) be an indecomposable summand of \( V_{PT} \) on which \( T_q \) does not act trivially (\( q \in \pi \)). Then, as in step (7), the proposition applies. We obtain that \( U \) belongs to a block with defect \( a_q \) and that \( U_p \) is indecomposable with dimension \( q \cdot 2^{n-a_q} \). Now write

\[
V_p = \bigoplus_{q \in \pi} V_q \oplus L,
\]
where $V_q$ is the direct sum of all indecomposable summands of $V_P$ with dimension $q \cdot 2^{n-a_q}$ ($q \in \pi$) and $L$ is the sum of the remaining components. Any $T_q$ respects this decomposition. In fact, $T_q$ acts faithfully on $V_q$ and trivially on the other components. From (11) we infer that $V_T$ cannot be irreducible. Consequently $T \neq H$ because $V_H$ is irreducible by (6).

We are now in a position to achieve the final contradiction: Fix $q \in \pi$. $T_{q_0} = [S_{q_0}, P_0]$ is not normal in $H$ for otherwise $H = T_{q_0}$ were a $q_0$-group by (9) and so $\pi = \{q_0\}$ in contrast to (11). On the other hand, by (14)

$$R_0 = N_H(T_{q_0}) \supseteq T.$$ 

Note that $R_0 \supseteq S_{q_0}$ and that $R_0$ is a proper $P$-invariant subgroup of $H$. From (13) it follows that $[R_0, P_0] \supseteq [T, P_0] = T$ is nilpotent. Using that $O_2([R_0, P_0])$ is contained in a $P$-invariant Sylow $q$-subgroup of $H$ and that the $P$-invariant Sylow $q$-subgroups of $H$ are conjugate under $C_H(P) \subseteq C_H(P_0)$, we deduce that

$$T_q = O_q([R_0, P_0])$$

for any $q \in \pi$. Consequently $T_q$ is normalized by $S_{q_0}$. By symmetry we get that any $S_q$ normalizes $T_{q_0}$. From (12) we obtain that $T_{q_0}$ is yet normal in $H$. This completes the proof of the theorem.

\textit{Note added in proof}. (Nov. 9, 1981). It might be worth mentioning that the module $V$ in the second alternative of Theorem B"{a} necessarily is completely reducible. We know that $N = [O_{\ast}(G), x^{e-1}]$ is a normal $q$-subgroup of $G$ for some prime $q \neq p$ and that all composition factors of $V$ are isomorphic (by block theory). Applying the Proposition we obtain that every indecomposable summand of $V_{p^N}$ is irreducible (of dimension $d$). Now use the fact that $V$ is relatively $PN$-projective.

\textbf{References}