



# The $\beta$ angle as the CP violating phase in the CKM matrix

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## ABSTRACT

The CKM matrix describing quark mixing with three generations can be parameterized by three Euler mixing angles and one CP violating phase. In most of the parameterizations, the CP violating phase chosen is not a directly measurable quantity and is parameterization dependent. In this work, we propose to use the most accurately measured CP violating angle  $\beta$  in the unitarity triangle as the phase in the CKM matrix, and construct an explicit  $\beta$  parameterization. We also derive an approximate Wolfenstein-like expression for this parameterization.

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## 1. Introduction

The mixing between different quarks is described by a unitary matrix in the charged current interaction of W-boson in the mass eigen-state of quarks, the Cabibbo–Kobayashi–Maskawa (CKM) matrix  $V_{CKM}$  [1,2], defined by

$$L = -\frac{g}{\sqrt{2}} \bar{U}_L \gamma^\mu V_{CKM} D_L W_\mu^+ + H.C., \quad (1)$$

where  $U_L = (u_L, c_L, t_L, \dots)^T$ ,  $D_L = (d_L, s_L, b_L, \dots)^T$ . For  $n$ -generations,  $V = V_{CKM}$  is an  $n \times n$  unitary matrix. With three generations, one can write

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (2)$$

A commonly used parameterization for mixing matrix with three generations of quark is given by [3]

$$V_{PDG} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CK}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CK}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CK}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CK}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CK}} & c_{23}c_{13} \end{pmatrix}, \quad (3)$$

where  $s_{ij} = \sin\theta_{ij}$  and  $c_{ij} = \cos\theta_{ij}$  with  $\theta_{ij}$  being angles rotating in flavor space and  $\delta_{CK}$  is the CP violating phase. We refer this as

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the CK parameterization. This form of parameterization was used by Particle Data Group as the standard parameterization [4].

There are a lot of experimental data on the mixing pattern of quarks. Fitting available data, the mixing angles and CP violating phase are determined to be [5]

$$\begin{aligned} \theta_{12} &= 13.015^\circ \pm 0.059^\circ, \\ \theta_{23} &= 2.376^\circ \pm 0.046^\circ, \\ \theta_{13} &= 0.207^\circ \pm 0.008^\circ, \\ \delta_{CK} &= 69.7^\circ \pm 3.1^\circ. \end{aligned} \quad (4)$$

The angles can be viewed as rotations in flavor spaces. But both the angles and the phase in the CKM matrix are not directly measurable quantities. There are different ways to parameterize the mixing matrix. In different parameterizations, the angles and phase are different. To illustrate this point let us study the original KM parameterization [2],

$$V_{KM} = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta_{KM}} & c_1c_2s_3 + s_2c_3e^{i\delta_{KM}} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta_{KM}} & c_1s_2s_3 - c_2c_3e^{i\delta_{KM}} \end{pmatrix}. \quad (5)$$

Using the observed values for the mixing matrix, one would obtain

$$\begin{aligned} \theta_1 &= 13.016^\circ \pm 0.003^\circ, \\ \theta_2 &= 2.229^\circ \pm 0.066^\circ, \\ \theta_3 &= 0.921^\circ \pm 0.036^\circ, \end{aligned} \quad (6)$$

and the central value of the CP violating phase angle is  $\delta_{KM} = 88.2^\circ$ .

We see that the angles and phases in the CK and KM parameterizations are indeed very different. The angles and phase are

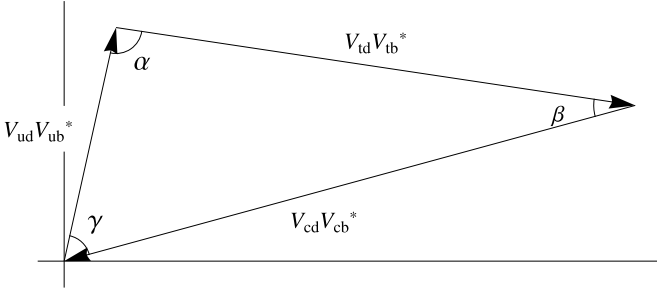


Fig. 1. The unitarity triangle.

parameterization dependent. One can use this freedom to choose a convenient parameterization to study. It is interesting to see whether all quantities used to parameterize the mixing matrix can have well-defined physical meanings, that is, all are experimentally measurable quantities, as have been done for several other quantities related to mixing matrices [6–10]. To this end we notice that the magnitudes of the CKM matrix elements are already experimentally measurable quantities, one can take them to parameterize the mixing matrix. Experimentally there are also several measurable angles which can signify CP violations. The famous ones are the angles  $\alpha$ ,  $\beta$  and  $\gamma$  in the unitarity triangle defined by the unitarity condition

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0. \quad (7)$$

In the complex plane, the above defines a triangle shown in Fig. 1. The unitarity of the CKM matrix actually defines six independent triangle relations through:  $\sum_j V_{ij}V_{kj}^* = 0$ , and  $\sum_j V_{ji}V_{jk}^* = 0$  for  $i$  not equal to  $k$ . Among them,  $i = d$  and  $k = b$  case is the best studied experimentally and the inner angles (phase angles) of the triangle independently measured.

The three inner angles defined by the triangle in Fig. 1 can be expressed as

$$\begin{aligned} \alpha &= \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \\ \beta &= \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \\ \gamma &= \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right). \end{aligned} \quad (8)$$

CP violation dictates that the area of this triangle be non-zero. This implies that none of the angles  $\alpha$ ,  $\beta$  and  $\gamma$  can be zero. Experimentally these three angles have been measured directly [4],  $\alpha = (89.0_{-4.2}^{+4.4})^\circ$ ,  $\beta = (21.1 \pm 0.9)^\circ$  and  $\gamma = (73_{-25}^{+22})^\circ$ . These numbers are consistent with that obtained using the numerical numbers in Eq. (4),  $\alpha = 88.14^\circ$ ,  $\beta = 22.20^\circ$  and  $\gamma = 69.67^\circ$ . Also the directly measured numbers are consistent with the SM prediction  $\alpha + \beta + \gamma = \pi$  in the CKM model with three generations. Notice that the values  $\alpha$ ,  $\gamma$  are very close to the two phases  $\delta_{KM}$ ,  $\delta_{CK}$ , respectively. Among  $\alpha$ ,  $\beta$  and  $\gamma$  angles,  $\beta$  angle is the most accurately measured one. It is therefore interesting to see if one can find a parameterization in which the CP violating phase is represented by the angle  $\beta$ . In the following we will discuss how one can obtain a parameterization using  $\beta$  angle as the phase in the CKM matrix.

## 2. The $\beta$ angle parameterization

Using Eq. (8), one can allocate the  $\beta$  angle at different place, for example the following four ways in which only one of the

$V_{cd,cb,td,tb}$  relevant to the definition of  $\beta$  is complex and all others are real and positive,

$$\begin{aligned} \beta_1): & (|V_{cd}|, |V_{cb}|, |V_{td}|, -|V_{tb}|e^{i\beta}), \\ \beta_2): & (|V_{cd}|, |V_{cb}|, -|V_{td}|e^{-i\beta}, |V_{tb}|), \\ \beta_3): & (|V_{cd}|, -|V_{cb}|e^{-i\beta}, |V_{td}|, |V_{tb}|), \\ \beta_4): & (-|V_{cd}|e^{i\beta}, |V_{cb}|, |V_{td}|, |V_{tb}|). \end{aligned} \quad (9)$$

The above defines four ways of parameterize the CKM matrix in which  $\beta$  is explicitly the CP violating phase. These parameterizations are all equivalent. We will use  $\beta_1$  for discussion. We have

$$V_{CKM}^{\beta_1} = \begin{pmatrix} |V_{ud}| & -\frac{(|V_{ud}|^2 - |V_{cb}|^2)|V_{cd}| + |V_{cb}||V_{td}||V_{tb}|e^{i\beta}}{|V_{cs}||V_{ud}|} & -\frac{|V_{cb}||V_{cd}| - |V_{td}||V_{tb}|e^{i\beta}}{|V_{cb}|} \\ |V_{cd}| & |V_{cs}| & |V_{ud}| \\ |V_{td}| & \frac{|V_{cb}||V_{tb}|e^{i\beta} - |V_{cd}||V_{td}|}{|V_{cs}|} & -|V_{tb}|e^{i\beta} \end{pmatrix}. \quad (10)$$

The CKM matrix is expressed explicitly in terms of modulus of matrix elements and the CP violating angle  $\beta$ .

For this case, we can use  $\beta$ ,  $|V_{cs}|$ ,  $|V_{cd}|$ ,  $|V_{td}|$  as independent variables, and express others as functions of them. We have

$$\begin{aligned} |V_{ud}| &= \sqrt{1 - |V_{cd}|^2 - |V_{td}|^2}, & |V_{cb}| &= \sqrt{1 - |V_{cd}|^2 - |V_{cs}|^2}, \\ |V_{tb}| &= \frac{|V_{cb}||V_{cd}||V_{td}|\cos\beta}{1 - |V_{cd}|^2} \\ &+ \sqrt{\left(\frac{|V_{cb}||V_{cd}||V_{td}|\cos\beta}{1 - |V_{cd}|^2}\right)^2 - \frac{|V_{cs}|^2(|V_{td}|^2 - 1) + |V_{cd}|^2|V_{td}|^2}{1 - |V_{cd}|^2}}. \end{aligned}$$

The CP violating Jarlskog parameter  $J$  [6] is given by

$$J = |V_{cb}||V_{tb}||V_{cd}||V_{td}|\sin\beta.$$

## 3. The $\beta$ and the Euler angle parameterizations

Numerically, one finds that the approximate relations  $\delta_{KM} \approx \alpha$  and  $\delta_{CK} \approx \gamma$ . These can be understood easily by noticing the relations between them [8,11],

$$\begin{aligned} \alpha &= \arctan\left(\frac{\sin\delta_{KM}}{x_\alpha - \cos\delta_{KM}}\right), \\ x_\alpha &= \frac{c_1 s_2 s_3}{c_2 c_3} = \frac{|V_{ud}||V_{td}||V_{ub}|}{|V_{cd}||V_{us}|} = 0.0006; \\ \gamma &= \arctan\left(\frac{\sin\delta_{CK}}{x_\gamma + \cos\delta_{CK}}\right), \\ x_\gamma &= \frac{c_{12} s_{23} s_{13}}{s_{12} c_{23}} = \frac{|V_{ud}||V_{cb}||V_{ub}|}{|V_{tb}||V_{us}|} = 0.0006. \end{aligned}$$

Therefore,  $\delta_{KM} + \alpha$  is approximately  $\pi$ , since  $\alpha$  is close to  $90^\circ$ ,  $\delta_{KM}$  must also be close to  $90^\circ$  and therefore  $\delta_{KM} \approx \alpha$ . It is also clear that  $\delta_{CK}$  is approximately equal to  $\gamma$ .

One may wonder if there is a parameterization with three Euler angles and a phase where the phase is close to  $\beta$ . We find indeed there are such parameterizations. An example is provided by the parameterization  $P4$  discussed in Ref. [12] where

$$V_{CKM}^{P4} = \begin{pmatrix} c_\theta c_\tau & c_\theta s_\sigma s_\tau + s_\theta c_\sigma e^{-i\varphi} & c_\theta c_\sigma s_\tau - s_\theta s_\sigma e^{-i\varphi} \\ -s_\theta c_\tau & -s_\theta s_\sigma s_\tau + c_\theta c_\sigma e^{-i\varphi} & -s_\theta c_\sigma s_\tau - c_\theta s_\sigma e^{-i\varphi} \\ -s_\tau & s_\sigma c_\tau & c_\sigma c_\tau \end{pmatrix}. \quad (11)$$

We have

$$\beta = \arctan\left(\frac{\sin\varphi}{x_\beta + \cos\varphi}\right),$$

$$x_\beta = \frac{s_\theta c_\sigma s_\tau}{c_\theta s_\sigma} = \frac{|V_{cd}||V_{tb}||V_{td}|}{|V_{ud}||V_{ts}|} = 0.0497. \quad (12)$$

#### 4. A Wolfenstein-like expansion

It has proven to be convenient to use approximate formula such as the Wolfenstein parameterization [13]. In the literature different approximate forms have been proposed [14,15]. We now derive an approximate Wolfenstein-like parameterization in which  $\beta$  is taken to the CP violating phase.

Set  $|V_{cd}| = \lambda$ ,  $|V_{td}| = b\lambda^3$ , and  $|V_{cb}| = c\lambda^2$  with  $\lambda = 0.2251 \pm 0.0010$ , and  $b = 0.7685 \pm 0.0250$ ,  $c = 0.8185 \pm 0.0176$ . Rotating the b-quark field by a phase  $\pi - \beta$ , we obtain to order  $\lambda^3$  for  $V_{CKM}^{\beta_1}$

$$V_{CKM}^{\beta_1} \approx \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & -\lambda & \lambda^3(ce^{-i\beta} - b) \\ \lambda & 1 - \frac{1}{2}\lambda^2 & -c\lambda^2 e^{-i\beta} \\ b\lambda^3 & c\lambda^2 e^{i\beta} & 1 \end{pmatrix}. \quad (13)$$

#### 5. Conclusion

To conclude, we have proposed a new parameterization using the most accurately measured CP violating angle  $\beta$  in the unitarity triangle as the CP violating phase in the CKM matrix. We find an Euler angle parameterization in which the CP violating phase is very close to the angle  $\beta$ . We also derived a new Wolfenstein-like parameterization. Since  $\beta$  is the most accurately measured among these three angles in the unitarity triangle, we therefore consider the  $\beta$  parameterization, the best one to use, to provide information for CP violation.

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