Maximal Orders over Valuation Rings

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In this paper we study maximal orders over commutative valuation rings in central simple algebras. We are particularly interested in maximal orders which are either Bézout or semihereditary. We construct a class of Bézout maximal orders and a class of semihereditary maximal orders, and show that for any valuation ring $V$ (resp. $V$ with value group $\mathbb{Z}^m$), any Bézout (resp. semihereditary) maximal order over $V$ belongs to the class constructed. Furthermore, we classify all maximal orders in $M_2(F)$ over a valuation ring with value group $\mathbb{Z}^m$ and in $M_n(F)$ given a mild "defectless" assumption.

1. INTRODUCTION

The subject of maximal orders over Noetherian integrally closed domains has a rich history and has been a major area of study in noncommutative ring theory. The case of orders over discrete valuation rings has been the focus of much of this work, and there is a great deal known about such rings. For example, any two maximal orders in a central simple algebra over a discrete valuation ring are isomorphic, and if the discrete valuation ring is complete then the maximal orders can be described completely (see [R]). Results such as these have been quite useful in determining the arithmetic of division algebras over certain classes of fields (such as local and global fields), and indicate that maximal orders merit the attention given to them.

The case of maximal orders over non-Noetherian valuation rings is much different. This subject seems to have been largely neglected, and little is

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apparently known about such rings. One reason for this is because much
is lost by not having the Noetherian condition. For instance, two
fundamental properties of a maximal order \( R \) over a discrete valuation ring
are that any right (or left) ideal of \( R \) is principal and projective as
an \( R \)-module. It is unreasonable to hope that maximal orders over
non-Noetherian valuation rings have these properties since non-Noetherian
valuation rings fail to do so. However, any finitely generated ideal of a
valuation ring is principal (and projective). Thus it is natural to consider
maximal orders whose finitely generated one-sided ideals are principal (the
Bézout property) or projective (semihereditary). In this paper we consider
maximal orders in central simple algebras over commutative valuation
rings, concentrating on maximal orders which are either Bézout or
semihereditary. Although we consider maximal orders over an arbitrary
valuation ring \( V \), we obtain more complete results when \( V \) is a generalized
discrete valuation ring, that is, if the value group of \( V \) is isomorphic to \( \mathbb{Z}^n \),
ordered antilexiographically. In Section 2 we give some preliminary results.
Section 3 is concerned with Bézout maximal orders. The main result of this
section is that an order \( R \) is Bézout iff \( R \) is a suitable intersection of
Dubrovin valuation rings. Furthermore, if \( V \) is a generalized discrete
valuation ring then \( R \) is Bézout iff \( RW \) is a maximal \( W \)-order for any
overring \( W \) of \( V \) in \( F \).

In Section 4 we consider semihereditary maximal orders. We almost
exclusively restrict to the case of orders in \( M_n(D) \) where \( D \) is a division
algebra containing an invariant valuation ring \( B \). Using \( B \) we construct a
class of "block matrix" orders, and prove these are semihereditary maximal
orders. If \( V \) is generalized discrete and \( S = M_n(D) \) as above, then any semi-
hereditary maximal order is isomorphic to one of these block matrix
orders.

Finally, in Section 5 we attempt to classify all maximal \( V \)-orders in
\( M_n(F) \) for \( V \) a generalized discrete valuation ring. We succeed in the case
\( n = 2 \). The ideas in the \( n = 2 \) proof can be used to classify maximal orders
in \( M_n(F) \) for some small \( n \), but the general structure for large \( n \) becomes
unwieldy. However, given a "defectless" assumption, we show that any
maximal order in \( M_n(F) \) is semihereditary, thus classifying all maximal
orders by the results in Section 4. This defectless assumption is mild, as it
occurs quite often. For instance, if \( \bar{V} \) is the residue field of \( V \), then if
\( \text{char}(F) = 0 \) or \( \text{char}(\bar{F}) > n \) then the defectless assumption holds. Also, if \( F \)
is maximally complete with respect to \( V \) then this assumption holds for any
\( n \). By using defective field extensions we construct another class of maximal
orders and indicate how all the above constructions can be combined to
give more complicated examples of maximal orders. I thank D. Haile and
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2. Preliminaries

In this section we briefly discuss some of the ring theoretic properties dealt with in the later sections and prove some basic results that will help us analyze maximal orders. Some of the notation used in this paper is as follows. For a ring $R$, $J(R)$ will denote the Jacobson radical of $R$, $Z(R)$ the center of $R$, $R^*$ the group of units of $R$ and $M_n(R)$ the ring of $n \times n$ matrices over $R$. If $A$, $B$, $C$, and $D$ are subsets of $R$ then the set $\{(a_{ij}) | a \in A, b \in B, c \in C, d \in D\} \subseteq M_2(R)$ will be denoted by $(\begin{array}{cc} A & B \\ C & D \end{array})$. For larger matrices, following [R, Ch. 9] we denote by

$$(A_{11} \ldots A_{1m})_{\{n_1, \ldots, n_t\}}$$

the set of all

$$\begin{pmatrix}
    a_{11} & \cdots & a_{1m} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nm}
\end{pmatrix},$$

where $a_{ij}$ is an $n_j \times n_i$ matrix with entries in $A_{ij}$. We now define maximal orders in the context that we will consider.

**Definition 2.1.** Let $S$ be a finite-dimensional $F$-algebra and $V$ a ring with quotient field $F$. A subring $R$ of $S$ is said to be an order in $S$ if $RF = S$. If $V \subseteq Z(R)$ then $R$ is said to be a $V$-order if in addition $R$ is integral over $V$. If $R$ is maximal with respect to inclusion among $V$-orders of $S$ then $R$ is called a maximal order over $V$.

In most definitions (e.g., [R, p. 108]) of maximal orders over discrete valuation rings, the ring $R$ is assumed to be a finitely generated $R$-module, which is equivalent to integrality (for $S$ a separable $F$-algebra) in that case. This equivalence is no longer true if $V$ is not Noetherian. The integrality hypothesis in the definition above is used to guarantee the existence of maximal orders for any $S$ and $V$, as an easy Zorn's lemma argument shows. If finite generation is required this existence may fail, as seen in Proposition 2.3.

In this paper we will be concerned with maximal orders inside central simple algebras that are either Bézout or semihereditary. A ring $R$ is said to be right (resp. left) Bézout if any finitely generated right (resp. left) ideal of $R$ is principal. The ring $R$ is said to be right semihereditary (resp. hereditary) if all finitely generated right ideals (resp. all right ideals) of $R$
are projective as $R$-modules. Left semihereditary (resp. hereditary) is defined similarly. If $R$ is both left and right Bézout (resp. semihereditary or hereditary) then $R$ is simply called Bézout (resp. semihereditary or hereditary). For prime $PI$ rings it is not hard to see that a Bézout ring is semihereditary. In Section 3 we shall see that there is a close connection between Bézout maximal orders and Dubrovin valuation rings. Definitions and properties of Dubrovin valuation rings can be found in [D$_1$, D$_2$, BG$_2$, M$_1$, W$_2$]. There are a few properties of Dubrovin valuation rings that we will be using repeatedly in this paper. Any Dubrovin valuation ring of a central simple algebra $S$ is a prime $PI$ Bézout ring and its two-sided ideals are linearly ordered by inclusion, as are its overrings in $S$. Any overring $A$ of a Dubrovin valuation ring $B$ in $S$ is itself a Dubrovin valuation ring and is a central localization $A = BZ(A)$ of $B$. The following type of Dubrovin valuation ring will occur frequently in Section 4. Recall that a subring $B$ of a division algebra $D$ is an invariant valuation ring if $B$ is the valuation ring of a Krull valuation on $D$. It is well known that $B$ is characterized by the two properties (1) if $d \in D - B$ then $d^{-1} \in B$, and (2) $dBd^{-1} = B$ for all $d \in D - \{0\}$. Furthermore, $B$ is the integral closure of $Z(B)$ in $D$ [W$_1$, Cor.], hence is the unique maximal $Z(B)$-order in $D$. Note that (1) implies that the two-sided ideals of $B$ are linearly ordered by inclusion and that (2) shows that the one-sided ideals of $B$ are two-sided. For further information on invariant valuation rings see [Sc].

Suppose $V$ is a discrete valuation ring of $F = Z(S)$. A $V$-order $B$ of $S$ is a maximal order iff $B$ is a Dubrovin valuation ring with $B \cap F = V$ [W$_2$, Ex. 1.15]. The following is a generalization of this example.

**Example 2.2.** Let $S$ be a central simple $F$-algebra and $B$ a Dubrovin valuation ring of $S$ with $V = B \cap F$. If $B$ is integral over $V$ then $B$ is a maximal order over $V$.

**Proof:** If $A$ is a proper overring of $B$ in $S$ then by [D$_2$, p. 495, (E), p. 499, Th. 1] $A$ is a Dubrovin valuation ring with center $W \supseteq V$. Since $V$ is integrally closed, $W$, and hence $A$, is not integral over $V$. Thus $B$ is maximal. □

It is well known that using the traditional definition of maximal order that maximal orders need not exist. For instance, if $K/F$ is a finite extension and $V$ is a discrete valuation ring of $F$ such that the integral closure $U$ of $V$ in $K$ is not a finite $V$-module, then there are no (finitely generated) maximal orders over $V$ in $K$. This does not happen inside separable $F$-algebras for $V$ a discrete valuation ring. However, by using Dubrovin valuation rings we show in the next proposition the necessity of integrality versus finite generation for maximal orders in central simple $F$-algebras over general valuation rings $V$. 
PROPOSITION 2.3. Let $S$ be a central simple $F$-algebra and $V$ a valuation ring of $F$.

(a) Let $B$ be a Bézout order of $S$ containing $V$. If $R$ is a subring of $S$ finitely generated as a $V$-module then $R \subseteq xBx^{-1}$ for some $x \in S^*$. In particular, $R$ lies in a Dubrovin valuation ring $B$ with $B \cap F = V$.

(b) Finitely generated maximal orders need not exist inside central simple algebras.

Proof. (a) Let $B$ be a Bézout order of $S$ with $V \subseteq B$. Since $R$ is a finitely generated $V$-module, $RB$ is a finitely generated right $B$-module in $S$. The finite generation implies that there is an $\alpha \in V$ with $\alpha RB \subseteq B$. Then $\alpha RB$ is a finitely generated right ideal of $B$, hence principal since $B$ is Bézout. Say $\alpha RB = yB$. Then $RB = \alpha^{-1}yB := xB$. Since $1 \in RB$, we have $x \in S^*$. Therefore $xB = RB = R(\alpha B) \supseteq Rx$, so $R \subseteq xBx^{-1}$. If $B$ is a Dubrovin valuation ring with $B \cap F = V$ then $B$ is Bézout. Hence $R$ lies in the Dubrovin valuation ring $xBx^{-1}$.

To prove (b), let $S$ be an $F$-central simple algebra and $A$ a Dubrovin valuation ring of $S$ with center $V$ such that $A$ is integral over $V$ but not finitely generated as a $V$-module. Suppose $R$ is a finitely generated $V$-order. By (a), $R \subseteq xAx^{-1} := A'$ for some $x \in S^*$. Since $A'$ is not finitely generated over $V$, $R \not\subseteq A'$. Take $a \in A' - R$. Then by Shirshov's theorem [Ro, Cor. 4.2.9], $R[a]$ is finitely generated as a $V$-module and $R \not\subseteq R[a]$. Thus $R$ is not maximal. Therefore there are no finitely generated maximal orders over $V$ in $S$.

Part (a) of this proposition (for $B$ a Dubrovin valuation ring) was discovered independently by M. Westmoreland, who uses it in his thesis [We]. This result is also utilized in [HM].

The next lemma will be used in the proof of Theorem 3.4, but is also of independent interest.

LEMMA 2.4. Let $R$ be an order in $S$ with $V = Z(R)$ a valuation ring. Then $R$ has only finitely many maximal ideals.

Proof. Suppose $M_1, \ldots, M_n$ are maximal ideals of $R$. Then $J(R) \subseteq \bigcap_i M_i$, so there is a surjection of $R/J(R)$ onto $R/\bigcap_i M_i \cong \bigoplus_i R/M_i$. Now $R/J(R)$ and each $R/M_i$ are $V/J(V)$-algebras. Thus $n \leq [\bigoplus_i R/M_i : V/J(V)] \leq [R/J(R) : V/J(V)] \leq [R/J(V) : V/J(V)]$. However, $[R/J(V) : V/J(V)] \leq [S : F] < \infty$ since elements of $R$ which are linearly independent mod $J(V)R$ over $V/J(V)$ are linearly independent over $F$. Therefore $n$ is bounded above, and hence finite.

Suppose $V$ is a valuation ring of a field $F$ of Krull dimension greater than one. Then there are proper overrings $W$ of $V$ in $F$. If $R$ is a $V$-order
in $S$ then $RW$ is a $W$-order in $S$. It is possible for $R$ to be maximal over $V$ and $RW$ not maximal over $W$ and vice-versa; see the comments after Proposition 4.3. The relationship between orders and overrings will be a major consideration in describing maximal orders over valuation rings.

Note that if $R$ is a $V$-order in $S$ and $W$ is an overring in $V$ in $F$, if $T = RW$ and $J(T) \subseteq R$ then $R/J(T)$ is a $V/J(W)$-order in $T/J(T)$. The last lemma of this section is easy to prove, yet will prove invaluable throughout this paper.

**Lemma 2.5.** Let $R \subseteq T \subseteq S$ with $R$ a $V$-order and $T$ a $W$-order with $W \subseteq V$.

(a) If $R$ is maximal then $J(T) \subseteq R$, and if $T = RW$ then $R/J(T)$ is a maximal order over $V/J(W)$ in $T/J(T)$.

(b) If $C$ is a subring of $T/J(T)$ integral over $V/J(W)$ and $C = \{ t \in T | t + J(T) \in C \}$, then $C$ is a $V$-order with $J(T) \subseteq C$.

(c) If $T = RW$ and $J(T) \subseteq R$ then $R$ is maximal over $V$ if $T$ is maximal over $W$ and $R/J(T)$ is maximal over $V/J(W)$.

**Proof.** We first show that $J(T)$ is integral over $V$. Since $T/J(W)T$ is finite-dimensional over $W/J(W)$, $T/J(W)T$ is Artinian, hence has nilpotent radical. Thus $J(T)^n \subseteq J(W)T$ for some $n$. Take $x \in J(T)$. Then $x^n = ax$ with $a \in J(W)$ and $a \in T$. Since $T$ is integral over $W$, there is a monic polynomial $f(t) \in W[t]$ such that $f(a) = 0$. If $\deg(f) = m$ then $x^m f(x^{-1}t)$ is a monic polynomial with $ax^n$ as a root. Since all nonleading coefficients of this polynomial lie in $J(W) \subseteq V$, $x^n$, and hence $x$ is integral over $V$. For the first part of (a), the argument above shows that the ring $R + J(T)$ is generated as a $V$-module by elements integral over $V$. Since $R + J(T)$ is a PI ring, $R + J(T)$ is integral over $V$ by [AS, Th. 2.3]. Maximality of $R$ then gives $J(T) \subseteq R$.

Before proving the second half of (a) we prove (b). Clearly $C$ is a subring of $T$ with $J(T) \subseteq C$. Since $J(T)F = S$, $CF = S$. To show $C$ is a $V$-order we thus need to show integrality. Take $c \in C$. Since $C$ is integral over $V/J(W)$ there is a monic polynomial $f(x) \in V[x]$ with $f(c) \equiv 0 \mod J(T)$, so $f(c) \in J(T)$. Thus $f(c)$ is integral over $V$ by the argument above. Because $f(x)$ is monic we see that $c$ is integral over $V$. This proves (b). To continue with (a), suppose $T = RW$. As noted before this lemma, $R/J(T)$ is a $V/J(W)$-order in $T/J(T)$. Suppose $R/J(T) \subseteq R'$, where $R'$ is an order over $V/J(W)$. If $R' = \{ t \in T | t + J(T) \in R' \}$ then by (b) $R' \subseteq R$ is a $V$-order. Maximality of $R$ shows $R' = R$, so $R' = R/J(T)$. Therefore $R/J(T)$ is maximal.

For the proof of (c), suppose $R' \supseteq R$ is integral over $V$. Then $R'W \supseteq RW = T$ is integral over $W$, hence $R'W = T$ by the maximality of $T$.
Therefore $R' \subseteq T$. Thus we obtain $R/J(T) \subseteq R'/J(T) \subseteq T/J(T)$. Since $R'$ is integral over $V$, $R'/J(T)$ is integral over $V/J(W)$, and so maximality of $R/J(T)$ gives $R/J(T) = R'/J(T)$. Since $J(T) \subseteq R \subseteq R'$, we obtain $R' = R$. Therefore $R$ is a maximal order over $V$.

In the following sections we will consider arbitrary valuation rings but will be especially concerned with the following type of valuation ring. We will call $V$ a generalized discrete valuation ring if the value group of $V$ is isomorphic to $\mathbb{Z}^n$, ordered antilexicographically. Since the rank of $\mathbb{Z}^n$ as an ordered group is $n$, the Krull dimension of $V$ is $n$, and there are exactly $n$ proper overrings of $V$ in $F$. We will refer to $n$ as the rank of $V$. Say $V = V_n \supseteq V_{n-1} \supseteq \ldots \supseteq V_1 \supseteq F$ are the overrings of $V$. Then $V_i$ is a generalized discrete valuation ring of rank $i$, and so $V_1$ is a discrete valuation ring. Also, $V_{i+1}/J(V_i)$ is a discrete valuation ring of the field $V_i/J(V_i)$ for each $i$. These valuation rings lend themselves to induction arguments. Many of our arguments in the following sections will use results about discrete valuation rings together with induction. It is because of the lack of such techniques that the author has as yet been unable to prove (or disprove) some of the theorems in this paper for general valuation rings.

3. Bézout Maximal Orders

In this section we consider maximal orders that are Bézout. By using recent results of Gräter we are able to classify these orders, showing that they are intersections of Dubrovin valuation rings. We start by briefly discussing such intersections. Let $S$ be a central simple $F$-algebra and $V$ a valuation ring of $F$. Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of $S$ and set $R = \bigcap B_i$. Following $[G_1]$ we say $B_1, \ldots, B_n$ have the intersection property (IP) if there is a well defined inclusion reversing correspondence between the Dubrovin valuation rings of $S$ containing $R$ and the prime ideals of $R$ given by $A \leftrightarrow J(A) \cap R$. By $[G_1$, Cor. 6.2, Prop. 6.3, Cor. 6.7] the intersection property is equivalent to the $B_i$ satisfying the hypotheses of the approximation theorem $[M_2$, Th. 2.3]. If the $B_i$ satisfy the IP then $R$ is Bézout by $[M_2$, Th. 3.4]. Furthermore, there exists $B_1, \ldots, B_n$ with $B_i \cap F = V$ satisfying the IP such that $R$ is integral over $V$; in this case $R$ is unique up to isomorphism $[G_1$, Th. 6.11, 6.12]. The main result of this section, Theorem 3.4, is that the Bézout maximal orders over $V$ are precisely the intersections of Dubrovin valuation rings satisfying the IP which are integral over $V$. We first prove one direction of this. The author thanks J. Gräter for showing him an early version of $[G_2]$. The results in that paper allowed the author to prove Theorem 3.4 for any valuation ring $V$. Previous to seeing $[G_2]$ the author could only prove Theorem 3.4 for $V$ a generalized discrete valuation ring, and required more work to do so.
Lemma 3.1. Let $R$ be a prime PI Bézout ring. If $S$ is the quotient ring of $R$ and $T$ is any overring of $R$ in $S$ then $T$ is Bézout.

Proof. Let $V = Z(R)$. If $I = a_1T + \cdots + a_nT$ is a finitely generated right ideal of $T$, then as $RF = S$, where $F$ is the quotient field of $V$, we can write $a_i = b_i\alpha^{-1}$ with the $b_i \in R$ and $\alpha \in V$. Set $K = b_1R + \cdots + b_nR$. Since $R$ is Bézout, $K = xR$ for some $x \in R$. Thus $KT = xT$. But $KT = \sum_i b_iR = aI$, so $I = (\alpha^{-1}x)T$. Therefore $I$ is principal. Hence $T$ is Bézout.  

Proposition 3.2. Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of an $F$-central simple algebra $S$ with $B_i \cap F = V$ satisfying the IP with $R = \bigcap_i B_i$ integral over $V$. Then $R$ is a Bézout maximal order over $V$. In addition, $RW$ is a maximal order over $W$ for all overrings $W$ of $V$ in $F$.

Proof. Let $R = B_1 \cap \cdots \cap B_n$, where the $B_i$ are Dubrovin valuation rings of $S$ satisfying the IP, $V = B_i \cap F$, and $R$ is integral over $V$. Then $R$ is a order in $S$ since each $B_i$ is an order in $S$. Suppose $R \subseteq T$ with $T$ a maximal order over $V$ in $S$. Then $T$ is Bézout by Lemma 3.1, and is semilocal by Lemma 2.4. Thus by [G_2, Cor. 3.2, Th. 3.6], $T = A_1 \cap \cdots \cap A_m$ with the $A_i$ Dubrovin valuation rings satisfying the IP. Hence by [G_1, Th. 6.12] $R$ and $T$ are conjugate, say $T = xRx^{-1}$ for some $x \in S^*$. Then $R = x^{-1}Tx \subseteq T$, so $T \subseteq xTx^{-1}$. Maximality of $T$ gives $T = xTx^{-1}$, so $T = x^{-1}Tx = R$. Thus $R$ is maximal.

Finally, to show $RW$ is maximal over $W$ for all overrings $W$ of $V$ in $F$, note that $RW = \bigcap_i B_i W = \bigcap_i A_i$. Since the $A_i$ have the IP by [G_1, Th. 6.8], the first part of this theorem shows $RW$ is maximal over $W$.

Given a maximal $V$-order $R$ and an overring $W$ of $V$ in $F$, $RW$ is integral over $W$ but need not be maximal (see the example after Proposition 4.3). However, if $R$ is Bézout and $V$ is a generalized discrete valuation ring then $RW$ is maximal for each $W$, as will be shown in Theorem 3.4. In order to prove Theorem 3.4 we need a preliminary lemma.

Lemma 3.3. Let $R$ be an order in $S$ and $T$ an overring of $R$ in $S$. Suppose $T = RZ(T)$ and that there is an ideal $J$ of both $T$ and $R$ with $J \subseteq J(T)$. Then $R$ is Bézout iff both $T$ and $R/J$ are Bézout.

Proof. Suppose $R$ is Bézout. It is clear that $R/J$ is Bézout, and $T$ is Bézout by Lemma 3.1. Conversely, suppose $T$ and $R/J := \hat{R}$ are Bézout. Let $I$ be a finitely generated right ideal of $R$. First suppose $I$ is regular (that is, $IS = S$). Then $IT = aT$ with $a \in S^*$ as $T$ is Bézout. By considering $a^{-1}I$ we may assume $IT = T$. Thus $J \subseteq I$ since if $1 = \sum_i x_i a_i$ with $x_i \in I$ and $a_i \in T$ then for $m \in J$ we have $m = \sum_i x_i(a_i m) \in IJ \subseteq IR = I$. Hence we obtain $I/J$ is a finitely generated right $\hat{R}$-submodule of $T/J$. So $I/J = \hat{x}\hat{R}$ for some $x \in T$.
as $\bar{R}$ is an order in $T/J$. As $IT = T$, $(I/J)(T/J) = T/J$, so $\bar{x} \in (T/J)^*$. Since $J \subseteq J(T)$ we have $x \in T^*$. Therefore $J \subseteq xR$, and so $I = xR$. For general $I$, if $I$ is not regular then [MR, 2.2.2, 2.3.51] there is a $y$ with $I + yR \subseteq I \oplus yR$ regular. By the argument above $I + yR = cR$ for some $c$. By replacing $I$ by $c^{-1}I$ we may assume $I + yR = R$. Then $1 = a + b$ with $a \in I$ and $b \in yR$. If $u \in I$ then $u = au + bu$. Since the sum $I + yR$ is direct this gives $u = au$, so $I \subseteq aI \subseteq aR \subseteq I$, therefore $I = aR$. Thus $R$ is Bézout.

We can now prove the main result of this section, a characterization of Bézout maximal orders over a valuation ring.

**Theorem 3.4.** Let $V$ be a valuation ring of a field $F$ and $S$ a central simple $F$-algebra. Suppose $R$ is a $V$-order in $S$. Then the following statements are equivalent.

(i) $R$ is a Bézout order.

(ii) $R$ is a Bézout maximal order.

(iii) $R$ is an intersection of Dubrovin valuation rings satisfying the intersection property.

If these statements hold then the following statements also holds:

(iv) $RW$ is maximal over $W$ for any overring $W$ of $V$ in $F$.

Furthermore, if $V$ is a generalized discrete valuation ring then (iv) is equivalent to each of the three statements above.

**Proof.** We have (iii) $\Rightarrow$ (ii) by Proposition 3.2 and (ii) $\Rightarrow$ (i) is clear. For (i) $\Rightarrow$ (iii), suppose $R$ is a Bézout order. Let $M$ be a maximal ideal of $R$ and set $G_R(M) = \{ r \in R \mid r + M$ is regular in $R/M \}$. Then by [G$_2$, Th. 2.3, Th. 3.4] $G_R(M)$ is a two-sided Ore set and the localization $R_{M}$ is a Dubrovin valuation ring with $J(R_{M}) \cap R = M$. If $Z(R_{M}) \not\subseteq V$ for some $M$ then there is an $\alpha \in J(V) - J(R_{M})$. Then $\alpha^{-1} \in R_{M}$. Writing $\alpha^{-1} = rs^{-1}$ with $r \in R$ and $s \in G_R(M)$ shows $s = rs \in J(V)R \subseteq M$, which is false. Therefore $Z(R_{M}) = V$ for all $M$. Let $M_1, \ldots, M_t$ be all the maximal ideals of $R$, a finite set by Lemma 2.4. By [G$_2$, Cor. 3.2] $R = \bigcap_i R_{M_i}$. The $R_{M_i}$ have the IP by [G$_2$, Th. 3.5]. Thus (iii) holds. This completes the equivalence of the first three statements.

If statement (iii) holds then so does (iv) by Proposition 3.2. Now suppose $V$ is a generalized discrete valuation ring and that (iv) holds. We prove that $R$ is a Bézout maximal order. To do this we use induction on $\text{rank}(V) = n$. If $n = 1$ then $V$ is a discrete valuation ring and so $R$ is a Dubrovin valuation ring [W$_2$, Ex. 1.15] and so is Bézout by [D$_1$, p. 276, Th. 4]. For $n > 1$ let $W \not\subseteq V$ be of rank $n - 1$ and set $T = RW$. By hypothesis $T$ is a maximal order over $W$, and $T$ is Bézout by induction. Since $R$ is
maximal by hypothesis, \( J(T) \subseteq R \) by Lemma 2.5. Thus \( R/J(T) := \overline{R} \) is a maximal order over \( \overline{V} := V/J(W) \) in the semisimple Artinian ring \( \overline{T} = T/J(T) \). Say \( T := \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_r \) with the \( \mathcal{S}_i \) simple. Let \( e_1, \ldots, e_r \) be the standard central idempotents of \( T \) with \( e_i T e_i = \mathcal{S}_i \). Set \( e_i \overline{R} e_i = \overline{\mathcal{S}_i} \), an order in \( \mathcal{S}_i \). Then \( \overline{\mathcal{S}_i} \) is integral over \( \overline{V} \) and \( \overline{R} \subseteq \overline{\mathcal{S}_1} \oplus \cdots \oplus \overline{\mathcal{S}_r} \). The maximality of \( \overline{R} \) then shows \( \overline{R} = \overline{\mathcal{S}_1} \oplus \cdots \oplus \overline{\mathcal{S}_r} \). If the \( \overline{\mathcal{S}_i} \) are not maximal over \( \overline{V} \), take \( \overline{C_i} \) a \( \overline{V} \)-order in \( \mathcal{S}_i \) containing \( \overline{\mathcal{S}_i} \). Then \( C_1 \oplus \cdots \oplus C_r \) would be a \( \overline{V} \)-order of \( \overline{T} \) larger than \( \overline{R} \), which is false. We will have shown (ii) as soon as we know each \( \overline{\mathcal{S}_i} \) is Bézout, since we will then obtain \( \overline{R} = \overline{\mathcal{S}_1} \oplus \cdots \oplus \overline{\mathcal{S}_r} \) is then Bézout, and thus \( R \) will be Bézout by Lemma 3.3. Therefore we may suppose \( S \) is central simple with \( F \subseteq Z(S) \) and \( V \) is a discrete valuation ring of \( F \). Let \( Y_1, \ldots, Y_r \) be the extensions of \( V \) to \( Z(S) \), and let \( C_i \) be a maximal order over \( Y_i \) containing \( R \). As \( \cap_i C_i \) is integral over \( \cap_i Y_i \), the integral closure of \( \cap \) in \( Z(S) \), we obtain \( R = \cap_i C_i \). However, as the \( Y_i \) are discrete valuation rings, they are pairwise independent, hence \( R = \cap_i C_i \) is Bézout by [M_2, Th. 3.4].

**Corollary 3.5.** (a) Suppose that \( B \) is a Dubrovin valuation ring integral over \( V \) in \( S \). Then any Bézout order over \( V \) in \( S \) is isomorphic to \( B \).

(b) Suppose \( V \) is a Henselian valuation ring of a field \( F \). If \( R \) is a Bézout order over \( V \) then \( R \) is a Dubrovin valuation ring.

**Proof.** (a) This follows from Theorem 3.4 since there cannot be a collection of Dubrovin valuation rings over \( V \) satisfying the IP whose intersection is integral over \( V \), unless the collection contains a single Dubrovin valuation ring, by [G_1, Th. 6.12]. For (b), this follows from (a) and the fact that any Dubrovin valuation ring of \( S \) is integral over \( V \).

The situation of maximal orders over valuation rings whose value group is not isomorphic to \( \mathbb{Z}^n \) is more complicated. For instance, any two maximal orders over a discrete valuation ring in a central simple algebra \( S \) are isomorphic, which is no longer true for a non-Noetherian valuation ring, even of rank one, as can be seen from the following example. This example also shows statement (iv) of Theorem 3.4 is not equivalent to the others in general.

**Example 3.6.** Let \( V \) be a (nondiscrete) valuation ring of a field \( F \) with value group \( \mathbb{Q} \). Let \( \nu \) be a valuation on \( F \) corresponding to \( V \) and set \( K = \{ x \in F | \nu(x) > \sqrt{2} \} \) and \( L = \{ x \in F | \nu(x) < -\sqrt{2} \} \). Then \( R = \{ (a, b) | a, d \in V, b \in K, c \in L \} \) is a maximal order over \( V \) in \( M_2(F) \). Furthermore, \( R \) is not Bézout, and so is not isomorphic to \( M_2(V) \), another maximal order in \( M_2(F) \) over \( V \).

**Proof.** Note that \( KL = J(V) \), \( K = \{ x \in F | Lx \subseteq V \} \), and \( L = \{ x \in F | \).
Because $KL \subseteq V$, $R$ is a ring, and is an order in $M_2(F)$ since $K$ and $L$ are nonzero. As the determinant and trace of any $x \in R$ lie in $V$, $R$ is integral over $V$. For maximality of $R$, suppose $R' \supsetneq R$ is integral over $V$. If $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in R'$, then as $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \in R$, we have $(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \in R$. Since $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ is integral over $V$, we obtain $a$ and $d$ are integral over $V$, hence $a, d \in V$. Also, as $(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \subseteq R$, $(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \cdot (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \subseteq R'$, giving $Lb \subseteq V$, so $b \in K$. Similarly $c \in L$, so $R' = R$. Therefore $R$ is maximal.

Suppose $R$ is Bézout. Let $T = M_2(V)$, a Dubrovin valuation ring which is integral over $V$. By Corollary 3.5, $R \cong T$. However, this forces $R$ to be primary, which is false, since it is easy to see that $(\begin{smallmatrix} n' & K \\ L \end{smallmatrix})$ and $(\begin{smallmatrix} L & K' \\ n \end{smallmatrix})$ are both maximal ideals of $R$. Therefore $R$ is not Bézout.

Examples of this type can be found in [D3, p. 489, Th. 1]. In Section 4 we will use a variant of the construction in Example 3.6 to obtain a class of semihereditary maximal orders.

4. Semihereditary Maximal Orders

Recall that a ring is right (resp. left) semihereditary if every finitely generated right (resp. left) ideal is projective. In this section we will construct a class of semihereditary maximal orders. By using the description of hereditary orders in [R, Ch. 9] we will be able to classify semihereditary maximal $V$-orders in $M_n(D)$, where $D$ is a division algebra containing an invariant valuation ring $B$ whose center $V$ is a generalized discrete valuation ring.

The examples we construct will be in block matrix form. We remind the reader that $(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ denotes the set of $\{ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) | a \in A, b \in B, c \in C, d \in D \} \subseteq M_2(S)$, where $A, B, C,$ and $D$ are subsets of a ring $S$. We will need the following lemma to prove that the examples given below are maximal.

**Lemma 4.1.** Let $B \subseteq A$ be invariant valuation rings of a division algebra $D$.

1. If $x \in D$ with $xA \subseteq B$ then $x \in J(A)$.
2. If $x \in D$ with $xJ(A) \subseteq B$ then $x \in A$.

**Proof.** Suppose $xA \subseteq B$. Then $x \in B \subseteq A$. If $x \notin J(A)$ then $x \in A^*$, giving $xA = A$, which is false as $B \neq A$. For (2), if $xJ(A) \subseteq B$ with $x \notin A$ then $x^{-1} \in J(A)$, so $J(A) \subseteq x^{-1}B \subseteq x^{-1}A \subseteq J(A)$. Therefore we obtain $x^{-1}B = x^{-1}A$, so $B = A$, a contradiction. Thus $x \in A$.

Let $B$ be an invariant valuation ring of a division algebra $D$. If $I$ is a $B$-submodule of $D$ we denote by $I^{-1}$ the $B$-submodule $\{ d \in D | dI \subseteq B \}$. This
notation does not reflect the dependence on $B$, but this should not be a cause of confusion. Lemma 4.1 shows that if $A$ is a proper overring of $B$ then $A^{-1} = J(A)$ and $J(A)^{-1} = A$.

We now describe a class of orders which will turn out to be precisely the class of semihereditary maximal orders in the situation of Theorem 4.12 below.

Let $S = M_n(D)$, where $D$ is a division algebra containing an invariant valuation ring $B$. Let $F = Z(S)$ and $V = B \cap F$.

**Definition 4.2.** Let $R$ be a subring of $S$. We say $R$ is of type $\mathcal{SH}$ if $R = (B_{ij})$, where the $B_{ij}$ satisfy the following properties.

(i) $B_{ij}$ is a nonzero $B$-submodule of $D$.

(ii) $B_{ii} = B$ for all $i$.

(iii) If $i \geq j$ then $B_{ij}$ is an overring of $B$.

(iv) $B_{ij}^{-1} = B_{ji}$.

(v) If $j \geq j'$ then $B_{ij} \subseteq B_{ij'}$.

(vi) For all $j, k, l$, $B_{ij} \cdot B_{kl} \subseteq B_{el}$.

It follows by (vi) that $R$ is a ring, and clearly $RF = RD = S$ as all the $B_{ij}$ are nonzero. Note that if $B_{ij}$ is a proper overring of $B$ then $B_{ji} = J(B_{ij})$ by Lemma 4.1 and (iv). Thus, if $x \in D$, $B_{ij}$ then $x^{-1} \subseteq B_{ji}$. Also, for $i \geq i'$, $B_{ij} \subseteq B_{ij'}$ by (iv) and (v). If $n = 2$ and $R$ is of type $\mathcal{SH}$ then either $R = M_2(B)$ or $R = (B \ (J(A)))$ for some $A \not\subseteq B$.

**Proposition 4.3.** If $R$ is of type $\mathcal{SH}$ then $R$ is a maximal order in $S$ over $V$.

**Proof.** To see that $R$ is integral over $V$, note that

$$R = \sum_i B e_{ii} + \sum_{i > j} B_{ij} e_{ij} + \sum_{i < j} B_{ij} e_{ij},$$

where the $e_{ij}$ are the usual matrix units. Since $B$ is integral over $V$, the elements of $\sum_i B e_{ii}$ are integral over $V$. Furthermore, anything in $\sum_{i > j} B_{ij} e_{ij}$ or $\sum_{i < j} B_{ij} e_{ij}$ is nilpotent, hence integral over $V$. Thus by [AS, Th. 2.3], $R$ is integral over $V$. Therefore $R$ is a $V$-order.

Suppose $R \subseteq T$ with $T$ an order over $V$. Take $\sum d_{kl} e_{kl} \in T$. As all $e_{kl} \in R$, $d_{ij} e_{ij} = e_i (\sum d_{kl} e_{kl}) e_j \in T$ for all $i, j$. Then as $B_{ji} e_{ji} \subseteq R \subseteq T$, $(d_{ij} e_{ij}) \cdot (B_{ji} e_{ji}) = (d_{ij} B_{ji}) e_{ji}$ is integral over $V$, hence $d_{ij} B_{ji} \subseteq B$. Hence by Lemma 4.1, $d_{ij} B_{ji}^{-1} = B_{ij}$. Thus $\sum d_{ij} e_{ij} \in R$, so $T = R$. Therefore $R$ is maximal.

Note that if $V \not\subseteq U \not\subseteq W$ are valuation rings of $F$ and $S = M_2(F)$, if
Then $R = (\bigcup_{V} \mathcal{A}(U))$ then $R$ is a maximal order over $V$, $RU = (\bigcup_{U} \mathcal{A}(U)) \subseteq M_2(U)$ is not a maximal order over $U$, and $RW = M_2(W)$ is a maximal order over $W$. Therefore it is possible to have a nonmaximal order $R$ over $V$ with $RW$ a maximal order over $W$, an vice-versa.

We now prove that the rings just constructed are semihereditary. We break this into a series of lemmas. To fix notation, let $R = (B_{ij})$ be of type $\mathcal{K}$.

**Lemma 4.4.** $M_m(R)$ is of type $\mathcal{K}$ for any $m$.

**Proof.** We have $M_m(R) = (M_m(B_{ij})) \subseteq M_m(D)$. For $t$ a positive integer let $\lfloor t/m \rfloor$ be the smallest integer $s$ with $s \geq t/m$. We then have $M_m(R) = (C_{ij})$, where $C_{ij} = B_{ig \lfloor g/m \rfloor}$. It is then easy to see that the $C_{ij}$ satisfy the conditions of Definition 4.2. Therefore $M_m(R)$ is of type $\mathcal{K}$. 

The next lemma is a technical result needed in the proof of Lemma 4.6.

**Lemma 4.5.** If $x_1, \ldots, x_n \in D$ then there is an $i$ with $x_i x_j^{-1} \in B_{ij}$ for all $j$.

**Proof.** Suppose the lemma is false. Then for each $i$ there is a $j$ with $x_i x_j^{-1} \notin B_{ij}$. Define $m(i)$ by $x_i x_j^{-1} \in B_{ij}$ for $j < m(i)$ but $x_i x_m^{-1} \notin B_{im(i)}$. Choose $i$ with $m(i)$ maximal. Set $k = m(i)$. Then we have $x_k x_i^{-1} \notin B_{ik}$, so $x_i x_k^{-1} \notin B_{ki}$. Thus for $j < k - m(i)$, $x_j x_k^{-1} = (x_j x_i^{-1}) \cdot (x_i x_k^{-1}) \in B_{ij} \cdot B_{ki} = B_{kj} \cdot B_{ij} \subseteq B_{kj}$. Also for $j = k$, $x_k x_k^{-1} = 1 \in B_i = B_{kk}$. Therefore we have $m(k) > k = m(i)$, contradicting the maximality of $m(i)$. This contradiction proves the lemma.

**Lemma 4.6.** $xR$ is projective as a $R$-module for all $x \in S$.

**Proof.** We first suppose $xR$ is projective for all $x \in e_i R$ for any $i$ and prove $xR$ is projective for any $x$. We do this by showing $e_i xR$ is projective, where $e_i = e_{i1} + \cdots + e_{in}$. We use induction on $i$, the case $i = 1$ is true by assumption. So suppose $e_{i-1} xR$ is projective for any $x$. We have the exact sequence of $R$-modules

$$0 \to e_i xR \cap (1 - e_{i-1}) R \to e_i xR \to e_{i-1} e_i xR \to 0.$$

Now $e_{i-1} e_i xR = e_{i-1} xR$ and $e_i xR \cap (1 - e_{i-1}) R \subseteq e_i R \cap (1 - e_{i-1}) R = e_i R$. Since $e_{i-1} xR$ is projective by induction, the sequence splits. So $e_i xR \cong e_{i-1} xR \oplus (e_i xR \cap (1 - e_{i-1}) R)$. Thus $e_i xR \cap (1 - e_{i-1}) R$ is principal and a submodule of $e_i R$, hence is projective by assumption. Therefore we obtain $e_i xR$ is a sum of two projective modules, hence is projective. Thus by induction $e_i xR$ is projective for all $i$. Setting $i = n$, we see $e_n xR = xR$ is projective.
We now show $xR$ is projective for $x \in e_i S$. Recall that $xR$ is projective iff the annihilator $\text{ann}_R(x) = eR$ for some idempotent $e \in R$. (This holds for $x \in S$, not just $x \in R$ as $RF = S$ and $\text{ann}_R(x) = \text{ann}_R(x\alpha)$ for any $\alpha \in F^*$.) Say $x = \sum_j x_j e_j \in e_i S$ with $x_j \in D$. If $\sum_{i,j} d_{ij} e_j \in \text{ann}_R(x)$ then $\sum_i x_i d_{ij} = 0$ for all $j$. By Lemma 4.5 there is an $i_0$ with $x_i x_{i_0}^{-1} \in B_{i_0,j}$ for all $j$. Now $d_{i_0,j} = \sum_{i \neq i_0} x_i x_{i_0}^{-1} d_{ij}$. Let

$$e = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-x_1 x_{i_0}^{-1} & \cdots & x_{i_0-1} x_{i_0}^{-1} & 0 & -x_{i_0+1} x_{i_0}^{-1} & \cdots & -x_n x_{i_0}^{-1} \\
0 & \cdots & 0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}.$$

We have $e \in R$ since $x_i x_{i_0}^{-1} \in B_{i_0,j}$. An easy computation shows $ex = 0$ and $ea = a$ for all $a \in \text{ann}_R(x)$. Thus $e^2 = e$, and so $\text{ann}_R(x) = eR$ is generated by an idempotent. Therefore $xR$ is projective.  

**THEOREM 4.7.** $R = (B_{ij})$ is a semihereditary maximal order, which is not Bézout unless $R = M,(B)$. 

**Proof.** By Lemmas 4.4 and 4.6, $xM_m(R)$ is projective for any $x \in M_m(R)$. Therefore $R$ is right semihereditary by [S. Prop.]. By analogous arguments $R$ is also left semihereditary, hence is semihereditary. By Proposition 4.3, $R$ is a maximal order.

Suppose $R \neq M_n(B)$. Let $C = \max_{i,j} \{B_{ij}\}$. Then $C \supsetneq B$. Also, $J(C) = \min_{i,j} \{B_{ij}\}$, and if $W = C \cap F$ then $RW = (B_{ij}W) \subsetneq M_n(C)$ as some component of $RW$ is equal to $J(C)$. If $R$ is Bézout then $R \cong M_n(B)$ by Corollary 3.5. But then $RW$ would be a Dubrovin valuation ring over $W$, which is false as $RW \subsetneq M_n(C)$. Thus $R$ is not Bézout.

The following two propositions give some properties of the orders of type $\mathcal{H}$. 

**Proposition 4.8.** $J(R) = J(B) R = (B'_{ij})$, where $B'_{ij} = B_{ij}$ if $B_{ij} \neq B$ and $B'_{ij} = J(B)$ if $B_{ij} = B$. Furthermore, 

$$R = \begin{pmatrix}
(B) & * & \cdots & * \\
* & (B) & * & * \\
\vdots & \ddots & \ddots & \ddots \\
* & \cdots & * & (B)
\end{pmatrix}^{n_1, \ldots, n_r}.$$
where the entries in the regions marked by * are never B. Therefore $R/J(R) \cong \bigoplus M_n(B)$.

Proof. We first note that for any $r < n$, if $C_j = B_{i-r}, j-r$, then the $C_j$ satisfy the conditions of Definition 4.2. Therefore $(C_j) \subseteq M_n(B)$ is of type $\mathscr{H}$. To start, we use induction on $n$ to show $R$ in this "block" form. Suppose in row one of $R$ the first $m$ entries are $B$ and the $(m+1)$st entry is $J(C)$. That is, $B_{1i} = B$ for $i \leq m$. Then $B_{1i} = B$ for $i \leq m$, so all $B_{ij} = B$ for $i, j \leq m$ by property (v) in Definition 4.2. Since $\sum_{i \leq m} B_{1i} B_{i,m+1} \subseteq B_{1,m+1} = J(C)$ we obtain $B_{i,m+1} \subseteq J(C)$ for $i \leq m$. As $B_{1,m+1} \subseteq B_{i,m+1}$, $B_{i,m+1} = J(C)$, and so $B_{m+1,i} = C$. Thus we have

$$R = \begin{pmatrix} M_m(B) & J(C) & \cdots & \ast \\ \vdots & \ast & \ddots & \cdots \\ C & \cdots & C & B \\ \ast & \ast & \cdots & B \end{pmatrix}.$$

Property (v) of Definition 4.2 shows that all the entries in the top right corner of $R$ lie in $J(C)$. Similarly all the entries in the bottom left corner contain $C$. The observation at the start of the proof shows the bottom right corner is of type $\mathscr{H}$. Thus by induction we see $R$ is of the desired form.

We now show $J(R)$ has the desired form. Set $J = (B_{ij})$. It is easy to see that $J$ is a two-sided ideal of $R$. Since $J(B) B_{ij} = B_{ij}$ whenever $B_{ij} \neq B$, we obtain $J = J(B) R$. The description of $R$ shows $R/J \cong \bigoplus M_n(B)$, a semisimple Artinian ring. Thus there are maximal ideals $M_1, \ldots, M_r$ of $R$ with $J = M_1 \cap \cdots \cap M_r$. Now as $R$ is a PI ring, $J(R)$ is the intersection of all the maximal ideals of $R$. Therefore $J(R) \subseteq J$. If $e$ is the ramification index of $B/V$ then $J(B)^e = J(V)R$, so $J^e = J(V) R \subseteq J(R)$, as $M \cap V = J(V)$ for any maximal ideal $M$ of $R$. Given $M$, we have $(M_1 \cdots M_r)^e \subseteq J^e \subseteq J(R) \subseteq M$, so $M_i \subseteq M$ for some $i$. Thus $M_1, \ldots, M_r$ are all the maximal ideals of $R$, so $J = J(R)$.

Proposition 4.9. (a) Finitely generated right ideals contained in $e_i R$ are principal.

(b) If $I$ is a finitely generated right ideal of $R$ then $I \cong \bigoplus I_j$ with $I_j \subseteq e_j R$. Thus $I$ can be generated by $n$ elements.
Proof. (a) We prove this for $i = 1$, the general case is analogous. We set up some notation that will be used later in the proof. Set

$$
\hat{B}_j = \begin{cases} 
B_{ij} & \text{if } B_{ij} \text{ is a ring;} \\
B_{ji} & \text{if } B_{ij} = J(B_{ji}).
\end{cases}
$$

Thus $\hat{B}_j$ is an overring of $B$. We see that if $\hat{B}_{j} \supsetneq \hat{B}_{j'}$ then $J(\hat{B}_j) \supsetneq J(\hat{B}_{j'})$, so $\hat{B}_{j'} \cap J(\hat{B}_j) = \hat{B}_j$. Thus $B_{ij} \cdot \hat{B}_j = \hat{B}_j$, and so $B_{ij} \cdot B_j = B_j$. Say

$$
I - \sum_k \begin{pmatrix} x_{ik} & \cdots & x_{nk} \\
0 & \cdots & 0 \\
\vdots & & \vdots 
\end{pmatrix} R = \begin{pmatrix} I_1 & \cdots & I_n \\
0 & \cdots & 0 \\
\vdots & & \vdots 
\end{pmatrix},
$$

where $I_j = \sum_{i,k} x_{ik} B_{ij}$. Since each $x_{ik} B_{ij}$ is a $B$-submodule of $D$, there is a maximum one, say $I_j = x_{ik_j} B_{ij}$. We define $Y = \sum_k y_k e_{1k}$ as follows. Given an $i$, if $i \neq j$ for any $j$, set $y_i = 0$. If $i = i_j$ for some $j$, then pick $j'$ among those $j$ with $i_j = i$ for which $\hat{B}_{j'}$ is minimal. Then set $y_i = x_{ik_j} = x_{i_k, j'}$. We then claim $I = Y R$. Now $Y R = \sum_k T_k e_{1k}$, where $T_j = \sum_i y_i B_{ij}$. Then for each $i$, $y_i B_j = x_{ik_j} B_{ij} \subseteq I_j$, so $T_j \subseteq I_j$ for each $j$. Thus $Y R \subseteq I$. Now given $j$, set $i = i_j$, and suppose $j_1, \ldots, j_r$ are those $j$ with $i = i_1 = \ldots = i_r$ and $j'$ is chosen with $\hat{B}_{j'} \subseteq \hat{B}_{i_1}$, for all $t$. Then $T_j \supseteq y_i B_j = x_{ik_j} B_{ij}$. But by choice of $i_j, j$, we have $x_{ik_j} B_{ij} \supseteq x_{ik_j} B_{ij}$ and $x_{ik_j} B_{ij} \supseteq x_{ik_j} B_{ij}$. If $B_{j'} = B_j$, then gives $x_{ik_j} B_{ij} = x_{ik_j} B_{ij}$. If $B_{j'} \supsetneq B_j$ then as $B_{ij} \cdot B_{ij} = B_j$, we have $x_{ik_j} B_{ij} \supseteq x_{ik_j} B_{ij}$, $x_{ik_j} B_{ij} \supseteq x_{ik_j} B_{ij}$, so again we obtain $x_{ik_j} B_{ij} = x_{ik_j} B_{ij}$. Hence $T_j \supseteq x_{ik_j} B_{ij} = x_{ik_j} B_{ij} = I_j$. Thus $Y R = I$.

For the proof of (b) we argue as in the proof of Lemma 4.6. We prove by induction on $i$ that if $I$ is a finitely generated right ideal of $R$ then $e_i I \cong \bigoplus_{j < i} I_j$, with $I_j \subseteq e_j R$, where $e_i = e_{11} + \cdots + e_{ii}$. Since $R$ is semihereditary by Theorem 4.7, the sequence

$$
0 \to e_i I \cap (1 - e_{i-1}) R \to e_i I \to e_{i-1} I \to 0
$$

splits, hence $e_i I \cong e_{i-1} I \bigoplus (e_i I \cap (1 - e_{i-1}) R)$. By induction $e_{i-1} I \cong \bigoplus_{j < i} I_j$ with $I_j \subseteq e_j R$. Because $e_i I$ is finitely generated so is $I_i := e_i I \cap (1 - e_{i-1}) R \subseteq e_i R$. Thus $e_i I \cong \bigoplus_{j < i} I_j$. Therefore by induction we have the result. Setting $i = n$ gives the first part of (b). Combining (a) and (b) shows that any finitely generated right ideal of $R$ can be generated by $n$ elements. 

We wish to characterize semihereditary maximal orders. We will prove a converse to Theorem 4.7 for $V$ a generalized discrete valuation ring. We first give two preliminary lemmas.
Lemma 4.10. Let $R$ be a semihereditary order in a central simple algebra $S$ and $T$ an overring of $R$ in $S$. Then $T$ is semihereditary.

Proof. We first prove $xT$ is $T$-projective for all $x \in T$. Since $xT$ is isomorphic to $xR$ a right $T$-modules for all $x \in S^*$, we may assume $x \in R$ as $R$ is an order in $S$. Since $R$ is semihereditary, $xR$ is projective over $R$. Therefore $\text{ann}_R(x) = eR$ for some idempotent $e \in R$. We claim $\text{ann}_S(x) = eS$. To see this, clearly $eS \subseteq \text{ann}_S(x)$. If $xy = 0$ for $y \in S$, then write $y = bc^{-1}$ with $b, c \in R$. Then $xb = 0$, so $b \in \text{ann}_R(x) = eR$, say $b = eb'$. Then $y = eb'c^{-1} \in eS$. Thus $\text{ann}_S(x) = eS$. If $a \in \text{ann}_T(x) \subseteq \text{ann}_S(x)$ then $a = es$ for some $s \in S$. Then $ea = e^2s = a$, so $a \in eT$. Hence $eT = \text{ann}_T(x)$, and so $xT$ is projective over $T$.

Since $M_n(R)$ is a semihereditary order in $M_n(S)$ for all $n$, the argument above shows that all principal right ideals of $M_n(T)$ are projective as right $M_n(T)$-modules. Therefore $T$ is semihereditary by [S, Prop. 1].

Lemma 4.11. Let $R$ be an order in a central simple algebra $S$ and $T = RW$ where $W$ is an overring of $Z(R)$ in $F$. If $J(T) \subseteq R$ and $R$ is semihereditary then $R/J(T)$ is semihereditary. If, in addition $T$ is Bézout then $R$ is semihereditary iff $R/J(T)$ is semihereditary.

Proof. Recall that a right ideal $I$ of $R$ is regular if $I$ contains a regular element of $S$, i.e., $IS = S$. To show $R$ (or $R/J(T)$) is right semihereditary it suffices to prove finitely generated regular right ideals are projective. For, if $I$ is a finitely generated right ideal of $R$ then since $R$ has a semisimple Artinian quotient ring, there exists an $x \in R$ with $I + xR$ regular and $I + xR \cong I \oplus xR$ [MR, 2.2.2, 2.3.5]. Thus if $I + xR$ is projective, then so is $I$.

For $I$ a regular right ideal of $R$, set $I^{-1} = \{x \in S \mid xI \subseteq R\}$ and $\mathcal{C}_I(I) = \{x \in S \mid xI \subseteq I\}$. Then $I$ is projective over $R$ iff $I^{-1} = \mathcal{C}_I(I)$ [MR, 3.1.15, 3.5.2]. Suppose $IT = T$. Then $J(T) \subseteq I$, since if $1 = \sum_i x_ia_i$ with $u_i \in I$ and $a_i \in T$, for $m \in J(T)$, $m = \sum_i u_i(a_im) \in IJ(T) \subseteq IR = I$. Similarly, $J(T) \subseteq I^{-1}$, $\mathcal{C}_I(I)$ and since $IT = T$, $I^{-1}$, $\mathcal{C}_I(I)$ $\subseteq T$. We then have

$$(I/J(T))^{-1} = I^{-1}/J(T) \quad \text{and} \quad \mathcal{C}_I(I/J(T)) = \mathcal{C}_I(I)/J(T),$$

where $(I/J(T))^{-1}$ and $\mathcal{C}_I(I/J(T))$ are defined accordingly. Thus (for $IT = T$) $I$ is projective over $R$ iff $I/J(T)$ is projective over $R/J(T)$.

First, take $I$ a finitely generated regular right ideal of $R$. If $T$ is Bézout then $IT = xT$ for some $x \in T$. As $I$ is regular, $x \in S^*$. Because $I \cong x^{-1}I$ as $R$-modules, we may replace $I$ by $x^{-1}I$ and assume $IT = T$. If $R/J(T)$ is semihereditary then $I/J(T)$ is a finitely generated right $R/J(T)$-module, hence projective. Therefore $I$ is projective over $R$ by the arguments above. Hence $R$ is semihereditary.

Conversely, assume $R$ is semihereditary, and suppose $\mathcal{S} = \sum_i \mathcal{F}_i(R/J(T))$
is a finitely generated regular right ideal of \( R/J(T) \). Since \( R/J(T) \) is an order in \( T/J(T) \) and \( \mathcal{J} \) is regular, \( \mathcal{J}(T/J(T)) = T/J(T) \). Thus we can write \( \mathcal{I} = \sum b_i \mathcal{A}_i \) with \( a_i \in T \). Therefore \( 1 = (\sum b_i a_i) + m \) for some \( m \in J(T) \). Set \( I = \sum b_i R + mR \). Then \( IT = T \) (and \( I \) is regular). Also \( J(T) \subseteq I \) and \( I/J(T) = \mathcal{J} \). Since \( I \) is a finitely generated right ideal of \( R \), \( I \) is projective over \( R \), and so \( \mathcal{J} \) is projective over \( R/J(T) \). Thus \( R/J(T) \) is semihereditary. 

We now classify semihereditary maximal orders inside \( M_n(D) \), where \( D \) is an \( F \)-central division algebra and \( V \) a generalized discrete valuation ring of \( F \) which extends to an invariant valuation ring \( B \) of \( D \).

**Theorem 4.12.** Suppose \( S = M_n(D) \), \( V \) a generalized discrete valuation ring of \( F = Z(S) \) and \( B \) an invariant valuation ring of \( D \) with \( B \cap F = V \). Let \( R \) be a semihereditary maximal order over \( V \) in \( S \). Then \( R \) is isomorphic to a ring of type \( \mathcal{J} \mathcal{H} \).

**Proof.** We use induction on rank(\( V \)) = \( m \), the case \( m = 1 \) is clear since then \( V \) is a discrete valuation ring and so \( R \cong M_n(B) \) by [R, 18.7]. So suppose \( m > 1 \). Let \( W \supseteq V \) be a discrete valuation ring and set \( T = RW \).

Then \( T \) is semihereditary by Lemma 4.10. Furthermore, \( T \) is Noetherian by [Ro, Cor. 5.4.1], hence \( T \) is hereditary. Thus by [R, Th. 39.14], if \( C = BW \) then

\[
T \cong \left( \begin{array}{cccc}
(C) & (J(C)) & \cdots & (J(C)) \\
(C) & (C) & (J(C)) & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
(C) & \cdots & \cdots & (C)
\end{array} \right)^{\{n_1, \ldots, n_r\}}
\]

By conjugating \( R \) by an appropriate element, we can assume \( T \) is equal to this matrix ring. Now \( J(T) \subseteq R \) by Lemma 2.5, and also by [R, Th. 39.14]}

\[
J(T) = \left( \begin{array}{ccccc}
(J(C)) & J(C) & \cdots & J(C) \\
(C) & (J(C)) & \cdots & (J(C)) \\
\vdots & \vdots & \ddots & \vdots \\
(C) & \cdots & (C) & (J(C))
\end{array} \right)^{\{n_1, \ldots, n_r\}}
\]

By Lemma 2.5 and Lemma 4.11, \( \bar{R} := R/J(T) \) is a semihereditary maximal order in \( \bar{T} := T/J(T) \) over \( \bar{V} := V/J(W) \). Now \( \bar{T} = M_{n_1}(\bar{C}) \oplus \cdots \oplus M_{n_r}(\bar{C}) \).

Let \( e_1, \ldots, e_r \) be the idempotents satisfying \( e_i \bar{T} e_i = M_{n_i}(\bar{C}) \). As \( e_i \bar{R} e_i \) is a \( \bar{V} \)-order in \( e_i \bar{T} e_i \) and \( \bar{R} \subseteq \oplus e_i \bar{R} e_i \), the maximality of \( R \neq \bar{R} \) shows \( \bar{R} = \oplus e_i \bar{R} e_i \) and each \( e_i \bar{R} e_i \) is maximal. Furthermore, each \( e_i \bar{R} e_i \) is semihereditary. Let \( Y = Z(\bar{R}) \). Since \( B/J(C) \) is an invariant valuation ring of \( \bar{C} \) which is integral
over \( \tilde{V} \), \( Y = (B/J(C)) \cap Z(\tilde{C}) \) is a valuation ring, and so as an extension to \( Z(\tilde{C}) \) of \( \tilde{V} \) we see \( Y \) is a generalized discrete valuation ring of rank \( m - 1 \). Thus by induction \( e_i\tilde{R}_i = x_i\tilde{R}_ix_i^{-1} \) with \( x_i \in (e_i \tilde{T}e_i)^* \) and each \( \tilde{R}_i = (B_{ki}(t)) \subseteq M_n(\tilde{C}) \) of type \( \mathcal{M} \). If \( t \in T \) with \( t + J(T) = \sum_i x_i \), then \( t \in T^* \) and \( tRt^{-1}/J(T) = \bigoplus \tilde{R}_i \). Thus we can again conjugate \( R \) and assume \( \tilde{R} = \bigoplus \tilde{R}_i \). Let \( B_{ki}^{(t)} \) be the preimage of \( B_{ki}^{(t)} \) in \( C \). We then have, as \( J(T) \subseteq R \) that

\[
R = \begin{pmatrix}
(B_{ki}^{(1)}) & \cdots & (J(C)) \\
(C) & \ddots & \vdots \\
\vdots & \cdots & (B_{ki}^{(r)})
\end{pmatrix} = (B_{ij}).
\]

Since each \( \tilde{R}_i \) is of type \( \mathcal{M} \), we see that \( R \) is also of type \( \mathcal{M} \).

It is also possible to construct semihereditary maximal orders inside division algebras (which are not Bézout). The following example uses the orders of type \( \mathcal{M} \) above to do this.

**Example 4.13.** \( D \) a division algebra over \( F \), \( V \) a valuation ring of \( F \) and \( B \) a semihereditary maximal order over \( V \) in \( D \) which is not Bézout.

*Proof.* Let \( D \) be an \( F \)-central division algebra, \( V \subseteq W \subseteq U \) generalized discrete valuation rings of \( F \) and \( A \) a Dubrovin valuation ring of \( D \) with center \( U \) and \( A/J(A) = M_2(U) \). Set \( \tilde{B} = (V_{jU} \cup W_{jU}) \cup J(U) \) and let \( B \) be the preimage of \( \tilde{B} \) in \( A \). Then since \( \tilde{B}U = A \), \( BU = A \), and \( J(A) \subseteq B \) by construction. By Lemma 4.11 and Theorem 4.7, \( B \) is semihereditary. Also, \( B \) is a maximal order over \( V \) by Lemma 2.5 and Proposition 4.3.

We end this section with an example involving maximal orders under the passage to Henselization, and mention some open questions relating to this example.

**Example 4.14.** A Bézout maximal order \( R \) over a valuation ring \( V \) such that \( R \otimes_V V_h \) is a semihereditary maximal order, but is not Bézout, where \( V_h \) is the Henselization of \( V \).

*Proof.* Let \( F = k(x, y) \), the rational function field in two variables over a field \( k \) of characteristic not 2. Let \( D = (1 + x, y)^F \), the quaternion algebra over \( F \) generated by \( i_0 \) and \( j_0 \) subject to the relations \( i_0^2 = 1 + x, j_0^2 = y, \) and \( i_0j_0 = -j_0i_0 \). Let \( V \) be the \((x, y)\)-adic valuation ring of \( F \) and \( W \) the \( y \)-adic valuation ring of \( F \). By [JW, Ex. 4.3], \( W \) extends to an invariant valuation ring \( A \) of \( D \) with \( A/J(A) = W/J(W)(\sqrt{1 + x}) \). The \( x \)-adic valuation ring \( V/J(W) \) of \( W/J(W) \) extends in two ways to \( A/J(A) \). Let \( \mathcal{U}_1, \mathcal{U}_2 \) be these extensions. If \( B_i = \{ a \in A \mid a + J(A) \in \mathcal{U}_i \} \), then \( B_1 \) is a total valuation ring of \( D \). Furthermore, \( \{ B_1, B_2 \} \) is the set of conjugates of \( B_1 \), so \( R = B_1 \cap B_2 \) is
the integral closure of $V$ in $D$ [BG1, Th. 5], hence is the unique maximal $V$-order in $D$. By Proposition 3.2 and [M2, Lemma 2.4] $R$ is a Bézout maximal order over $V$.

Let $(F_h, V_h)$ be the Henselization of $(F, V)$. There is an $a \in V^*$ with $a^2 = 1 + x$ by Hensel's lemma since $1 + x$ is a square mod $J(V)$ and $V_h/J(V_h) = V/J(V) = k$ has characteristic not 2. Then $D \otimes F_h = (1 + x, y)_{F_h} \simeq (1, y)_{F_h} \cong M_2(F_h)$. Let $i = a^{-1}i_0$ and $j = j_0$. Then $i^2 = 1$, $j^2 = y$ and $ij = -ji$. Set

$$e_{11} = \frac{1 - i}{2}, \quad e_{22} = \frac{1 + i}{2}, \quad e_{12} = \frac{j - ij}{4y}, \quad e_{21} = j + ij.$$ 

An easy (but tedious) computation shows that the $e_{ij}$ are matrix units for $D \otimes F_h$. Clearly $V_h e_{rr} \subseteq R \otimes V_h$ for $r = 1, 2$. We want to show $J(W')e_{12} \subseteq R \otimes V_h$ and $W'e_{21} \subseteq R \otimes V_h$, where $W' = W/V_h$. But $W = V[1/x]$, so $W' = V_h[1/x]$. Now $(1/x^m)e_{21} = (1/x^m)(j + ij) = (1/x^m)j_0 + (1/x^m)a^{-1}i_0i_0$. Since $((1/x^m)i_0)^2 = y/x^m \in V$ and $((1/x^m)i_0i_0)^2 = -y/x^2m \in V$, we see that both these elements are integral over $V$, hence lie in $R$. Therefore $(1/x^m)e_{21} \in R \otimes V_h$. Similarly $(y/x^m)e_{12} \in R \otimes V_h$. This gives

$$R \otimes V_h \cong V_h e_{11} + J(W')e_{12} + W'e_{21} + V_h e_{22} = \begin{pmatrix} V_h & J(W') \\ W' & V_h \end{pmatrix}.$$ 

However, $R \otimes V_h$ is integral over $V_h$ since $R$ is integral over $V$. By Proposition 4.3, we see $R \otimes V_h - (W, J(W'))$, a semidirect maximal order. That $R \otimes V_h$ is not Bézout follows from Proposition 4.7.

Two questions that arise naturally from this example are the following. Suppose $R$ is a maximal $V$-order in a central simple $F$-algebra $S$. Let $(F_h, V_h)$ be the Henselization of $(F, V)$. Then $R \otimes V_h$ is a $V_h$-order in $S \otimes F_h$. Is $R \otimes V_h$ a maximal order? If $R$ is semidirect then is $R \otimes V_h$ a $V_h$-order in $S \otimes F_h$. Is $R \otimes V_h$ semidirect? If the answer to both these questions is yes, then one would have a description of all semidirect maximal orders by applying Theorem 4.12, since the underlying division algebra of $S \otimes F_h$ has an invariant valuation ring extending $V_h$ as $V_h$ is Henselian, and $R = S \cap (R \otimes V_h)$.

5. MAXIMAL ORDERS INSIDE MATRICES

In this section we consider arbitrary maximal orders in $M_n(F)$ over a generalized discrete valuation ring $V$ of $F$. For $n = 2$ we are able to fully classify all such orders. We will use defective field extensions to construct
a class of maximal orders that along with the class of semihereditary orders given in the previous section yields all maximal orders. If \( n \) is large the structure of general maximal orders is unwieldy. Furthermore, the necessity in [D₃, Sect. 3, Th. 2] for \( V \) to be almost maximal and rank one indicates the difficulty of characterizing maximal orders over general valuation rings. However, given sufficient “defectless” (as will be clarified below), any maximal order in \( M_n(F) \) is semihereditary, and so the results of Section 4 classifies all orders. We now make this more precise.

Suppose \( V \) is a generalized discrete valuation ring of a field \( F \) of rank \( n \).

Let \( V = V_m \subseteq V_{m-1} \subseteq \cdots \subseteq V_1 \subseteq F \) be the collection of overrings of \( V \). Let \( n \) be a positive integer. We will say \( V \) is \( n \)-defectless if for any \( i \) and any field extension \( \mathcal{L} \) of \( V_i := V_i/J(V_i) \) with \( [\mathcal{L} : V_i] \leq n \) then \( \mathcal{L}/V_i \) is defectless with respect to the valuation ring \( V_{i+1}/J(V_i) \). This is equivalent to the integral closure \( \mathcal{R} \) of \( V_{i+1}/J(V_i) \) being a finite module \([E, 18.6]\). For instance, if \( \text{char}(\mathcal{V}) = 0 \) or \( \text{char}(\mathcal{V}) > n \) then \( V \) is \( n \)-defectless (see Corollary 5.3).

In this section we will find it necessary to look at more general types of rings than orders. If \( S \) is a ring, then a proper subring \( T \) of \( S \) is said to be a maximal subring of \( S \) if \( T \subseteq T' \subseteq S \) implies either \( T' = T \) or \( T' = S \).

For \( S = M_n(F) \), Dubrovin in [D₃] classifies all maximal subrings of \( S \). This classification will be quite useful to us in classifying maximal orders in \( M_n(F) \). For a second definition, suppose \( S \) is an \( F \)-algebra and \( V \) is a subring of \( F \). A \( V \)-subalgebra \( R \) of \( S \) is said to be \( V \)-maximal in \( S \) if \( R \) is integral over \( V \) and not properly contained in any subring of \( S \) which is integral over \( V \). Note that we do not require \( R \) to be an order in \( S \). Such rings will come up in the following way. Suppose \( R \) is a maximal \( V \)-order, \( W \) an overring of \( V \) and \( T \supseteq R \) is a maximal \( W \)-order. It is quite possible that \( RW \not\subseteq T \), so \( R/J(T) \) is not an order in \( T/J(T) \). However, it is easy to see from the maximality of \( R \) that \( R/J(T) \) is \( V/J(W) \)-maximal in \( T/J(T) \).

Recall that if \( V \) is a Dedekind domain of \( F \) and \( K \) is a finite extension of \( F \) then \( K/F \) is defectless with respect to \( V \) if \( K/F \) is defectless with respect to the discrete valuation ring \( V_M \) for all maximal ideals \( M \) of \( V \).

**Lemma 5.1.** Let \( V \) be a semilocal Dedekind domain of \( F \) and \( S = M_n(F) \).

Suppose \( K/F \) is defectless with respect to \( V \) for all fields \( K \) with \([K:F] \leq n\).

If \( R \) is \( V \)-maximal in \( S \) then either \( R \cong M_n(V) \) or

\[
R \cong \begin{pmatrix}
(V) & (0) & \cdots & (0) \\
(F) & (V) & (0) & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
(F) & \cdots & (F) & (V)
\end{pmatrix}^{[n_1, \ldots, n_r]}
\]

**Proof.** First suppose \( R \) is an order in \( S \). Then \( R \) is a maximal order,
hence $R \cong M_n(V)$ by [R, 21.7]. Thus we may assume $R$ is not an order, that is, $RF \not\subseteq S$. Set $S_0 = RF$. Now $S_0$ lies in a maximal subring $S_1$ of $S$. We have $F \subseteq S_1$, so by [D$_3$, Sect. 3, Th. 3] there are two possibilities for $S_1$.

Case 1. $S_1 \cong \left(\begin{array}{c}(F) \\ (F) \end{array}\right)^{\{n_1, n_2\}}$. In fact, the proof of [D$_3$, Sect. 3, Th. 3] shows that $S_1 = x \left(\begin{array}{c}(F) \\ (F) \end{array}\right) x^{-1}$ for some $x \in M_n(F)^*$. Thus by replacing $R$ by $x^{-1}Rx$ we may assume $S_1 = \left(\begin{array}{c}(F) \\ (F) \end{array}\right)$. By maximality of $R$, the argument of Lemma 2.5 shows $J(S_1) \subseteq R$. Now $J(S_1) = \left(\begin{array}{c}(0) \\ (0) \end{array}\right)$. Thus $R/J(S_1)$ is integral over $V$ in $S_1/J(S_1) \cong M_n(F) \oplus M_{n_2}(F)$. Furthermore, $R/J(S_1)$ is $V$-maximal since $R$ is. We see that $R/J(S_1) = \mathcal{R}_1 \oplus \mathcal{R}_2$, where $\mathcal{R}_i$ is $V$-maximal in $M_n(F)$. By induction,

\[
\mathcal{R}_i \cong \left(\begin{array}{c}(V) \\ (F) \end{array}\right)^{\{i_1, i_2\}}
\]

so

\[
R \cong \left(\begin{array}{c}\mathcal{R}_1 \\ \mathcal{R}_2 \end{array}\right)^{\{n_1, n_2\}}.
\]

Case 2. $S_1$ is central simple over a field $L \supseteq F$. By the double centralizer theorem, $S_1$ is a tensor factor of $C_S(L)$, the centralizer of $L$ in $S$, hence we must have $S_1 = C_S(L)$ by maximality of $S_1$. Note that $C_S(L) \cong M_r(L)$ where $r[L : F] = n$. By the Noether–Skolem theorem, we can assume $S_1 = C_S(L)$. Let $U$ be the integral closure of $V$ in $L$. Then by [ZS, Ch. V, Sect. 8, Th. 19] $U$ is a Dedekind domain. Furthermore $U$ is semilocal, since the number of maximal ideals of $U$ contracting to a given maximal ideal of $V$ is bounded by $[K : F]$, by [E, 13.7]. Since $L$ commutes with $R$, $RU$ is integral over $V$, hence $RU = R$, so $U \subseteq R$. Then $R$ is $U$-maximal in $M_r(L)$, so by induction we have

\[
R \cong \left(\begin{array}{c}(U) \\ (L) \\ (L) \\ \end{array}\right)^{\{n_1, n_2\}}.
\]

We claim that $U$ is a finite $V$-module. To see this, if $M$ is a maximal ideal of $V$ then $U_M$ is the integral closure of $V_M$ in $L$. Then by the defectless assumption $U_M$ is a finite $V_M'$-module. Say $U_M = \sum_i V_M \alpha_{i,M}$. We may suppose each $\alpha_{i,M} \in U$. If $U_0 := \sum_i V_M \alpha_{i,M}$ then $U_0$ is a finite $V$-module since $V$ is semilocal. But $U_0 \subseteq U$ and $U_0M = U_M$ for each $M$. 


Thus \( U_0 = U \), so \( U \) is a finite \( V \)-module. Therefore \( U \) embeds in \( M_s(V) \) by Proposition 2.3, where \( s = [L : F] \). By an appropriate conjugation we may assume equality holds in (*) \( U \subseteq M_s(V) \) and \( L = M_s(F) \). Since \( C_s(L) = M_s(L) \subseteq M_s(M_s(F)) = M_n(F) \) we have

\[
R \cong \begin{pmatrix}
(M_s(V)) & (0) & \cdots & (0) \\
(M_s(F)) & (M_s(V)) & (0) & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
(M_s(F)) & \cdots & (M_s(F)) & (M_s(V))
\end{pmatrix}^{[m, \ldots, n]}
\]

Therefore this case cannot happen, and so we are done.

**Theorem 5.2.** Let \( V \) be a generalized discrete valuation ring of \( F \) and \( S = M_n(F) \). Suppose \( V \) is \( n \)-defectless. If \( R \) is a maximal order over \( V \) in \( S \) then \( R \cong (V_W) \) is of type \( \mathcal{S} \mathcal{H} \). Thus \( R \) is semihereditary.

**Proof.** We use induction on \( \text{rank}(V) = m \). The case \( m = 1 \) follows by [R, 18.7] as \( V \) is then a discrete valuation ring. So suppose \( m > 1 \). Let \( W = V_{m-1} \) the overring of \( V \) of rank \( m-1 \), and let \( T \) be a maximal \( W \)-order containing \( R \). By induction \( T \cong (W_W) \) is of type \( \mathcal{S} \mathcal{H} \). By conjugating \( R \) we may assume \( T \cong (W_W) \). By Lemma 2.5, \( J(T) \subseteq R \). Since \( R \) is a maximal \( V \)-order, \( R/J(T) \) is \( V/J(W) \)-maximal in \( T/J(T) \). By Proposition 4.8 we have \( T/J(T) = \bigoplus_i M_n(W_i) \). As in the proof of Theorem 3.4 we have \( R/J(T) = \bigoplus_i \mathcal{R}_i \), where \( \mathcal{R}_i \) is \( V/J(W) \)-maximal in \( M_n(W_i) \). By Lemma 5.1 each \( \mathcal{R}_i \) is conjugate to a ring in the block matrix form of that lemma. By an appropriate conjugation of \( R \) we may assume each \( \mathcal{R}_i \) is in this block matrix form. As \( J(T) \subseteq R \) we see that \( R \cong (V_W) \), where \( V_W = V_{W} \) if \( W_W \neq W \) and \( V_W = V \) if \( W_W = W \). Since \( T \cong (W_W) \) is of type \( \mathcal{S} \mathcal{H} \), it follows that \( R \) is of type \( \mathcal{S} \mathcal{H} \). Thus we have proven the theorem.

**Corollary 5.3.** Let \( R \) be a maximal \( V \)-order in \( M_n(F) \). Then \( R \) is semihereditary provided that either (i) \( \text{char}(\overline{V}) = 0 \), (ii) \( \text{char}(\overline{V}) > n \), (iii) \( F \) is maximally complete with respect to \( V \), or (iv) \( \overline{V}_i \) is perfect for all \( i \).

**Proof.** In all of these cases \( V \) is \( n \)-defectless, and so this follows from Theorem 5.2. To see that \( V \) is \( n \)-defectless, for (i) and (ii), if \( [\mathcal{L} : V_i] \leq n \) then \( \text{char}(V_i) \) does not divide \( [\mathcal{L} : V_i] \), so \( \mathcal{L}/V_i \) is separable. Thus \( \mathcal{L}/V_i \) is defectless with respect to the discrete valuation ring \( V_{i+1}/J(V_i) \) by [E, 18.7]. If \( F \) is maximally complete with respect to \( V \) then each \( V_i \) is complete with respect to \( V_{i+1}/J(V_i) \) by [Ri, Ch. D, Prop. 2, 3; Sc, Remark 2, p. 37]. Hence \( \mathcal{L}/V_i \) is defectless with respect to \( V_{i+1}/J(V_i) \) for any finite extension \( \mathcal{L} \) by [E, 18.8]. Last, if each \( V_i \) is perfect then this also follows from [E, 18.7].
Note that if $F$ is maximally complete, then $F$ is Henselian, and so if $S = M_n(D)$ with $D$ a division algebra then $D$ contains an invariant valuation ring extending $V$. We can therefore use the arguments above to show any maximal order in $S$ over $V$ is semihereditary.

We now look at maximal orders in $M_n(F)$, where $V$ is not assumed to be $n$-defectless. The following construction uses defective field extensions to construct a class of maximal orders.

Let $F$ be a field and $V \subset W$ be generalized discrete valuation rings of $F$ with $V/J(W)$ a discrete valuation ring of $\overline{W} := W/J(W)$. Let $A$ be an Azumaya algebra over $W$ and $S = AF$. Then $S$ is $F$-central simple. Set $\overline{A} = A/J(A)$, a central simple $\overline{W}$-algebra. Let $\mathcal{L}$ be a subfield of $\overline{A}$ containing $\overline{W}$ and let $C = C_{\overline{A}}(\mathcal{L})$. Let $\mathcal{U}$ be the integral closure of $V/J(W)$ in $\mathcal{L}$. Thus $\mathcal{U}$ is a semilocal Dedekind domain. Let $\mathcal{R}$ be a maximal $\mathcal{U}$-order in $C$. Then $\mathcal{R}$ is also a maximal $V/J(W)$-order. Set

$$R = \{ a \in A | a + J(A) \in \mathcal{R} \}.$$

By Lemma 2.5, $R$ is a $V$-order in $S$. In most cases $R$ fails to be a maximal order, in particular, if $\mathcal{L}/\overline{W}$ is defectless, since then $\mathcal{R}$ embeds in a maximal $V/J(W)$-order of $\overline{A}$. However, we will see exactly when $R$ is a maximal order. We need two preliminary lemmas. We would like to thank A. Wadsworth for the following proof, a much simpler proof than the authors original one.

**Lemma 5.4.** Let $S$ be a central simple $F$-algebra and $L$ a subfield of $S$ containing $F$. If $R$ is an overring of $C_{S}(L)$ in $S$ then $R = C_{S}(K)$ for some subfield $K$ of $L$.

**Proof:** Suppose $C_{S}(L) \subseteq R$. Then $C_{S}(R) \subseteq C_{S}(C_{S}(L)) = L$, so $C_{S}(R) = K$ is a subfield of $L$. If $R$ is simple then by the double centralizer theorem $R = C_{S}(C_{S}(R)) = C_{S}(K)$. Thus we need to show $R$ is simple. First suppose $S = M_n(F) = \text{End}_F(\mathcal{V})$, where $\mathcal{V}$ is an $n$-dimensional $F$-vector space. Set $C = C_{S}(L)$. If $I$ is the unique simple $C$-module, then as $C \cong M_s(L)$ for $s = n/[L:F]$, we have $\dim_L(I) = s$. Thus $\dim_F(I) = n$. Since $\mathcal{V}$ is a $C$-module, $\mathcal{V} \cong \mathcal{V}^r$ for some $r$. By comparing dimensions we obtain $r = 1$, so $\mathcal{V}$ is a simple $C$-module. Thus $\mathcal{V}$ is a simple $R$-module. Furthermore, $\mathcal{V}$ is a faithful $R$-module as $V$ is faithful over $S$. Thus $R$ is a primitive ring. Since $R$ is Artinian as $\dim_F(R) < \infty$, $R$ is simple.

For general $S$, we have $C_{S \otimes_F S^{op}}(L \otimes 1) = C_{S}(L) \otimes_F S^{op} \subseteq R \otimes_F S^{op} \subseteq S \otimes_F S^{op} = M_t(F)$ for $t = \dim_F(S)$. By the argument above we see that $R \otimes_F S^{op}$ is simple, hence $R$ is simple. □

**Lemma 5.5** Let $A$ be an Azumaya algebra over $W$, where $W$ is a generalized discrete valuation ring of $F$. Suppose $A_0$ is a subring of $A$
containing \( J(A) \) such that \( A_0/J(A) = C_{A/J(A)}(\mathcal{L}) \), the centralizer in \( A/J(A) \) of the subfield \( \mathcal{L} \). If \( A_0 \subseteq xAx^{-1} \) for some \( x \in (AF) \) then \( xAx^{-1} = A \).

**Proof.** Suppose \( A_0 \subseteq A' := xAx^{-1} \). Then \( A_0 \subseteq A \cap A' \). Set \( C = A \cap A' \). Then \( J(A) \subseteq C \), and by Lemma 5.4, \( C/J(A) = C_{A/J(A)}(\mathcal{K}) \) for some subfield \( \mathcal{K} \) of \( \mathcal{L} \). It therefore suffices to assume \( A_0 = A \cap A' \). We have \( J(A) \subseteq A_0 \subseteq A' \). Since \( J(A) = J(W)A \) and \( W \) is a generalized discrete valuation ring, \( J(W) = \pi W \) for some \( \pi \in W \). Thus \( \pi A \subseteq xAx^{-1} \), so \( x^{-1} \pi Ax \subseteq A \). By replacing \( x \) by \( x \alpha \) for some \( \alpha \in F \) we may assume \( x \in A - J(A) \), and so \( AxA = A \). Then \( x^{-1} \pi Ax = x^{-1} \pi A \subseteq A \), so \( x^{-1} \pi A \subseteq A \).

Hence \( J(A') = xJ(A)x^{-1} = xAx\pi x^{-1} \subseteq A \). This gives \( J(A') \subseteq A \cap A' = A_0 \), so \( J(A') \) is an ideal of \( A_0 \). Now \( J(A) \) is a maximal ideal of \( A_0 \) since \( A_0/J(A) \) is a simple ring. Since \( J(A) \) is the unique maximal ideal of \( A \), by [AS, Th. 2.5] we see that \( J(A) \) is also the unique maximal ideal of \( A_0 \). Hence \( J(A') \subseteq J(A) \), or \( \pi A' \subseteq \pi A \). Thus \( A' \subseteq A \), so \( A' = A \).

**Proposition 5.6.** With the notation preceding Lemma 5.4, suppose \( V \) is a generalized discrete valuation ring. Then \( R \) is a maximal order over \( V \) iff \( \mathcal{L}/\mathcal{K} \) is defectless with respect to \( \mathcal{U}/\mathcal{U} \cap \mathcal{K} \) for all fields \( \mathcal{K} \) with \( \mathcal{W} \subseteq \mathcal{U} \subseteq \mathcal{L} \).

**Proof.** Suppose there is a subfield \( \mathcal{K} \) of \( \mathcal{L} \) with \( \mathcal{L}/\mathcal{K} \) defectless. If \( \mathcal{U}_{\mathcal{K}} = \mathcal{U} \cap \mathcal{K} \), then \( \mathcal{U} \) is the integral closure of \( \mathcal{U}_K \) in \( \mathcal{L} \). By [E, 18.6], \( \mathcal{U} \) is a finite module over \( \mathcal{U}_\mathcal{K} \) of rank \( [\mathcal{L} : \mathcal{K}] \). Now \( C_\mathcal{K}(\mathcal{L}) = C_\mathcal{K}(\mathcal{K}) \). Since \( \mathcal{R} \) is a finite \( \mathcal{U} \)-module, \( \mathcal{R} \) is a finite \( \mathcal{U}_\mathcal{K} \)-module, hence embeds in a (maximal) order \( \mathcal{T} \) in \( C_\mathcal{K}(\mathcal{K}) \). Thus \( \mathcal{T} \) is integral over \( V/J(W) \) and \( \mathcal{R} \subseteq \mathcal{T} \). If \( T = \{ a \in A \mid a + J(A) \in \mathcal{T} \} \) then by Lemma 2.5, \( T \) is integral over \( V \) and larger than \( R \). Thus \( R \) is not maximal.

Conversely, suppose \( \mathcal{L}/\mathcal{K} \) is defectless for all subfields \( \mathcal{K} \) of \( \mathcal{L} \) which contain \( \mathcal{W} \). Then \( \mathcal{U} \) is not a finite \( \mathcal{U} \cap \mathcal{K} \)-module by [E, 18.6]. Therefore \( \mathcal{R} \) does not lie inside any \( \mathcal{U} \cap \mathcal{K} \)-order in \( C_\mathcal{K}(\mathcal{K}) \) since such an order is a finite module over the Noetherian ring \( \mathcal{U} \cap \mathcal{K} \), hence any submodule is also a finite module. Now suppose \( R \subseteq T \) with \( T \) integral over \( V \). First, let us suppose that \( T \subseteq A \). Then \( \mathcal{R} \subseteq T/J(A) \subseteq A \). Set \( \mathcal{C} = (T/J(A)) W \). Then \( \mathcal{C} \) is an overring of \( C_\mathcal{K}(\mathcal{L}) \), so by Lemma 5.4, \( \mathcal{C} = C_\mathcal{K}(\mathcal{K}) \) for some subring \( \mathcal{K} \) of \( \mathcal{L} \), and \( T/J(A) \) is an order in \( \mathcal{C} \). As we have just seen, \( \mathcal{R} \) does not lie in any order in such an algebra, so this implies that \( T/J(A) = \mathcal{R} \), so \( T = R \).

Now in general, \( T \) lies in a maximal \( W \)-order \( A' \). As \( J(A) \subseteq R, RU = AU \), a Dubrovin valuation ring over \( U \), for any proper overring \( U \) of \( W \). Thus \( A'U \) is maximal over \( U \) for any \( U \supseteq W \). Therefore by Corollary 3.5, \( A' \) is a Dubrovin valuation ring over \( W \), hence conjugate to \( A \). Let \( A_0 = RW \). Then \( A_0/J(A) = C_\mathcal{K}(\mathcal{L}) \). We have \( A_0 \subseteq A \cap A' = A \cap xAx^{-1} \) for some \( x \).
Then by Lemma 5.5 we obtain \( A' = A \), so \( T \subseteq A \). Thus the argument above shows that \( T = R \), so \( R \) is maximal.

We will now classify maximal orders in \( M_2(F) \). In this case we discuss our constructions of orders, and set up some terminology to help clarify the proof of the next theorem. Set \( S = M_2(F) \) and \( V = V_n \), a generalized discrete valuation ring of dimension \( n \). Let \( V_m \) be the overring of \( V \) of dimension \( m \). Let

\[
A_m = M_2(V_m),
\]

the unique up to isomorphism Bézout maximal order over \( V_m \) by Corollary 3.5. For \( m < p \), let

\[
B_{m,p} = \begin{pmatrix} V_m & J(V_p) \\ V_p & V_m \end{pmatrix}.
\]

By Theorem 4.12, any semihereditary maximal order over \( V_m \) is isomorphic to \( B_{m,p} \) for some \( p \). Next, let

\[
C_p = \{ x \in A_p \mid x + J(A_{p-1}) \in \mathcal{U} \},
\]

where \( \mathcal{L} \) is a quadratic extension of \( \overline{V_{p-1}} \), and \( \mathcal{U} \) is the integral closure of \( V_p/J(V_{p-1}) \) in \( \mathcal{L} \) such that \( \mathcal{U} \) is not a finite module over \( V_p/J(V_{p-1}) \). Hence by [E, 18.7, 13.8], \( \mathcal{U} \) is a discrete valuation ring and \( C_p/J(C_p) = \mathcal{U} = \overline{V_p} \). Finally, if \( m \leq p \), let

\[
C_{m,p} = \{ z \in C_p \mid z + J(C_p) \in V_m/J(V_p) \}.
\]

The rings \( C_p \) and \( C_{m,p} \) depend on the field \( \mathcal{L} \), but for simplicity of notation we will not worry about this. The ring \( C_p \) is precisely the type of example dealt with in Proposition 5.6 for \( V = V_p \). Thus \( C_p \) is maximal over \( V_p \). Since \( C_{m,p} V_p = C_p \), we see that \( C_{m,p} \) is maximal over \( V_m \) by Lemma 2.5. We point out some simple properties of \( C_{m,p} \) that will be helpful in the next theorem. We have \( C_{m,p}/J(C_{m,p}) = \overline{V_m} \). Hence \( C_{m,p} \) is a primary ring. Furthermore, \( C_p V_{p-1} = A_{p-1} \) as \( \mathcal{U} \overline{V_{p-1}} = \mathcal{L} \overline{A_{p-1}} \). Also, as \( J(A_{p-1}) \subseteq C_p \), we obtain \( C_p V_{p-2} = A_{p-2} \), and thus \( C_p V_r = A_r \) for \( r \geq p-2 \). Note that \( C_{m,p} \) is not a Dubrovin valuation ring since \( C_{m,p} V_{p-1} \not\subseteq A_{p-1} \), hence is not semihereditary by [D, Sect. 1, Th. 4]. Each of these three classes is then mutually disjoint.

**Theorem 5.7.** Let \( R \) be a maximal \( V_n \)-order in \( S = M_2(F) \). Then there are three possibilities for the isomorphism class of \( R \):
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(1) \( R \cong A_n \) (if \( R \) is Bézout),

(2) \( R \cong B_{n,m} \) for some \( m \) (if \( R \) is semihereditary but not Bézout), or

(3) \( R \cong C_{n,m} \) for some \( m \) (if \( R \) is primary but not Bézout).

Proof. We use induction on \( n \), the case \( n = 1 \) given by [R, 18.7], as then \( R \cong M_2(V_1) \). Suppose \( n > 1 \) and that the theorem holds for \( V_{n-1} \). Let \( T \supseteq R \) be a maximal order over \( V_{n-1} \). By [D₃, Sect. 3, Th. 3] we have three cases.

Case 1. \( T \cong A_{n-1} \). Let \( W = V_{n-1} \) and \( A = A_{n-1} \). We have \( T = xA_{n-1} \) for some \( x \), so by replacing \( R \) by \( x^{-1}R \) we may assume \( T = A \). If \( RV_{n-1} = A \) then by Theorem 3.4 \( R \cong A_n \). Thus we may suppose that \( RV_{n-1} \not\subseteq A \). Thus \( \bar{R} := R/J(A) \) is not an order in \( \bar{A} = M_2(\bar{W}) \). However, \( \bar{R} \) is \( \bar{P} \)-maximal in \( \bar{A} \). Let \( \mathcal{A} \) be a maximal subring of \( \bar{A} \) containing \( \bar{R} \). By [D₃, Sect. 3, Th. 3] we have three possibilities for \( \mathcal{A} \).

(i) \( \mathcal{A} \) is a maximal order over \( \bar{V} \). This cannot happen by the maximality of \( \bar{R} \) together with the fact that \( \bar{R} \) is not an order in \( \bar{A} \).

(ii) \( \mathcal{A} \) is a quadratic (i.e., maximal) subfield of \( \bar{A} \). It then follows from the maximality of \( \bar{R} \) that \( \bar{R} = \mathcal{A} \), the integral closure of \( \bar{V} \) in \( \mathcal{A} := \mathcal{P} \). If \( \mathcal{A} \) is a finite \( \bar{V} \)-module then \( \mathcal{A} \) embeds in a \( \bar{V} \)-order of \( \bar{A} \), which is false from the maximality of \( \bar{R} \). Thus \( \mathcal{P}/\bar{W} \) is a defective extension, and so we have \( R = C_n \).

(iii) \( \mathcal{A} \cong (\begin{array}{cc} W & 0 \\ W & \bar{W} \end{array}) \). As in the proof of Lemma 5.1, by conjugating \( R \) we may assume equality here. Now the integral closure of \( \bar{V} \) in \( (\begin{array}{cc} W & 0 \\ W & \bar{W} \end{array}) \) is easily seen to be \( (\begin{array}{cc} W & 0 \\ 0 & \bar{W} \end{array}) \), a ring. By maximality of \( \bar{R} \) we obtain \( \bar{R} = (\begin{array}{cc} V & 0 \\ W & \bar{W} \end{array}) \), hence \( R = (\begin{array}{cc} V & 0 \\ W & J(V) \end{array}) = B_{n,n-1} \).

Case 2. \( T \cong B_{n-1,m} \). Again by conjugating \( R \) we may assume \( T = B_{n-1,m} \). Then by Lemma 2.5, \( J(B_{n-1,m}) = (\begin{array}{cc} J(V_{n-1}) & J(V) \\ V & J(V) \end{array}) \subseteq R \subseteq (\begin{array}{cc} V_{n-1} & J(V) \\ V_{m} & J(V) \end{array}) \). This implies that if \( (a \ b) \in R \) then \( (a \ 0) \in R \). Integrality over \( V \) implies that \( a, d \in V \). Thus \( R = (\begin{array}{cc} V & J(V) \\ V & J(V) \end{array}) = B_{n,m} \). Therefore \( R = B_{n,m} \).

Case 3. \( T \cong C_{n-1,m} \). We assume \( T = C_{n-1,m} \). Then \( R \subseteq C_m \). As \( J(C_m) \subseteq R \) we have \( V/J(V_m) \subseteq R/J(C_m) \subseteq C_m/J(C_m) = \bar{V}_m \). Thus \( R/J(C_m) = V_m/J(V_m) \), and so \( R = C_{n,m} \). This completes the proof of the theorem.

For larger matrices one can combine the various constructions to obtain more types of maximal orders. For instance, it is not hard to show that

\[
\begin{pmatrix}
C_{n,p} & J(W) \\
W & W
\end{pmatrix}
\begin{pmatrix}
J(W) \\
J(W) \\
V
\end{pmatrix}
\]
is a maximal $V$-order, where $W \supseteq V$. The arguments in the proof of Theorem 5.7 can be used to show that any maximal order over $V$ in $M_3(F)$ is either of type $\mathcal{K}$, is of the type in Proposition 5.6, or is isomorphic to

$$
\begin{pmatrix}
T & J(W) \\
W & J(W)
\end{pmatrix}
$$

with $W \supseteq V$ and $T$ a maximal $V$-order in $M_2(F)$.

REFERENCES


