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Characters of p'-Degree of p-Solvable Groups

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1. INTRODUCTION

Fix a prime p and a Sylow p-subgroup P of a finite group G, and write $N = \mathbf{N}_G(P)$. The (still unproved) "McKay conjecture" asserts that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N)|$, or in other words that G and N have equal numbers of irreducible characters with degrees not divisible by p. (J. McKay proposed his conjecture in 1971, but only for simple groups and only for $p = 2$; the first formal statement of the general form of the conjecture was given in 1975 by Alperin in [1].)

The McKay conjecture is now known to be valid for many classes of groups. In 1973, the first author proved the conjecture for (solvable) groups of odd order and for all solvable groups when $p = 2$. (This result appears [3].) Subsequently, T. R. Wolf gave a proof valid for every solvable group and E. C. Dade verified the conjecture for all p-solvable groups. A simpler proof for p-solvable groups was then given by Okuyama and Wajima in [10]. (Since our concern in this paper is only with p-solvable groups, we will not mention explicitly the many other known cases.)

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In this paper, we show that even stronger results hold for p-solvable groups. We will show that if G is p-solvable and $\chi \in \mathrm{Irr}_{p'}(G)$, then there are certain linear characters of the Sylow subgroup P that are naturally associated with χ. Also, we shall see that the linear characters associated with χ constitute a single N-orbit and that every linear character of P is associated with at least one member of $\mathrm{Irr}_{p'}(G)$. (This generalizes the case where $P \triangleleft G$, where the linear characters of P that are associated with a character $\chi \in \mathrm{Irr}_{p'}(G)$ are just the irreducible constituents of the restriction χ_P.) Our stronger form of the Okuyama–Wajima theorem can be stated in terms of these associated linear characters. (The precise definition of "associated" is somewhat technical, and so it seems best to defer it until it is actually needed.)

THEOREM A. *Let G be p-solvable and write $N = \mathbf{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$. Then for each linear character λ of P, there are equal numbers of characters in $\mathrm{Irr}_{p'}(G)$ and in $\mathrm{Irr}_{p'}(N)$ that are associated with λ.*

Effectively, what we prove in this paper is that if G is p-solvable, then the sets $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N)$ can be naturally partitioned into "families" of irreducible characters, where each family is associated with an N-orbit of linear characters of P. There is, therefore, a natural bijection between the set of families in $\mathrm{Irr}_{p'}(G)$ and the set of families in $\mathrm{Irr}_{p'}(N)$, and Theorem A tells us that corresponding families have equal size. (In general, however, there does not exist a natural bijection from a family in $\mathrm{Irr}_{p'}(G)$ onto the corresponding family in $\mathrm{Irr}_{p'}(N)$. Of course, by summing over the various families, we can recover from Theorem A the fact that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N)|$ when G is p-solvable.)

For each linear character λ of P, there is a natural numerical invariant: the order of λ as an element of the group of linear characters of P. It is natural, therefore, to ask how this invariant is reflected in the characters $\chi \in \mathrm{Irr}_{p'}(G)$ that are associated with λ. To answer this question, we need to discuss field automorphisms.

Consider the cyclotomic field $\mathbb{Q}_{|G|}$, which contains the values of all characters of G. Given a positive integer e, there exists a unique automorphism σ_e of this field such that σ_e fixes all p'-roots of unity and $\sigma_e(\epsilon) = \epsilon^{1+p^e}$ for every p-power root of unity ϵ. It is easy to see that σ_e has p-power order in the Galois group $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$. Furthermore, since σ_e fixes a p-power root of unity ϵ if and only if $\epsilon^{p^e} = 1$, we see that σ_e fixes a linear character λ of a p-subgroup of G if and only if the order of λ divides p^e. The following result applies to these field automorphisms σ_e. It allows us to determine from the values of $\chi \in \mathrm{Irr}_{p'}(G)$ the order of the linear characters of P that are associated with χ.

THEOREM B. *Suppose that G is p-solvable and that $P \in \mathrm{Syl}_p(G)$. Let σ be an automorphism of the cyclotomic field $\mathbb{Q}_{|G|}$ and assume that σ has p-power order and that it fixes all p'-roots of unity. Then σ fixes $\chi \in \mathrm{Irr}_{p'}(G)$ if and only if σ fixes the linear characters associated with χ.*

We observe that since the linear characters of P associated with a given character $\chi \in \mathrm{Irr}_{p'}(G)$ are conjugate under $N = \mathbf{N}_G(P)$, it follows that a field automorphism that fixes any one of these linear characters necessarily fixes all of them.

The following is immediate from Theorems A and B.

COROLLARY C. *In the situation of Theorem B, let $N = \mathbf{N}_G(P)$. Then the field automorphism σ fixes equal numbers of characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N)$.* ∎

It is tempting to conjecture that the conclusion of Corollary C might remain true even if we drop the hypothesis that G is p-solvable.

Using the automorphisms σ_e, we obtain the following.

COROLLARY D. *Let G be p-solvable. Then the character table of G determines the exponent of P/P', where $P \in \mathrm{Syl}_p(G)$.*

Proof. The exponent of P/P' is the smallest positive integer e such that the field automorphism σ_e fixes all characters in the set $\mathrm{Irr}_{p'}(G)$. To see why this is true, observe that the exponent of P/P' is equal to the exponent of the group of linear characters of P, and this exponent divides p^e if and only if the field automorphism σ_e fixes every linear character of P. By Theorem B, this happens if and only if σ_e fixes all members of $\mathrm{Irr}_{p'}(G)$. ∎

We can also say which members of $\mathrm{Irr}_{p'}(G)$ are associated with the principal character of P.

THEOREM E. *Suppose that G is p-solvable and let $P \in \mathrm{Syl}_p(G)$. Then the members of $\mathrm{Irr}_{p'}(G)$ that are associated with the principal character 1_P are exactly the p'-special characters of G.*

We shall see that our association of linear characters of a Sylow p-subgroup with the members of $\mathrm{Irr}_{p'}(G)$ respects the normal (and subnormal) structure of G. For example, we have the following.

THEOREM F. *Suppose that $S \lhd\lhd G$, where G is p-solvable, and let Q be a Sylow p-subgroup of S. Then a linear character μ of Q has an extension to some Sylow p-subgroup of G containing Q if and only if there exist characters $\theta \in \mathrm{Irr}_{p'}(S)$ and $\chi \in \mathrm{Irr}_{p'}(G)$ such that θ is a constituent of χ_S and μ is associated with θ.*

The McKay conjecture was extended by Alperin so as to apply to individual p-blocks. The Alperin–McKay conjecture asserts that if B is a p-block of G with defect group D and b is the block of $\mathbf{N}_G(D)$ corresponding to B according to Brauer's "first main" theorem, then $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$ contain equal numbers of height-zero characters. (The Okuyama–Wajima paper [10], to which we referred earlier, established the Alperin–McKay conjecture for p-solvable groups.)

Our results can also be extended to work for p-blocks. For example, we prove the following block-theoretic version of Corollary C. (Note that, in general, field automorphisms permute the p-blocks of a group G, but an automorphism that fixes p'-roots of unity fixes all Brauer characters of G, and thus it fixes all p-blocks.)

THEOREM G. *Suppose that G is p-solvable and that B is a p-block of G with Brauer correspondent b. Let σ be an automorphism of the cyclotomic field $\mathbb{Q}_{|G|}$ and assume that σ has p-power order and fixes p'-roots of unity. Then* $\mathrm{Irr}(B)$ *and* $\mathrm{Irr}(b)$ *contain equal numbers of σ-fixed height-zero characters.*

It seems reasonable to conjecture that Theorem G might remain true if the hypothesis that G is p-solvable were dropped. Such a conjecture, of course, would be a strengthened form of the Alperin–McKay conjecture.

Finally, we mention the following variation on Corollary D.

THEOREM H. *Let G be p-solvable and suppose that B is a p-block of G with defect group D. Then D/D' has exponent dividing p^e if and only if the field automorphism σ_e fixes all height-zero members of* $\mathrm{Irr}(B)$.

2. THE MAP Ψ

Although it may be true that our results are most interesting in the case of p-solvable groups, almost everything goes through without extra effort for π-separable groups, where π is some set of primes. For that reason, we do most of our work in this somewhat more general context. (To obtain the results stated in the Introduction, it suffices to specialize to the case where $\pi = \{p\}$.) We write $\mathrm{Irr}_{\pi'}(X)$ to denote the set of all irreducible characters of π'-degree of a group X.

Let G be π-separable, fix a Hall π-subgroup H of G, and let $N = \mathbf{N}_G(H)$. Instead of starting with a character in $\mathrm{Irr}_{\pi'}(G)$ and constructing the N-orbit of linear characters of H that are associated with it, it is more convenient to work backwards, starting with a linear character of H. Accordingly, we construct a natural map Ψ from the set of linear characters of H into the set $\mathrm{Irr}_{\pi'}(G)$.

We begin by recalling some facts from the character theory of π-separable groups.

(2.1) THEOREM. *Let H be a Hall π-subgroup of a π-separable group G and suppose that $\lambda \in \mathrm{Irr}(H)$. Then there is a unique subgroup $W \supseteq H$ maximal with the property that λ extends to W, and there is a unique π-special extension γ of λ to W. Furthermore, if λ is quasiprimitive, then γ^G is irreducible.*

Proof. The uniqueness of the subgroup W is given by Theorem A of [5]. Since λ has an extension to W, it follows by Theorem G of [7] that λ has a π-special extension. That this extension is unique follows from Proposition 6.1 of Gajendragadkar's paper [2].

Now suppose that λ is quasiprimitive. By Theorem B of [6], we know that λ extends to some subgroup $R \supseteq H$ and that every extension of λ to R induces irreducibly to G. By the uniqueness of W, we know that $R \subseteq W$, and thus γ_R is an extension of λ to R. Then $(\gamma_R)^G$ is irreducible, and it follows that $R = W$ and that γ^G is irreducible, as claimed. ∎

In the situation of Theorem 2.1, if $\lambda \in \mathrm{Irr}(H)$ is quasiprimitive, we write $\Psi(\lambda)$ to denote the uniquely defined character $\psi = \gamma^G \in \mathrm{Irr}(G)$. Note that if λ is linear, then it is certainly quasiprimitive, and so $\Psi(\lambda)$ is defined for each linear character λ of H. In this case, we see that the degree of $\Psi(\lambda)$ is equal to $|G : W|$, which is a π'-number, and thus Ψ maps the linear characters of H into the set $\mathrm{Irr}_{\pi'}(G)$. We mention that in the case where λ is linear, the π-special extension γ of λ to W is the unique extension of λ whose order in the group of linear characters of W is a π-number.

Next, we work to determine the precise range of the map Ψ. Before we do this, however, we offer a little more review. If G is π-separable, we consider pairs (S, δ), where $S \lhd\lhd G$, $\delta \in \mathrm{Irr}(S)$, and δ is π-factored. (The latter condition means that δ is a product of a π-special character and a π'-special character of S.) The set of these **subnormal factored pairs** in G is naturally partially ordered, and we focus on the maximal members of this set. All of the details that underlie the following brief summary of facts can be found in the paper [4].

Let $\psi \in \mathrm{Irr}(G)$ be arbitrary, where G is π-separable. Then there exists a unique (up to G-conjugacy) maximal subnormal factored pair (S, δ) such that δ lies under ψ. Of course, if ψ is π-factored, then $S = G$ and $\delta = \psi$, but otherwise, one can show that if T is the stabilizer in G of the pair (S, δ), then $T < G$. Furthermore, induction defines a bijection from the set of irreducible characters of T that lie over δ onto the set of irreducible characters of G that lie over δ. In particular, there is a unique character $\eta \in \mathrm{Irr}(T)$ lying over δ such that $\eta^G = \psi$. If ψ is not π-factored, then we can replace G by T and ψ by η and repeat this process. Since $T < G$, we see that, eventually, we must reach a pair (W, γ), where γ is π-factored and, of course, $\gamma^G = \psi$. The pair (W, γ) is said to be a **nucleus** for ψ, and it is uniquely determined up to G-conjugacy. Finally, we recall that the set $\mathrm{B}_\pi(G)$ is exactly the set of characters $\psi \in \mathrm{Irr}(G)$ such that the nucleus character γ of ψ is π-special, and in particular, $\mathrm{B}_\pi(G)$ contains all of the π-special irreducible characters of G.

(2.2) THEOREM. *Let G, H, λ, W, and γ be as in Theorem* 2.1. *Suppose that λ is quasiprimitive and write $\psi = \gamma^G$, so that $\psi = \Psi(\lambda)$. Then the pair (W, γ) is a nucleus for ψ and $\psi \in \mathrm{B}_\pi(G)$.*

Proof. If $W = G$, then $\psi = \gamma$ is π-special, and in this case $\psi \in \mathrm{B}_\pi(G)$ and $(W, \gamma) = (G, \psi)$ is a nucleus for ψ, as desired. We can assume, therefore, that $W < G$, and we work by induction on $|G|$.

Let S be the unique largest subnormal subgroup of G contained in W. Note that $S \triangleleft W$, and thus since $H \subseteq W$, we see that $S \cap H \triangleleft H$ and $S \cap H$ is a Hall π-subgroup of S. By assumption, $\lambda \in \mathrm{Irr}(H)$ is quasiprimitive, and thus $\gamma_{S\cap H} = \lambda_{S\cap H}$ is homogeneous. Each irreducible constituent of γ_S is π-special, and the restriction of each of these constituents to the Hall π-subgroup $S \cap H$ of S is exactly the unique irreducible constituent of $\gamma_{S\cap H}$. It follows that γ_S has a unique irreducible constituent, which we call δ, and we observe that δ is invariant in W.

Since δ is π-special, it is certainly π-factored, and we claim that (S, δ) is a maximal subnormal factored pair in G. Otherwise, there exists $U \triangleleft\triangleleft G$ with $U > S$, and a π-factored character $\beta \in \mathrm{Irr}(U)$ lying over δ. In fact, the π-special factor of β must lie over δ, and so it is no loss to assume that β is π-special. Furthermore, we can assume that $S \triangleleft U$ and that U/S is a composition factor of G, and so it is either a π-group or a π'-group.

Since $H \subseteq W$ and $H \cap U$ is a Hall π-subgroup of U (because $U \triangleleft\triangleleft G$), we see that $|U : U \cap W|$ is a π'-number. Also, $S \subseteq U \cap W < U$, and it follows that U/S cannot be a π-group, and hence it is a π'-group. We conclude that $\beta_S = \delta$, and thus δ is invariant in U. We now know that U and W are both contained in the stabilizer T of the pair (S, δ), and U/S is a subnormal π'-subgroup of T/S. We conclude that the normal closure V/S of U/S in T/S is a π'-group. Since the π-special character δ of S is invariant in V, it follows that δ has a unique π-special extension $\hat{\delta}$ to V, and $\hat{\delta}$ is invariant in T. We see that each irreducible constituent of $\gamma_{V\cap W}$ is a π-special character lying over δ. Since $\hat{\delta}_{V\cap W}$ is the unique π-special character of $V \cap W$ that lies over δ, we conclude that $\hat{\delta}_{V\cap W}$ is an irreducible constituent of $\gamma_{V\cap W}$. Of course, this constituent extends to $\hat{\delta} \in \mathrm{Irr}(V)$, which is invariant in VW. We conclude that γ extends to some character of VW lying over $\hat{\delta}$. (This follows from Corollary 4.2 of [4].) But by the definition of W, we know that γ cannot extend to any subgroup of G that properly contains W.

We conclude that $V \subseteq W$, and thus $U \subseteq W$, which is not the case. This contradiction proves that (S, δ) is a maximal factored pair in G, as claimed.

Now $S < G$, and it follows that $T < G$, where T is the stabilizer of the pair (S, δ). Since $W \subseteq T$, the inductive hypothesis guarantees that (W, γ) is a nucleus for the character γ^T, and thus by the definition of a nucleus for ψ, we see that (W, γ) is a nucleus for $(\gamma^T)^G = \psi$, as required. Also, since the nucleus character γ for ψ is π-special, it follows by definition that $\psi \in \mathrm{B}_\pi(G)$. ∎

(2.3) THEOREM. *Let $H \subseteq G$ be a Hall π-subgroup, where G is π-separable. Then Ψ maps the linear characters of H onto the set* $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$.

Proof. We have already observed that if $\lambda \in \mathrm{Irr}(H)$ is linear, then $\Psi(\lambda)$ lies in $\mathrm{Irr}_{\pi'}(G)$, and by the previous theorem, we know that $\Psi(\lambda) \in \mathrm{B}_\pi(G)$. Conversely, we choose an arbitrary character $\psi \in \mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$ and we work to show that $\psi = \Psi(\lambda)$ for some linear character λ of H.

Let (W, γ) be a nucleus for ψ. Then γ is π-special and since $\gamma^G = \psi \in \mathrm{Irr}_{\pi'}(G)$, we see that γ has π'-degree, and hence γ is linear. Also, $|G : W|$ is a π'-number in this case, and so if we replace (W, γ) by a suitable conjugate, we can assume that $H \subseteq W$, and thus γ is a π-special extension to W of the linear character $\lambda = \gamma_H$. If $M \supseteq W$ is maximal such that λ extends to M and we let δ be the unique π-special extension of λ to M, we see that δ_W is a π-special extension of λ, and thus $\delta_W = \gamma$. Since $\gamma^G = \psi$ is irreducible, however, it follows that $M = W$ and $\delta = \gamma$. We see now that $\Psi(\lambda) = \gamma^G = \psi$, and the proof is complete. ∎

(2.4) THEOREM. *Let $H \subseteq G$ be a Hall π-subgroup, where G is π-separable, and let $N = \mathbf{N}_G(H)$. If $\psi \in \mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$, then the preimage of ψ under the map Ψ is an N-orbit of linear characters of H.*

Proof. If $n \in N$ and λ is a linear character of H, then since the map Ψ is canonically defined, it follows that $\Psi(\lambda^n) = \Psi(\lambda)^n = \Psi(\lambda)$, and thus the preimage of ψ under the map Ψ is a union of N-orbits.

Suppose now that λ and μ are linear characters of H such that $\Psi(\lambda) = \psi = \Psi(\mu)$. Let (W, γ) and (V, δ) be maximal π-special extensions of λ and μ, respectively, as in Theorem 2.1. Then by Theorem 2.2, we know that (W, γ) and (V, δ) are nuclei for ψ, and thus these pairs must be conjugate in G and we can write $(W, \gamma)^g = (V, \delta)$ for some element $g \in G$. Then H^g and H are Hall π-subgroups of V, and hence $H^{gv} = H$ for some element $v \in V$. We see now that $gv \in N$ and $\gamma^{gv} = \delta^v = \delta$. It follows that $\lambda^{gv} = \mu$, as required. ∎

We can summarize the results of this section as follows.

(2.5) COROLLARY. *Let $N = \mathbf{N}_G(H)$, where H is a Hall π-subgroup of the π-separable group G. Then the map Ψ defines a bijection from the set of N-orbits of linear characters of H onto the set* $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$.

3. SATELLITE CHARACTERS

We begin by quoting some more of the general character theory of π-separable groups.

(3.1) THEOREM. *Let* $\psi \in \mathrm{B}_\pi(G)$, *where* G *is* π*-separable, and suppose* (W, γ) *is a nucleus for* ψ. *Then the map* $\alpha \mapsto (\alpha\gamma)^G$ *is an injection from the set of* π'*-special characters* $\alpha \in \mathrm{Irr}(W)$ *into* $\mathrm{Irr}(G)$.

Proof. This is precisely Theorem C of [9]. ∎

We shall refer to the irreducible characters of the form $(\alpha\gamma)^G$ in Theorem 3.1 as the **satellites** of the character $\psi \in \mathrm{B}_\pi(G)$, and we note that the set of satellites of ψ depends only on ψ, and not on the choice of the particular nucleus (W, γ) for ψ. (This, of course, is because all of the nuclei for ψ are G-conjugate.) Also, we note that ψ is a satellite of itself since we can take α to be the principal character of W.

(3.2) THEOREM. *Let* G *be* π*-separable. Then the sets of satellites of distinct members of* $\mathrm{B}_\pi(G)$ *are disjoint.*

Proof. This is an immediate consequence of Theorem 4.2 of [9]. ∎

It may be tempting to conjecture that if G is π-separable, then every member of $\mathrm{Irr}(G)$ is a satellite of some character in $\mathrm{B}_\pi(G)$, but unfortunately, that is not true in general. In fact, as we shall see in Section 6, every irreducible character is a satellite if and only if G has a normal Hall π-subgroup.

Note that if $\chi \in \mathrm{Irr}(G)$ is a satellite of $\psi \in \mathrm{B}_\pi(G)$, then the π-parts of the degrees of χ and ψ are equal. In particular, it follows that the satellites of the members of $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$ all lie in $\mathrm{Irr}_{\pi'}(G)$. If $\chi \in \mathrm{Irr}_{\pi'}(G)$ is a satellite of $\psi \in \mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$, then ψ is unique, and we know (in the notation of Section 2) that there is a unique N-orbit of linear characters λ of H such that $\Psi(\lambda) = \psi$.

(3.3) DEFINITION. Let H be a Hall π-subgroup of a π-separable group G and suppose that $\chi \in \mathrm{Irr}_{\pi'}(G)$. Then a linear character λ of H is **associated** with χ if χ is a satellite of $\Psi(\lambda)$.

Although it is not true in general that that every irreducible character of G is a satellite, we shall see that each character in $\mathrm{Irr}_{\pi'}(G)$ is a satellite of some (necessarily unique) member of $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$. When this is established, it will follow that every character in $\mathrm{Irr}_{\pi'}(G)$ is associated with some linear characters of H. (We already know, of course, that every linear character of H is associated with some characters $\chi \in \mathrm{Irr}_{\pi'}(G)$, and we also know that the linear characters of H that are associated with a character χ constitute an N-orbit, where $N = \mathbf{N}_G(H)$.)

We begin by quoting some counting results.

(3.4) THEOREM. *Let $N = \mathbf{N}_G(H)$, where H is a Hall π-subgroup of the π-separable group G. Then*

(a) $|\mathrm{Irr}_{\pi'}(G)| = |\mathrm{Irr}(N/H')| = |\mathrm{Irr}_{\pi'}(N)|$.

(b) *The number of π'-special characters of G is equal to* $|\mathrm{Irr}(N/H)|$.

Proof. Since it is clear that $\mathrm{Irr}(N/H') = \mathrm{Irr}_{\pi'}(N)$, we see that the first assertion is just the π-version of the McKay conjecture for π-separable groups. A proof (modeled on that of Okuyama and Wajima for p-solvable groups) was given by Wolf, in [12], where our conclusion (a) is a special case of Wolf's Theorem A. Similarly, our conclusion (b) appears as a special case of Wolf's Corollary 1.16 in [12]. ∎

(3.5) THEOREM. *Let $N = \mathbf{N}_G(H)$, where H is a Hall π-subgroup of the π-separable group G, and let λ be a linear character of H. Then the number of satellites of the character $\psi = \Psi(\lambda)$ is equal to the number of characters of N that lie over λ.*

Proof. Let $W \supseteq H$ be the maximum subgroup to which λ extends and let γ be the π-special extension of λ to W. Then (W, γ) is a nucleus for ψ and the number of satellites of ψ is equal to the number of π'-special characters of W. By Theorem 3.4(b), this is exactly $|\mathrm{Irr}((W \cap N)/H)|$.

In fact, $W \cap N$ is exactly the stabilizer in N of λ. To see why this is true, observe first that since $\gamma_{W \cap N}$ is an extension of λ, it follows that λ is invariant in $W \cap N$. Also, if T is the stabilizer of λ in N, then since H is a normal Hall subgroup of T, it follows that λ extends to T. But W is the unique maximum subgroup containing H to which λ extends, and therefore $T \subseteq W$. Thus $W \cap N = T$, as claimed.

Since λ extends to $W \cap N$, we can conclude by Gallagher's theorem that $|\mathrm{Irr}((W \cap N)/H)| = |\mathrm{Irr}((W \cap N)|\lambda)|$, and by the Clifford correspondence, this is equal to $|\mathrm{Irr}(N|\lambda)|$, as desired. ∎

We point out that the our proof of Theorem 3.5 uses Theorem 3.4(b), but not Theorem 3.4(a). We will exploit this by observing that Theorem 3.5 can be used to give an alternative proof of 3.4(a).

(3.6) THEOREM. *Every character in $\mathrm{Irr}_{\pi'}(G)$ is a satellite of some unique member of $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$.*

Proof. Every irreducible character of N/H' lies over a unique N-orbit of linear characters of H. Furthermore, these N-orbits are in bijective correspondence (via the map Ψ) with the members of $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$. It follows by Theorem 3.5 that the total number of satellites of all of the members of $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$ is equal to $|\mathrm{Irr}(N/H')| = |\mathrm{Irr}_{\pi'}(N)|$. By Theorem 3.4(a), this number is equal to $|\mathrm{Irr}_{\pi'}(G)|$. It follows from the fact that the sets of satellites of the various members of $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$ are disjoint subsets of $\mathrm{Irr}_{\pi'}(G)$ that every member of the latter set is a satellite, as required. ∎

The counting argument that we have just presented actually shows that if we know Theorem 3.5, then the fact that every member of $\mathrm{Irr}_{\pi'}(G)$ is a satellite is equivalent to the fact (asserted in Theorem 3.4(a)) that $|\mathrm{Irr}_{\pi'}(G)| = |\mathrm{Irr}_{\pi'}(N)|$. In Section 4, we will give a different proof of Theorem 3.6, and thus the above argument can be viewed as providing a new proof of Theorem 3.4(a). This alternative proof, however, relies on Theorem 3.4(b) since that result was used in the proof of Theorem 3.5.

As promised, we have now established that each member of $\mathrm{Irr}_{\pi'}(G)$ is associated with a unique N-orbit of linear characters of H and that every linear character λ of H is associated with at least one member of $\mathrm{Irr}_{\pi'}(G)$. In fact, by Theorem 3.5, we know that the number of characters associated with λ is equal to the number of characters of N that lie over λ.

In the case where $H \lhd G$, it is especially easy to determine the linear characters of H that are associated with a character $\chi \in \mathrm{Irr}_{\pi'}(G)$: They are exactly the irreducible constituents of χ_H. To see why this is so, observe that in general, a linear character λ of H is a constituent of the restriction ψ_H, where $\psi = \Psi(\lambda)$. If $H \lhd G$, then H is contained in the kernel of every π'-special character of every subgroup of G that contains H, and thus λ is a constituent of χ_H, where χ is an arbitrary satellite of $\psi = \Psi(\lambda)$. In other words, if $H \lhd G$, then a linear character λ of H is a constituent of χ_H whenever λ is associated with χ.

It is now easy to prove the π-versions of Theorems A, B, and E.

(3.7) COROLLARY. *Let H be a Hall π-subgroup of a π-separable group G and write $N = \mathbf{N}_G(H)$. Then for each linear character λ of H, the numbers of members of* $\mathrm{Irr}_{\pi'}(G)$ *and of* $\mathrm{Irr}_{\pi'}(N)$ *that are associated with λ are equal.*

Proof. We know by Theorem 3.5 that the number of characters $\chi \in \mathrm{Irr}_{\pi'}(G)$ that are associated with a linear character λ of H is equal to the number of characters of $N = \mathbf{N}_G(H)$ that lie over λ. But the characters of N that lie over λ are exactly the members of $\mathrm{Irr}_{\pi'}(N)$ that are associated with λ, and this completes the proof. ∎

(3.8) COROLLARY. *Let H be a Hall π-subgroup of a π-separable group G and let σ be an automorphism of the cyclotomic field $\mathbb{Q}_{|G|}$. Assume that the order of σ is a π-number and that σ fixes all π'-roots of unity in $\mathbb{Q}_{|G|}$. If $\chi \in \mathrm{Irr}_{\pi'}(G)$, then σ fixes χ if and only if σ fixes any one of* (*or equivalently all of*) *the linear characters of H that are associated with χ.*

Proof. There is a unique character $\psi \in \mathrm{B}_{\pi}(G) \cap \mathrm{Irr}_{\pi'}(G)$ such that χ is a satellite of ψ, and thus if σ fixes χ, it must also fix ψ. Since the map Ψ is natural, it follows that σ permutes the preimage Λ of ψ under this map. We know, however, that Λ is an N-orbit of linear characters of H, where $N = \mathbf{N}_G(H)$. Since the actions of σ and N on Λ commute, it follows that the $\langle\sigma\rangle$-orbits in Λ are transitively permuted by N, and thus they all have the same size, which must divide both $o(\sigma)$ and $|\Lambda|$. But $o(\sigma)$ is a π-number and $|\Lambda|$ divides $|N : H|$, which is a π'-number. It follows that σ fixes all of the members of Λ, as required.

Conversely, suppose that σ fixes some linear character λ of H. Let $W \supseteq H$ be the maximum subgroup to which λ extends and let γ be the unique π-special extension of λ to W, so that $\Psi(\lambda) = \gamma^G$. By uniqueness, σ fixes γ and also σ fixes all of the π'-special characters $\alpha \in \mathrm{Irr}(W)$ since each of these characters has values in the subfield of $\mathbb{Q}_{|G|}$ generated by π'-roots of unity, and by hypothesis, σ acts trivially on this subfield. It follows that σ fixes each of the characters $(\alpha\gamma)^G$, and thus σ fixes all satellites of $\Psi(\lambda)$. These, of course, are exactly the members of $\mathrm{Irr}_{\pi'}(G)$ that are associated with λ. The proof is now complete. ∎

(3.9) COROLLARY. *Let H be a Hall π-subgroup of a π-separable group G. Then the members of* $\mathrm{Irr}_{\pi'}(G)$ *that are associated with the principal character of H are exactly the π'-special characters of G.*

Proof. The maximum subgroup to which the principal character of H can be extended is, of course, the group G, and the corresponding π-special extension of 1_H is 1_G. The characters associated with 1_H are therefore the satellites of 1_G, and these are exactly the π'-special characters α of G. ∎

4. SUBNORMAL SUBGROUPS

In this section, we discuss some connections between the normal and subnormal structure of a π-separable group and our results concerning satellites and associated linear characters. Note that if $S \lhd\lhd G$ and $\chi \in \mathrm{Irr}_{\pi'}(G)$, then the irreducible constituents of χ_S all lie in $\mathrm{Irr}_{\pi'}(S)$. Also, if $\psi \in \mathrm{B}_\pi(G)$, then the irreducible constituents of ψ_S all lie in $\mathrm{B}_\pi(S)$. (The first of these assertions is immediate because the degrees of the irreducible constituents of χ_S all divide $\chi(1)$; the second assertion follows from the results of [4].)

(4.1) LEMMA. *Suppose that $S \lhd\lhd G$, where G is π-separable. Let H be a Hall π-subgroup of G and write $K = S \cap H$, so that K is a Hall π-subgroup of S. Let λ be a linear character of H and let $\psi = \Psi(\lambda) \in \mathrm{B}_\pi(G)$ and $\eta = \Psi(\lambda_K) \in \mathrm{B}_\pi(S)$. Then η is a constituent of ψ_S.*

Proof. Let $W \supseteq H$ be the maximal subgroup to which λ can be extended and let γ be the π-special extension of λ to W, so that $\psi = \gamma^G$. Now write $\mu = \lambda_K$ and let V be the maximum subgroup of S to which μ extends. If δ is the π-special extension of μ to V. then both $\delta_{V\cap W}$ and $\gamma_{V\cap W}$

are π-special extensions of μ, and therefore these linear characters are equal since $K \subseteq V \cap W \subseteq S$ and K is a Hall π-subgroup of S. It follows that δ is a constituent of $(\gamma_{V\cap W})^V$, which, by Mackey's theorem, is a constituent of $(\gamma^G)_V = \psi_V$. It follows that $0 \neq [\delta, \psi_V] = [\delta^S, \psi_S] = [\eta, \psi_S]$, as required. ∎

(4.2) THEOREM. *Suppose that $S \lhd\lhd G$, where G is π-separable, and let K be a Hall π-subgroup of S. Let $\theta \in \mathrm{Irr}_{\pi'}(S)$ and assume that θ is associated with a linear character μ of K. Suppose that θ is a constituent of χ_S, where $\chi \in \mathrm{Irr}_{\pi'}(G)$. Then μ extends to a linear character λ of some Hall π-subgroup H of G, where H contains K and λ is associated with χ.*

Before we proceed with the proof of Theorem 4.2, we mention some consequences. First, we observe that if we combine Lemma 4.1 and Theorem 4.2, we obtain the following.

(4.3) COROLLARY. *Suppose that $S \lhd\lhd G$, where G is π-separable, and let $\chi \in \mathrm{Irr}_{\pi'}(G)$ and $\theta \in \mathrm{Irr}_{\pi'}(S)$, where θ is a constituent of χ_S. If θ is a satellite of $\eta \in \mathrm{B}_\pi(S)$, then χ is a satellite of some character $\psi \in \mathrm{B}_\pi(G)$, where η is a constituent of ψ_S.*

Alternative proof of Theorem 3.6 We are given $\chi \in \mathrm{Irr}_{\pi'}(G)$, where G is π-separable, and we need to find $\psi \in \mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$ such that χ is a satellite of ψ. The existence of ψ follows from Corollary 4.3 by taking S to be the trivial subgroup of G and $\theta = 1_S = \eta$. ∎

As we remarked following the proof of Theorem 3.6 in Section 3, the fact that 3.6 can be proved without using the counting result of Theorem 3.4 provides a new proof of Theorem 3.4(a). But as we pointed out, this proof relies on Theorem 3.4(b).

Proof of Theorem 4.2. Working by induction on the index $|G : S|$, it is easy to see that we can assume that S is maximal among proper subnormal subgroups of G, and thus $S \lhd G$ and G/S is simple, and hence G/S is either a π-group or a π'-group. Let V be the maximum subgroup of S to which μ can be extended and let δ be the π-special extension of μ to V. Then $\eta = \delta^S$ lies in $\mathrm{B}_\pi(S)$, and because μ is associated with θ by assumption, we know that θ is a satellite of η. It follows that there exists a π'-special character α of V such that $(\alpha\delta)^S = \theta$.

We suppose first that G/S is a π-group, and thus χ_S is irreducible since χ has π'-degree. In this case, we have $\theta = \chi_S$, and in particular, θ is G-invariant. It follows that η is G-invariant since θ uniquely determines η. Let W be the stabilizer of the pair (V, δ) in G. Since η is G-invariant, it follows that every G-conjugate of its nucleus (V, δ) is also a nucleus for η, and hence every such pair is conjugate to (V, δ) in S. We conclude from this that $WS = G$. Also, since δ is invariant in $W \cap S$ and δ induces irreducibly to S, it follows that $W \cap S = V$. We conclude that $|W : V| = |G : S|$ is a π-number.

Now $[\chi_V, \alpha\delta] = [\chi_S, (\alpha\delta)^S] = [\chi_S, \theta] = 1$, and thus there is a unique irreducible constituent τ of χ_W that lies over $\alpha\delta$, and we have $[\tau_V, \alpha\delta] = 1$. Also, we know that the pair (V, δ) and the character θ together determine α uniquely, and thus since W stabilizes both (V, δ) and θ, it follows that α is invariant in W. Thus $\alpha\delta$ is invariant in W and we deduce that $\tau_W = \alpha\delta$. Also, since α is π'-special and is invariant in W, and we know that W/V is a π-group, it follows that α has a π'-special extension $\beta \in \mathrm{Irr}(W)$. Furthermore, since δ is π-special, we see that all irreducible constituents of δ^W are π-special, and thus since $(\alpha\delta)^W = \beta\delta^W$, we conclude that we can write $\tau = \beta\gamma$, where $\gamma \in \mathrm{Irr}(W)$ lies over δ. Since τ extends $\alpha\delta$, it follows that γ extends δ, and thus γ is a linear π-special character of W.

Since $|G : W| = |S : V|$ is a π'-number, we can choose a Hall π-subgroup H of G such that $K \subseteq H \subseteq W$, and thus $K = S \cap H$. If we write $\lambda = \gamma_H$, we see that λ is an extension of μ. Also, W is the maximum subgroup of G to which λ can be extended since $V = S \cap W$ is the maximum subgroup of S to which μ can be extended. Furthermore, γ is the π-special extension of λ to W, and it follows that $\psi = \gamma^G$ lies in $\mathrm{B}_\pi(G)$ and that (W, γ) is a nucleus for ψ. Since $\beta \in \mathrm{Irr}(W)$ is π'-special, we know that $(\beta\gamma)^G$ is irreducible and this character is a satellite of ψ. But $\beta\gamma = \tau$, which lies under χ. Thus $\chi = \tau^G$ is a satellite of ψ, and hence λ is associated with χ, as desired.

We can assume now that G/S is a π'-group, and thus K is a Hall π-subgroup of G and we take $H = K$ and $\lambda = \mu$. Let W be the maximum subgroup of G to which μ can be extended and let γ be the π-special extensions of μ to W. Note that $V \subseteq W$ and $\gamma_V = \delta$. Also, if we write $\psi = \gamma^G$, then $\psi = \Psi(\mu)$ computed in G, and we must show that χ is a satellite of ψ.

Now χ lies over $\theta = (\alpha\delta)^S$, and thus some irreducible constituent τ of χ_W lies over $\alpha\delta$. It follows that τ is an irreducible constituent of $(\alpha\delta)^W = \alpha^W\gamma$. But $|W : V|$ divides $|G : K|$, which is a π'-number, and thus every irreducible constituent of α^W is π'-special. It follows that $\tau = \beta\gamma$, where β is a π'-special character of W. Then $\tau^G = (\beta\gamma)^G$ is irreducible and is a satellite of ψ. But χ lies over τ, and thus $\chi = \tau^G$ is a satellite of ψ, as desired. This completes the proof. ∎

Our next result is the π-version of Theorem F.

(4.4) THEOREM. *Suppose that $S \lhd\lhd G$, where G is π-separable, and let μ be a linear character of a Hall π-subgroup K of S. Then μ has an extension to some Hall π-subgroup of G containing K if and only if there exist characters $\theta \in \mathrm{Irr}_{\pi'}(S)$ and $\chi \in \mathrm{Irr}_{\pi'}(G)$ such that θ is a constituent of χ_S and μ is associated with θ.*

Proof. If θ and χ have the stated properties, then by Theorem 4.2, we know that μ extends to some Hall π-subgroup of G. Conversely, suppose that μ extends to $\lambda \in \mathrm{Irr}(H)$, where H is some Hall π-subgroup of G containing K. Let $\theta = \Psi(\mu)$ and $\chi = \Psi(\lambda)$, so that $\theta \in \mathrm{Irr}_{\pi'}(S)$ and $\chi \in \mathrm{Irr}_{\pi'}(G)$. Note that μ is associated with θ since θ is a satellite of itself. By Lemma 4.1, we know that θ is a constituent of χ_S, and this completes the proof. ∎

5. BLOCKS

If we wished to continue to work with π-separable groups, we could do so by referring to the π-block theory developed by Slattery in [11]. We have decided, however, to limit our discussion to the classical theory of p-blocks, and so in what follows, we will consider p-solvable groups.

We begin with some review. (All of this is discussed in Slattery's paper [11], to which we refer the reader for details.) Suppose that G is p-solvable and let $M = \mathrm{O}_{p'}(G)$. If B is a p-block of G, then there exists $\theta \in \mathrm{Irr}(M)$ such that all members of $\mathrm{Irr}(B)$ lie over θ (and θ is uniquely determined up to G-conjugacy). Let T be the stabilizer of θ in G.

If $T = G$ (so that θ is invariant in G) then a Sylow p-subgroup P of G is a defect group for B, and in this case, the set $\mathrm{Irr}(B)$ is the full set $\mathrm{Irr}(G|\theta)$ of irreducible characters of G that lie over θ. Also, in this case, if b is the block of $N = \mathbf{N}_G(P)$ that corresponds to B according to Brauer's theorem, then $\mathrm{Irr}(b) = \mathrm{Irr}(N \mid \theta^*)$, where $\theta^* \in \mathrm{Irr}(N \cap M)$ corresponds to θ under the Glauberman correspondence with respect to the action of P on M. (Note that $N \cap M = \mathbf{C}_M(P)$, so that this makes sense.) In this case, the characters of height zero in B and b are exactly the characters of p'-degree in these sets.

In the case where $T < G$, there is a p-block B_0 of T such that all of the members of $\mathrm{Irr}(B_0)$ lie over θ and these characters are exactly the Clifford correspondents of the members of $\mathrm{Irr}(B)$ with respect to the character θ. (It follows that induction from T to G defines a bijection from $\mathrm{Irr}(B_0)$ onto $\mathrm{Irr}(B)$.) In this case, a defect group D for B_0 is also a defect group for B and the characters of height zero in $\mathrm{Irr}(B)$ are exactly the characters obtained by inducing the characters of height zero in B_0 to G.

To describe the Brauer correspondent b of B in this case, write $N = \mathbf{N}_G(D)$ and let b_0 be the block of $N \cap T$ corresponding to B_0. Then $N \cap T$ is the stabilizer in N of the Glauberman correspondent θ^* of θ with respect to the action of the defect group D, and the members of $\mathrm{Irr}(b_0)$ are exactly the Clifford correspondents of the characters in $\mathrm{Irr}(b)$ with respect to the character $\theta^* \in \mathrm{Irr}(N \cap M)$. In this situation, the characters of height zero in b are exactly the characters obtained by inducing the characters of height zero in b_0 to N.

Given a p-block B of G with defect group D, let $N = \mathbf{N}_G(D)$. By analogy with our earlier construction, We would like to associate to each height-zero character $\chi \in \mathrm{Irr}(B)$ a unique N-orbit of linear characters of D. We construct this **block-association** map recursively, as follows. Let M, θ, and T be as in the foregoing discussion, where D is a defect group of the block B_0 of T. If $T = G$, then D is a Sylow p-subgroup of G and the height-zero characters of B are exactly the characters of p'-degree in $\mathrm{Irr}(B)$. To each of these, we have previously (in Section 3) associated an N-orbit of linear characters of D, and these are the linear characters of D that we now block-associate with χ.

If $T < G$, then we can suppose D is a defect group of the block B_0 of T, and by the inductive hypothesis, we can assume that we have already block-associated with each height-zero character $\eta \in \mathrm{Irr}(B_0)$ an $(N \cap T)$-orbit of linear characters of D. If $\chi \in \mathrm{Irr}(B)$ has height zero, we know that $\chi = \eta^G$ for some unique character η in $\mathrm{Irr}(B_0)$, and furthermore, η has height zero in B_0. We block-associate with χ the full N-orbit of linear characters of D that contains the $(N \cap T)$-orbit that was block-associated with η.

Of course, the construction that we have just given appears to depend on the choice of θ. We now show, however, that the N-orbit Λ of linear characters of D block-associated with the height-zero character $\chi \in \mathrm{Irr}(B)$ is unambiguously determined; it is independent of the choice of θ. Write $\chi = \eta^G$, where $\eta \in \mathrm{Irr}(B_0)$, and let Δ be the $(N \cap T)$-orbit of linear characters of D that is block-associated with η. (Working by induction on $|G|$, we can suppose that Δ is unambiguously determined by η.) Now suppose that we replace θ by θ^g. Then T is replaced by T^g and the block B_0 of T is replaced by the block $(B_0)^g$ of T^g, which we are assuming also has defect group D. Since D^g is also a defect group of this block, we know that $D^{gt} = D$ for some element $t \in T^g$. Since $\theta^g = \theta^{gt}$, we can replace g by gt and assume that g normalizes D. When we carry out our construction with θ^g in place of θ, therefore, the $(N \cap T)$-orbit Δ will be replaced by the $(N \cap T^g)$-orbit Δ^g. But since $g \in N$, both Δ and Δ^g are contained in the same N-orbit Λ, and this shows that, as claimed, Λ depends only on χ and is independent of the choice of θ. Block-association therefore yields an unambiguous assignment of linear characters of the defect group D of B to the height-zero characters in $\mathrm{Irr}(B)$.

We should mention that even in the case where the defect group D is a full Sylow p-subgroup of G, and hence a height-zero character $\chi \in \mathrm{Irr}(B)$ lies in $\mathrm{Irr}_{p'}(G)$, it is not clear that the linear characters of D that we have just block-associated with χ are the same as the linear characters that we associated with χ in Section 3. (Of, course, in the case where $T = G$, we do get the same linear characters.)

(5.1) LEMMA. *Let B be a p-block of a p-solvable group G and suppose that a defect group D of B is normal in G. If $\chi \in \mathrm{Irr}(B)$ has height zero,* *then the linear characters of D that are block-associated with χ are exactly the irreducible constituents of χ_D.*

Proof. Choose θ and T as usual. If $T = G$, then $D \in \mathrm{Syl}_p(G)$ and the linear characters of D that are block-associated with χ are just the associated linear characters in the sense of Section 3. Since $D \lhd G$, we know that these are exactly the irreducible constituents of χ_D.

We can suppose, therefore, that $T < G$, and using our previous notation, we write $\chi = \eta^G$, where η has height zero in B_0. Also, D is a normal defect group of B_0, and so by our inductive hypothesis, the linear characters of D that are block-associated with η are the irreducible constituents of η_D. Since $\chi = \eta^G$, we see that the G-orbit containing these linear characters consists exactly of the set of irreducible constituents of χ_D. ∎

We also need the following result, which is a refinement of Theorem 3.4(b).

(5.2) THEOREM. *Suppose that G is p-solvable and let $P \in \mathrm{Syl}_p(G)$. Write $N = \mathbf{N}_G(P)$ and let $M \lhd G$ be a p'-subgroup. Suppose that $\theta \in \mathrm{Irr}(M)$, where θ is G-invariant and let $\theta^* \in \mathrm{Irr}(N \cap M)$ be the Glauberman correspondent of θ with respect to the action of P. Then the number of p'-special characters in $\mathrm{Irr}(G \mid \theta)$ is equal to the number of p'-special characters in $\mathrm{Irr}(N \mid \theta^*)$.*

Proof. In a p-solvable group X, the restriction map to p-regular elements defines a bijection from the set of p'-special characters of X onto the set of irreducible Brauer characters of X having p'-degree. These are exactly the irreducible Brauer characters of X for which a full Sylow p-subgroup of X is a vertex.

We want to apply the results of [8] in the case where the prime set π of that paper is p'. In this case, the quantity $n(G, \theta, P)$ of [8] is the number of irreducible Brauer characters of G that lie over θ and have vertex P. By Theorem 6.3 of [8], we have $n(G, \theta, P) = n(NM, \theta, P)$. Also, we can apply Theorem 6.4 of that paper to the group NM, and we deduce that $n(NM, \theta, P) = n(N, \theta^*, P)$. It follows that $n(G, \theta, P) = n(N, \theta^*, P)$. The proof is now complete. ∎

Now let λ be a linear character of D, where D is a defect group of the p-block B of G, where G is p-solvable. We want to count the number of height-zero characters of B that are block-associated with λ. Our result is the block analog of Theorem A.

(5.3) THEOREM. *Let B be a p-block of the p-solvable group G, let D be a defect group of B, and write $N = \mathbf{N}_G(D)$. Let b be the block of N that is the Brauer correspondent of B and let λ be an arbitrary linear character of D. Then the number of height-zero characters in $\mathrm{Irr}(B)$ that are block-associated to λ is nonzero and is equal to the number of height-zero characters in $\mathrm{Irr}(b)$ that are block-associated with (or equivalently, lie over) λ.*

Proof. Let θ, T, and B_0 be as before, where D is also a defect group of the block B_0. Assume first that $T = G$, so that $\mathrm{Irr}(B) = \mathrm{Irr}(G \mid \theta)$ and $\mathrm{Irr}(b) = \mathrm{Irr}(N \mid \theta^*)$, where θ^* is the Glauberman correspondent of θ with respect to the action of D on $M = \mathbf{O}_{p'}(G)$. In this case, $D \in \mathrm{Syl}_p(G)$ and the height-zero characters $\chi \in \mathrm{Irr}(B)$ and $\theta \in \mathrm{Irr}(b)$ that are block-associated with λ are exactly the members of $\mathrm{Irr}_{p'}(G \mid \theta)$ and $\mathrm{Irr}_{p'}(N \mid \theta^*)$ that are associated with λ in the sense of Section 3.

Let $W \supseteq D$ be maximal such that λ extends to W and let γ be the p-special extension of λ to W. The members of $\mathrm{Irr}_{p'}(G)$ that are associated with λ are exactly the characters of the form $\chi = (\alpha\gamma)^G$, where α runs over the p'-special characters of W.

Now $M \subseteq W$ and $M \subseteq \ker \gamma$, and thus $\chi = (\alpha\gamma)^G$ lies over θ if and only if α lies over θ. In other words, the number of members of $\mathrm{Irr}_{p'}(G \mid \theta)$ that are associated with λ is equal to the number of p'-special characters α of W that lie over θ, and we must show that this number is equal to the number of members of $\mathrm{Irr}(N)$ that lie over both θ^* and λ. (Note that since D is a full Sylow p-subgroup of N in this case, all irreducible characters of N that lie over the linear character λ necessarily have p'-degree.)

Observe that $(N \cap M)D \triangleleft N$ and that this group is the direct product of $N \cap M$ and D. Also, the irreducible characters of N that lie over both θ^* and λ are exactly the members of $\mathrm{Irr}(N)$ that lie over $\theta^* \times \lambda$, and of course, this number is nonzero. Furthermore, the stabilizer of $\theta^* \times \lambda$ in N is exactly the stabilizer of λ in N, and this stabilizer is $N \cap W$. By the Clifford correspondence, therefore, the number of irreducible characters of N that lie over both θ^* and λ is equal to the number of irreducible characters of $N \cap W$ that lie over both θ^* and λ. These characters of $N \cap W$ are exactly the characters of the form $\gamma_{N \cap W}\beta$, where β runs over the characters of $N \cap W$ that lie over θ^* and over the principal character of D. Finally, we observe that since D is a Sylow p-subgroup of $N \cap W$, the characters β that we must count are exactly the p'-special characters of $N \cap W$ that lie over θ^*. We know that this number is equal to the number of p'-special irreducible characters α of W that lie over θ. (This is Theorem 5.2, applied to the group W.) The result now follows in this case. We can now assume that $T < G$, and thus the number of height-zero characters in $\mathrm{Irr}(B)$ that are block-associated with λ is equal to the number of height-zero characters in $\mathrm{Irr}(B_0)$ that are block-associated with some N-conjugate of λ. Working by induction on $|G|$, we can assume that this number is nonzero and is equal to the number of height-zero characters of b_0 that lie over some N-conjugate of λ, where b_0 is the block of $T \cap N$ that is the Brauer correspondent of B_0. Since induction defines a bijection from the set of height-zero characters in $\mathrm{Irr}(b_0)$ onto the set of the height-zero characters in $\mathrm{Irr}(b)$, and this map carries the characters lying over some N-conjugate of λ to the characters lying over λ, the result follows. ∎

In order to prove Theorem G of the introduction, we need the following block analog of Theorem B.

(5.4) THEOREM. *Let G be p-solvable and suppose B is a block of G with defect group D. Let σ be an automorphism of the cyclotomic field $\mathbb{Q}_{|G|}$ such that σ has p-power order and fixes all p'-roots of unity. Then σ fixes a height-zero character $\chi \in \mathrm{Irr}(B)$ if and only if it fixes the linear characters of D that are block-associated with χ.*

Proof. We observe first that a linear character λ of D is σ-fixed if and only if every linear character in the N-orbit of λ is σ-fixed. (This is because the actions of N and $\langle\sigma\rangle$ on the set of linear characters of D commute.) Let θ, T, and B_0 be as usual, and observe that if $T = G$, then the result follows by Theorem B. We can assume, therefore, that $T < G$ and we write $\chi = \eta^G$ where $\eta \in \mathrm{Irr}(B_0)$ has height zero.

Since θ is a character of the p'-subgroup $M = \mathbf{O}_{p'}(G)$, we see that θ is σ-invariant, and thus σ fixes χ if and only if it fixes its Clifford correspondent η. Working by induction on $|G|$, we know that σ fixes η if and only if it fixes the linear characters of D that are block-associated with η. But these are among the linear characters of D that are block-associated with χ, and the result follows. ∎

We can now prove Theorem G of the Introduction, which we restate here for convenience.

(5.5) THEOREM. *Suppose that G is p-solvable and that B is a p-block of G with Brauer correspondent b. Let σ be an automorphism of the cyclotomic field $\mathbb{Q}_{|G|}$ and assume that σ has p-power order and fixes p'-roots of unity. Then $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$ contain equal numbers of σ-fixed height-zero characters.*

Proof. Let D be a defect group of B such that b is a block of $\mathbf{N}_G(D)$. By Theorem 5.3, we know that for each linear character λ of D, the sets $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$ contain equal numbers of height-zero characters that are block-associated with λ. Also, by Theorem 5.4, we know that the height-zero characters (of both sets) that are block-associated with λ are σ-fixed if and only if λ is σ-fixed. The result now follows. ∎

Finally, we prove Theorem H, which we also restate.

(5.6) THEOREM. *Let G be p-solvable and suppose that B is a p-block of G with defect group D. Then D/D' has exponent dividing p^e if and only if the field automorphism σ_e fixes all height-zero members of* $\mathrm{Irr}(B)$.

Proof. By Theorem 5.4, we know that the field automorphism σ_e fixes all of the height-zero characters in $\mathrm{Irr}(B)$ if and only if it fixes all of the linear characters of D. (We are using the fact, established in Theorem 5.3, that every linear character of D is block-associated with some height zero character in $\mathrm{Irr}(B)$.) We also know that a linear character λ of D is fixed by σ_e if and only if its order divides p^e. Since the exponent of D/D' is equal to the exponent of the group of linear characters of D, the result follows. ∎

6. MORE ON SATELLITES

We conclude with one additional result about satellites of characters in $\mathrm{B}_\pi(G)$.

(6.1) THEOREM. *Suppose that G is π-separable. Then every member of* $\mathrm{Irr}(G)$ *is a satellite of some character in* $\mathrm{B}_\pi(G)$ *if and only if G has a normal Hall π-subgroup.*

Proof. Suppose first that all characters $\chi \in \mathrm{Irr}(G)$ are satellites. We claim that this hypothesis is inherited by all factor groups of G. To see this, let $N \triangleleft G$ and suppose that $N \subseteq \ker(\chi)$, where $\chi \in \mathrm{Irr}(G)$. Suppose that χ is a satellite of $\psi \in \mathrm{B}_\pi(G)$, and let (W, γ) be a nucleus for ψ, so that we can write $\chi = (\alpha\gamma)^G$, where $\alpha \in \mathrm{Irr}(W)$ is π'-special. It follows that $N \subseteq \ker(\alpha\gamma)$, and hence since γ is π-special and α is π'-special, it follows that $N \subseteq \ker(\gamma)$ and $N \subseteq \ker(\alpha)$. In particular, since $\gamma^G = \psi$, we see that $N \subseteq \ker(\psi)$ and we can view both ψ and χ as characters of G/N.

It is not hard to see from the construction of a nucleus that if we view γ as a character of W/N, then the pair $(W/N, \gamma)$ is a nucleus for ψ, viewed as a character of G/N, and in fact, $\psi \in \mathrm{B}_\pi(G/N)$. Also, α can be viewed as a π'-special character of W/N, and it follows that $\chi = (\alpha\gamma)^G$ is a satellite of ψ, viewed as characters of G/N.

Working by induction on $|G|$, we can assume that every proper homomorphic image of G has a normal Hall π-subgroup, and so we can suppose that $\mathbf{O}_\pi(G) = 1$ and that $1 < M < K$, where $M = \mathbf{O}_{\pi'}(G)$ and K/M is the normal Hall π-subgroup of G/M. We work to derive a contradiction.

Let H be a Hall π-subgroup of G. Then $1 < H \subseteq K$, but H is not normal in K since $\mathbf{O}_\pi(K) \subseteq \mathbf{O}_\pi(G) = 1$. It follows that H does not centralize M, and it is well known that this implies that there exists $\beta \in \mathrm{Irr}(M)$ such that H does not stabilize β, and in particular, β is not invariant in K. Since $K \triangleleft G$, it follows that no irreducible character of K that lies over any G-conjugate of β can have π'-degree.

Now let $\chi \in \mathrm{Irr}(G)$ lie over β. By hypothesis, χ is a satellite of some member $\psi \in \mathrm{B}_\pi(G)$, and thus we can write $\chi = (\alpha\gamma)^G$, where α is some π'-special character of W and (W, γ) is a nucleus for ψ. Since $\psi \in \mathrm{B}_\pi(G)$ and $M \triangleleft G$ is a π'-group, we know that $M \subseteq \ker(\psi)$, and thus since K/M

is a π-group, the irreducible constituents of ψ_K are π-special. It follows from the construction of a nucleus that $K \subseteq W$. Since $\chi = (\alpha\gamma)^G$ and χ lies over β, we see that $\alpha\gamma$ lies over some G-conjugate of β.

Since γ is π-special, we know that $M \subseteq \ker(\gamma)$ and it follows that α lies over a G-conjugate of β. But α is π'-special, and hence it has π'-degree. The irreducible constituents of α_K, therefore, have π'-degree and lie over G-conjugates of β, and this is the desired contradiction.

Conversely now, assume that the Hall π-subgroup H of G is normal. Let $\chi \in \mathrm{Irr}(G)$ be arbitrary and let δ be an irreducible constituent of χ_H. Since δ is π-special, we know by the general theory in [4] that there exists a character $\psi \in \mathrm{B}_\pi(G)$ such that ψ lies over δ, and we let (W, γ) be a nucleus for ψ. It is easy to see that $H \subseteq W$. Also, since $|W : H|$ is a π'-number, we see that the π-special character γ restricts irreducibly to H. If we replace δ by an appropriate G-conjugate, we can therefore assume that $\gamma_H = \delta$, and thus W is contained in the stabilizer T of δ in G.

Since H is a normal Hall π-subgroup, it follows that δ extends to some π-special character $\hat{\delta}$ of T, and we see that $\hat{\delta}_W$ is π-special, and thus $\hat{\delta}_W = \gamma$. But $\gamma^G = \psi$ is irreducible, and thus γ cannot be extended to any subgroup properly larger than W. We deduce that $T = W$, and thus χ is induced from some character of W lying over δ. But every such character has the form $\alpha\gamma$, where α is a character of W/H, and hence α is π'-special. It follows that $\chi = (\alpha\gamma)^G$ is a satellite of ψ, and the proof is complete. ∎

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