European Journal of Combinatorics 33 (2012) 336–339

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Ramsey-type constructions for arrangements of segments

Jan Kynčl

Department of Applied Mathematics and Institute for Theoretical Computer Science (ITI)[1] *, Charles University, Faculty of Mathematics and Physics, Malostranské nám. 25, 118 00 Prague, Czech Republic*

ARTICLE INFO

Article history:
Available online 10 October 2011

ABSTRACT

Improving a result of Károlyi, Pach and Tóth, we construct an arrangement of n segments in the plane with at most $n^{\log 8/\log 169}$ pairwise crossing or pairwise disjoint segments. We use the recursive method based on flattenable arrangements which was established by Larman, Matoušek, Pach and Törőcsik.

1. Introduction

An *arrangement of segments* is a finite set of compact straight-line segments in the plane in general position (i.e., no three endpoints are collinear). We study the following Ramsey-type problem [6]: what is the largest number $r(k)$ such that there exists an arrangement of $r(k)$ segments with at most k pairwise crossing and at most k pairwise disjoint segments?

Larman et al. [6] proved that $k^5 \geq r(k) \geq k^{\log 5/\log 2} > k^{2.3219}$. The upper bound has remained unchanged since then. Károlyi et al. [3] improved the lower bound to $r(k) \geq k^{\log 27/\log 4} > k^{2.3774}$.

We improve the construction for the lower bound even further and prove the following theorem.

Theorem 1. *For infinitely many positive integers k, there exists an arrangement of* $k^{\log 169/\log 8} > k^{2.4669}$ *segments with at most k pairwise crossing and at most k pairwise disjoint segments.*

Similar questions were studied by Fox et al. [2] for string graphs, a class of graphs generalizing intersection graphs of segments. They proved, as a consequence of a stronger result, that for each positive integer k there is a constant $c(k) > 0$ such that in any system of n curves in the plane where every two curves intersect in at most k points, there is a subset of $n^{c(k)}$ curves that are pairwise disjoint or pairwise crossing.

In an extended version of this paper [5], we also show examples of arrangements of segments that cannot be flattened.

E-mail address: kyncl@kam.mff.cuni.cz.

[1] ITI is supported by project 1M0545 of the Ministry of Education of the Czech Republic.

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2011.09.006

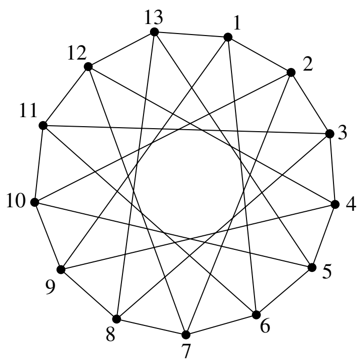

Fig. 1. A Cayley graph $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$.

2. Proof of Theorem 1

Both previous constructions for the lower bound [3,6] use the same approach. The starting configuration is an arrangement M_0 of n_0 segments with at most k_0 pairwise crossing or pairwise disjoint segments. In the i-th step, an arrangement M_i of n_0^{i+1} segments is constructed from the arrangement M_{i-1} by replacing each of its segments by a flattened copy (a precise definition will follow) of M_0, which acts as a "thick segment". Then two segments from different copies of M_0 cross if and only if the two corresponding segments in M_{i-1} cross. Our new arrangement M_i has then at most k_0^{i+1} pairwise crossing or pairwise disjoint segments. This gives a lower bound $r(k) \geq k^{\log n_0 / \log k_0}$ for infinitely many values of k.

We improve the construction by making a better starting arrangement. Unlike the previous constructions, our basic pieces will be arrangements with different maximal numbers of pairwise crossing and pairwise disjoint segments. By putting them together, we obtain our starting arrangement M_0.

Let $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ denote the Cayley graph of the cyclic group $\mathbb{Z}_{13}$ corresponding to the generators 1 and 5. That is, $V(\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)) = \{1, 2, \ldots, 13\}$ and $E(\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)) = \{\{i, j\}; 1 \leq i < j \leq 13, (j - i) \in \{1, 5, 8, 12\}\}$. See Fig. 1.

Lemma 2. *The graph* $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ *contains no clique of size 3 and no independent set of size 5.*

Proof. Suppose that $a < b < c$ are three vertices of $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ inducing a clique. Then the numbers $k = c - a$, $l = c - b$ and $m = b - a$ belong to the set $\{1, 5, 8, 12\}$, but this set contains no triple k, l, m satisfying the equation $k = l + m$; a contradiction.

Now suppose that $A = \{a < b < c < d < e\}$ is an independent set of $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$. By the pigeon-hole principle, A contains two vertices with difference 2 (modulo 13). Thus, we can, without loss of generality, assume that $a = 1$ and $b = 3$. It follows that $\{c, d, e\} \subseteq \{5, 7, 10, 12\}$. But A cannot contain both 5 and 10, neither both 7 and 12. Hence $|A \cap \{5, 7, 10, 12\}| \leq 2$; a contradiction. □

A (k, l)-*arrangement* is an arrangement of segments with at most k pairwise crossing and at most l pairwise disjoint segments.

An *intersection graph* $G(M)$ of an arrangement M is a graph whose vertices are the segments of M and two vertices are joined by an edge if and only if the corresponding segments intersect.

An arrangement M of segments is *flattenable* if for every $\varepsilon > 0$ there is an arrangement M_ε with $G(M_\varepsilon) = G(M)$ and two discs D_1, D_2 of radius ε whose centers are at unit distance, such that each segment from M_ε has one endpoint in D_1 and the second endpoint in D_2. A *flattened copy* of M is the arrangement M_ε with sufficiently small ε.

The key result is the following lemma.

Lemma 3. (1) *There exists a flattenable* $(2, 4)$-*arrangement of 13 segments.*
(2) *There exists a flattenable* $(4, 2)$-*arrangement of 13 segments.*

Table 1
Arrangement $M_a(\varepsilon)$.

	Left x	Left y	Right x	Right y
1	$-\varepsilon$	0	$1-2\varepsilon$	$2\varepsilon^2+2\varepsilon^6$
2	ε^2	$\varepsilon-\varepsilon^3$	$1-\varepsilon^2$	ε^3
3	0	$\varepsilon^4+\varepsilon^6$	1	$\varepsilon^3+3\varepsilon^4$
4	0	$\varepsilon^4-\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2-\varepsilon^6$
5	$-\varepsilon+\varepsilon^2$	0	$1-2\varepsilon^2$	$2\varepsilon^3-2\varepsilon^4$
6	$-\varepsilon$	$2\varepsilon^6$	$1-\varepsilon$	$2\varepsilon^6$
7	0	ε^6	1	$\varepsilon^3+2\varepsilon^4$
8	0	ε	$1+\varepsilon^3$	0
9	0	ε	$1-2\varepsilon^2$	$2\varepsilon^3-\varepsilon^4$
10	$-\varepsilon^2+3\varepsilon^3$	$3\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2+\varepsilon^6$
11	$-\varepsilon^2$	ε^6	$1-2\varepsilon^2$	$2\varepsilon^3-3\varepsilon^4$
12	0	ε^4	1	0
13	$-\varepsilon$	0	$1+\varepsilon$	0

Table 2
Arrangement $M_b(\varepsilon)$.

	Left x	Left y	Right x	Right y
1	ε	$\varepsilon^2-\varepsilon^3+\varepsilon^4-2\varepsilon^5$	$1+\varepsilon^2$	$-\varepsilon^4+\varepsilon^6$
2	0	$\varepsilon^2+3\varepsilon^5$	$1-\varepsilon^3$	ε^7
3	0	$\varepsilon^2+4\varepsilon^5$	$1+\varepsilon$	$-\varepsilon^3$
4	0	$2\varepsilon^3$	$1+3\varepsilon^4$	$-\varepsilon^8$
5	$\varepsilon-\varepsilon^2+\varepsilon^3$	$\varepsilon^2-\varepsilon^3+\varepsilon^4-\varepsilon^8$	$1+\varepsilon$	$-\varepsilon^4$
6	0	$\varepsilon^2+\varepsilon^5$	$1+\varepsilon$	$-\varepsilon^3$
7	0	$\varepsilon^2+5\varepsilon^5$	$1+3\varepsilon^4$	$-3\varepsilon^7$
8	$\varepsilon-\varepsilon^2+\varepsilon^3+\varepsilon^4+2\varepsilon^5$	$\varepsilon^2-\varepsilon^3+\varepsilon^4+\varepsilon^5+\varepsilon^6$	$1+\varepsilon-\varepsilon^4$	$-\varepsilon^3$
9	0	ε^2	$1+\varepsilon$	$-\varepsilon^4$
10	0	0	$1+5\varepsilon^3$	0
11	0	$\varepsilon^2+2\varepsilon^5$	$1+3\varepsilon^4-2\varepsilon^5$	ε^8
12	$\varepsilon-\varepsilon^3$	$\varepsilon^3-\varepsilon^4$	$1+\varepsilon$	$-\varepsilon^4$
13	0	0	1	ε

Note that 13 is the largest possible number of segments for these two types of arrangements since every graph with more than 13 vertices contains either a clique of size 5 or an independent set of size 3 [7].

Both previous constructions [3,6] used *convex* starting arrangement, i.e., an arrangement of segments with endpoints in convex position. Convex arrangements are flattenable by a relatively simple argument [3]. However, Kostochka [4] proved that any convex (k, k)-arrangement has at most $(1 + o(1)) \cdot k^2 \log k$ segments. He also gave a construction of a convex (k, k)-arrangement with $\Omega(k^2 \log k)$ segments (see also [1]). Černý [1] investigated convex (k, l)-arrangements for small values of k. He showed, in particular, that any convex $(2, 4)$-arrangement has at most 12 segments, and any convex $(4, 2)$-arrangement has at most 11 segments.

Our starting arrangements thus cannot be convex. Hence their flattening will require a special approach.

Proof. For each sufficiently small $\varepsilon > 0$, we construct an arrangement $M_a(\varepsilon)$ with intersection graph $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ and an arrangement $M_b(\varepsilon)$ whose intersection graph is the complement of $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$. See Fig. 2 for an illustration.

In Tables 1 and 2, we provide precise coordinates of the endpoints of all the 13 segments, as functions of ε. To achieve general position of the segments, which is required by our definition, we can slightly perturb the endpoints while preserving the intersection graph of the arrangement.

Since the coordinates of all the left endpoints converge to $(0, 0)$ and the coordinates of all the right endpoints converge to $(1, 0)$, it remains to verify that for sufficiently small $\varepsilon > 0$, each of these two described arrangements has the desired intersection graph. This is a straightforward calculation, which can be done by the following simple algorithm.

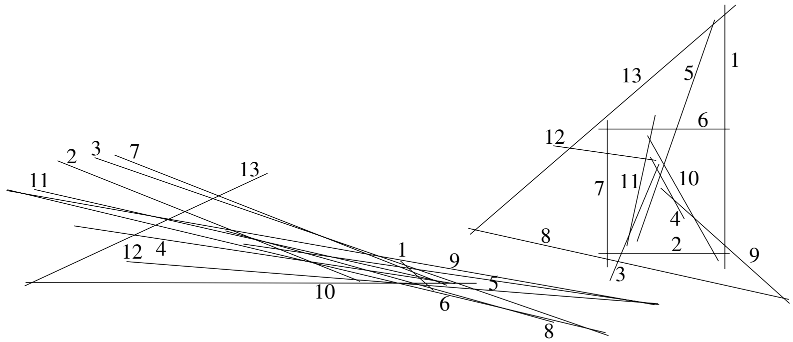

Fig. 2. A partially flattened (4, 2)-arrangement of 13 segments (left) and a (2, 4)-arrangement of 13 segments (right).

We use the fact that the functions describing the coordinates are polynomials in ε. For $i \in 1, 2, \ldots, 13$, let s_i be the i-th segment of the arrangement and let $l_x(i), l_y(i), r_x(i), r_y(i)$ be the polynomials representing the coordinates of the left and the right endpoint of s_i. For each pair $i < j$, we need to determine whether s_i and s_j cross if ε is small enough.

Let s be a segment with endpoints (l_x, l_y) and (r_x, r_y) and let s' be a segment with endpoints (l'_x, l'_y) and (r'_x, r'_y). Let p be the line containing s, and let p' be the line containing s'. The segments s and s' intersect if and only if $s' \cap p \neq \emptyset$ and $s \cap p' \neq \emptyset$. We have $p = \{(x, y); ax+by+c = 0\}$, where $a = r_y - l_y$, $b = r_x - l_x$ and $c = r_x l_y - l_x r_y$. Thus, $s' \cap p \neq \emptyset$ if and only if $(al'_x + bl'_y + c)(ar'_x + br'_y + c) \leq 0$. The relation $s \cap p' \neq \emptyset$ can be expressed similarly.

The algorithm now follows. For each i, compute the polynomials $a_i = r_y(i) - l_y(i)$, $b_i = r_x(i) - l_x(i)$ and $c_i = r_x(i) l_y(i) - l_x(i) r_y(i)$. Then for each pair $i \neq j$, compute the polynomial $d_{i,j} = (a_i l_x(j) + b_i l_y(j) + c_i)(a_i r_x(j) + b_i r_y(j) + c_i)$. Now s_i and s_j intersect if and only if each $d_{i,j}$ and $d_{j,i}$ is nonpositive in some positive neighborhood of 0. That is, the polynomial is either zero or the coefficient by the non-zero term of the smallest order is negative.

A program verifying both constructions can be downloaded from the following webpage: http://kam.mff.cuni.cz/~kyncl/programs/segments. $\square$

Now we are ready to finish the proof of Theorem 1. Take a sufficiently flattened arrangement $M_a(\varepsilon)$ and replace each of its segments by a copy of a sufficiently flattened arrangement $M_b(\delta)$. In this way, we obtain our starting flattenable $(8, 8)$-arrangement M_0 of 169 segments. Then we proceed by the method described at the beginning of this section.

Acknowledgments

The author is grateful to Géza Tóth and Jakub Černý for helpful and inspiring discussions on this problem.

The author was partially supported by the Phenomena in High Dimensions project, in the framework of the European Community's "Structuring the European Research Area" program.

References

[1] J. Černý, Combinatorial and computational geometry, Ph.D. thesis, Charles University, Prague, 2008.
[2] J. Fox, J. Pach, Cs.D. Tóth, Intersection patterns of curves, J. London Math. Soc. 83 (2) (2011) 389–406.
[3] G. Károlyi, J. Pach, G. Tóth, Ramsey-type results for geometric graphs, I, Discrete Comput. Geom. 18 (3) (1997) 247–255, ACM Symposium on Computational Geometry Philadelphia, PA, 1996.
[4] A. Kostochka, Upper bounds on the chromatic number of graphs, Trudy Inst. Mat. (Novosibirsk) 10 (Modeli i Metody Optim) (1988) 204–226.
[5] J. Kynčl, Ramsey-type costructions for arrangements of segments,http://arxiv.org/abs/1010.2167.
[6] D. Larman, J. Matoušek, J. Pach, J. Törőcsik, A Ramsey-type result for convex sets, Bull. London Math. Soc. 26 (2) (1994) 132–136.
[7] S. Radziszowski, Small Ramsey numbers, Electron. J. Combin. 1 (Dynamic Survey 1) (1994) 30 pp (electronic).