Stability criteria in terms of two measures for functional differential equations

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Abstract

Results involving uniform stability and uniform asymptotic stability in terms of two measures of delay differential equations by using Liapunov functional techniques are established.

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1. Introduction

Movchan’s theory of stability in terms of two measures [1] was introduced in 1960. It has been further developed by Lakshmikantham and Leela [2–4] and unifies several known stability concepts. Its importance and usefulness have been highlighted in the study of the stability for different classes of equations [5–10].

Our aim is to establish stability results for delay differential equations in terms of two different measures by employing Liapunov functionals and incorporating the ideas involved in recent advances relating to the concepts of normal and controllable functionals introduced in [11].
This paper is presented as follows. In Section 2, we introduce notations and concepts. In Section 3, conditions for uniform stability and uniform asymptotic stability in measure of delay differential equations are established.

2. Preliminaries

For \( r \geq 0 \), let \( C = C([-r, 0], \mathbb{R}^n) \) be the space of continuous functions taking \([-r, 0]\) into \( \mathbb{R}^n \) with \( \| \phi \|, \phi \in C \), defined by \( \| \phi \| = \sup \{ |\phi(\theta)|, \theta \in [-r, 0]\} \).

If \( x \in C([-t_0 - r, 0], \mathbb{R}^n), t_0 \in \mathbb{R}_+ \), we define, as usual, \( x_t \in C \) by \( x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0 \).

Consider the delay differential equation of the form

\[
x'(t) = f(t, x_t), \quad x_0 = \phi \in C,
\]

where \( f \in C(\mathbb{R}_+ \times C, \mathbb{R}^n) \), \( f \) takes bounded sets into bounded sets, then for each \((t_0, \phi)\), there exists a unique solution \( x(t) = x(t_0, \phi)(t) \) of (1) which can be continued for all future time.

To unify several different concepts of stability studied in the literature such as partial stability, conditional stability, and eventual stability, it is convenient to introduce stability concepts in terms of two measures [2–4]. We shall discuss the stability properties of (1) with respect to two measures. We need the following definitions and the classes of functions:

\[
\Gamma = \left\{ h \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) : \inf_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n} h(t, x) = 0 \right\},
\]

\[
\Gamma_0 = \left\{ h_0 \in C(\mathbb{R}_+ \times C, \mathbb{R}_+) : \inf_{(t, \phi) \in \mathbb{R}_+ \times C} h_0(t, \phi) = 0 \right\}.
\]

Let \( X \) be a Banach space. A function \( h \in C(\mathbb{R}_+ \times X, \mathbb{R}_+) \) is called a measure in \( X \) (denoted by \( h \in \Gamma \)) if \( \inf_{(t, x) \in \mathbb{R}_+ \times X} h(t, x) = 0 \).

Let \( h_0 \in \Gamma_0, h \in \Gamma \) and \( x(t_0, \phi)(t) \) be a solution of (1).

Definition 1. Eq. (1) is said to be:

(I) \( (h_0, h) \)-uniformly stable, if for each \( \epsilon > 0 \) and \((t_0, \phi), (t_0, \psi) \in \mathbb{R}_+ \times C \), there exists a \( \delta = \delta(\epsilon) > 0 \) such that if \( h_0(t_0, \phi - \psi) < \delta \), then \( h(t, (x(t_0, \phi) - x(t_0, \psi))(t)) < \epsilon, t \geq t_0 \).

(II) \( (h_0, h) \)-uniformly asymptotically stable, it is \( (h_0, h) \)-uniformly stable, and there is a \( \delta_0 > 0 \) such that for each \( \epsilon > 0 \), there exists a \( T = T(\epsilon) > 0 \), such that if \((t_0, \phi), (t_0, \psi) \in \mathbb{R}_+ \times C \) and \( h_0(t_0, \phi - \psi) < \delta_0 \), then \( h(t, (x(t_0, \psi) - x(t_0, \phi))(t)) < \epsilon, t \geq t_0 + T \).

Remark. Definition 1. (I) reduces to uniform stability of the prescribed solution \( y(t) = y(t_0, \psi)(t) \) of Eq. (1) if \( h(t, x) = |x - y(t)| \) and \( h_0(t_0, \phi) = |\phi - \psi| \).

A very complete study about stability in terms of two measures may be found in [2–4].

Definition 2. A function \( W : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is said to be positive definite, if \( W \) is continuous with \( W(0) = 0 \) and \( W(x) > 0 \) for any \( x \neq 0 \).

Definition 3. A functional \( V : \mathbb{R}_+ \times C \times C \rightarrow \mathbb{R}_+ \) is said to be an \( h \)-normal functional, \( h \in \Gamma \), if \( V \) is continuous and there exists a positive definite function \( W(x) \) such that \( V(t, \phi, \psi) \geq W(h(t, (\phi - \psi)(t))), \forall t \in \mathbb{R}_+ \) and \( \phi, \psi \in C \).
**Definition 4.** A functional \( V : \mathbb{R}_+ \times C \times C \to \mathbb{R}_+ \) is said to be an \( h_0 \)-controllable functional, \( h_0 \in \Gamma_0 \), if \( V \) is continuous and there exists a positive definite function \( W(x) \) such that \( V(t, \phi, \psi) \leq W(h_0(t_0, \phi - \psi)), \forall t \in \mathbb{R}_+ \) and \( \phi, \psi \in C \).

3. Main results

Our results are achieved with the following lemmas.

**Lemma 1.** A functional \( V(t, \phi, \psi) \) is an \( h \)-normal functional if and only if for any bounded set \( \Omega \subset C \) and any \( \epsilon > 0 \), there exists a \( \delta = \delta(\epsilon) > 0 \) such that if \( \phi - \psi \in \Omega \) and \( h(t, (\phi - \psi)(0)) \geq \epsilon, h \in \Gamma \), then \( V(t, \phi, \psi) \geq \delta, \forall t \in \mathbb{R}_+. \)

**Proof.** *Necessity.** Since \( V(t, \phi, \psi) \) is an \( h \)-normal functional, there exists a positive definite function \( W(x) \) such that \( V(t, \phi, \psi) \geq W(h(t, (\phi - \psi)(0))). \)

Suppose, by contradiction, that the conclusion is not valid. Then there exist a bounded set \( D \subset C \), an \( \epsilon_0 > 0 \) and a positive sequence \( \{t_n\} \) such that for any \( \delta_n = 1/n \), if \( \phi_n - \psi_n \in D \) and \( h(t_n, (\phi_n - \psi_n)(0)) \geq \epsilon_0 \), then \( V(t_n, \phi_n, \psi_n) < \delta_n \). Thus \( W(h(t_n, (\phi_n - \psi_n)(0))) < \delta_n \).

We can assume that \( \{\alpha_n\} = h(t_n, (\phi_n - \psi_n)(0)) \) is convergent (otherwise, we can select a subsequence); i.e., there exists an \( x_0 \in \mathbb{R}_+ \), such that \( \alpha_n = h(t_n, (\phi_n - \psi_n)(0)) \to x_0 \), as \( n \to \infty \). Since \( W(x) \) is a continuous function, \( W(h(t_n, (\phi_n - \psi_n)(0))) \to W(x_0) \), as \( n \to \infty \). Also \( W(h(t_n, (\phi_n - \psi_n)(0))) \) → 0, as \( n \to \infty \). It follows that \( W(x_0) = 0 \). Since \( W(x) \) is a positive definite function, we have \( x_0 = 0 \), which contradicts the fact that \( x_0 \geq \epsilon_0 \), since \( h(t_n, (\phi_n - \psi_n)(0)) \geq \epsilon_0 \).

**Sufficiency.** It is possible to choose a positive sequence \( \{\epsilon_n\} \) such that \( \epsilon_{n+1} < \epsilon_n \) and \( \epsilon_n \to 0 \), as \( n \to \infty \). For each \( \epsilon_n \), there exists a \( \delta_n > 0 \) such that if \( \phi - \psi \in \Omega \) and \( h(t, (\phi - \psi)(0)) \geq \epsilon_n \), then \( V(t, \phi, \psi) \geq \delta_n, \forall t \in \mathbb{R}_+. \) Without loss of generality we can assume that \( \delta_{n+1} < \delta_n \) and \( \delta_n \to 0 \) as \( n \to \infty \) (one only needs to replace \( \delta_n \) with \( \tilde{\delta}_n = \min\{\frac{\delta_n}{2}, \delta_n\}; n \geq 1 \), and \( \tilde{\delta}_0 = \delta_0 \)).

We define \( W(x) \) as

\[
W(0) = 0,
W(x) = \delta_{i+1} + \frac{\delta_i - \delta_{i+1}}{\epsilon_i - \epsilon_{i+1}}(x - \epsilon_i); \quad x \in [\epsilon_i, \epsilon_{i-1}],
W(x) = \delta_1; \quad x \geq \epsilon_0.
\]

\( W(x) \) is continuous, positive definite and satisfies \( W(h(t, (\phi - \psi)(0))) \leq \delta_i \leq V(t, \phi, \psi) \). Therefore \( V(t, \phi, \psi) \geq W(h(t, (\phi - \psi)(0))) \) and \( V(t, \phi, \psi) \) is an \( h \)-normal functional. \( \square \)

**Lemma 2.** A functional \( V(t, \phi, \psi) \) is an \( h_0 \)-controllable functional if and only if for any \( \epsilon > 0 \), there exists a \( \delta = \delta(\epsilon) > 0 \) such that if \( h_0(t, \phi - \psi) < \delta, h_0 \in \Gamma_0 \), then \( V(t, \phi, \psi) \leq \epsilon, \forall t \in \mathbb{R}_+. \)

**Proof.** *Necessity.** Since \( V(t, \phi, \psi) \) is an \( h_0 \)-controllable functional, there exists a positive definite function \( W(x) \) such that \( V(t, \phi, \psi) \leq W(h_0(t, \phi - \psi)), t \in \mathbb{R}_+. \) Since \( W(0) = 0 \) and \( W(x) \) is continuous, it follows that \( \lim_{a \to (\phi - \psi)} W(h_0(t, \phi - \psi)) = 0 \), i.e., for any \( \epsilon > 0 \), there is a \( \delta = \delta(\epsilon) > 0 \) so that if \( h_0(t, \phi - \psi) < \delta \), then \( W(h_0(t, \phi - \psi)) \leq \epsilon, \forall t \in \mathbb{R}_+. \) Thus \( V(t, \phi, \psi) \leq \epsilon, \forall t \in \mathbb{R}_+. \)

**Sufficiency.** We can choose a positive sequence \( \{\epsilon_n\} \) such that \( \epsilon_{n+1} < \epsilon_n \) and \( \epsilon_n \to 0 \) as \( n \to \infty \). For each \( \epsilon_n \), there exists a \( \delta_n > 0 \) such that if \( h_0(t, \phi - \psi) < \delta_n \), then \( V(t, \phi, \psi) \leq \epsilon_n, t \in \mathbb{R}_+. \)

We may assume that \( \delta_{n+1} < \delta_n \) and \( \delta_n \to 0 \), as \( n \to \infty \) (it only needs to replace \( \delta_n \) with \( \tilde{\delta}_n = \min\{\frac{\delta_n}{2}, \delta_n\}; n \geq 1 \), and \( \tilde{\delta}_0 = \delta_0 \)).
We define $W(x)$ as:

\[
W(0) = 0,
\]

\[
W(x) = \epsilon_i + \frac{\epsilon_{i-1} - \epsilon_i}{\sigma_i - \sigma_{i+1}}(x - \sigma_{i+1}); \quad x \in [\sigma_{i+1}, \sigma_i],
\]

\[
W(x) = \epsilon_0; \quad x \geq \sigma_1.
\]

$W(x)$ is continuous and positive definite.

The functional $V(t, \phi, \psi)$ satisfies $V(t, \phi, \psi) \leq \epsilon_i < W(h_0(t, \phi - \psi)), \forall t \in \mathbb{R}_+, \text{which implies that}$

\[
V(t, \phi, \psi) \text{ is an } h_0\text{-controllable functional.}
\]

**Theorem 1.** Let $h_0 \in \Gamma_0$ and $h \in \Gamma$. Eq. (1) is $(h_0, h)$-uniformly stable if and only if there exists an $h$-normal and $h_0$-controllable functional $V(t, \phi, \psi)$ such that $V(t, x(t_0), (t))$ is monotonically nonincreasing with respect to $t$ for any $t_0 \in \mathbb{R}_+$ and $\phi, \psi \in C$.

**Proof.** **Necessity.** We define the following functional:

\[
V(t, \phi, \psi) = \inf_{0 \leq t \leq T} h_0(\tau, x(\tau, \phi) - x(t, \psi)).
\]

By the definition, $V(t, \phi, \psi) \leq h_0(t, x(t, \phi) - x(t, \psi)) = h_0(0, \phi - \psi)$. Therefore $V(t, \phi, \psi)$ is $h_0$-controllable. Since Eq. (1) is $(h_0, h)$-uniformly stable, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $h_0(\tau, \phi - \psi) < \delta$, then $h(t, x(\tau, \phi) - x(t, \psi)) < \epsilon$, for any $t \geq \tau$.

We claim that if $h(t, (\phi - \psi)(0)) \geq \epsilon$, then $h_0(\tau, x(t, \phi) - x(t, \psi)) \geq \delta$, for any $t \geq \tau$.

In fact, if this is not true, then there are $\phi_0, \psi_0 \in C$ and $t_0, \tau_0$ with $t_0 \geq \tau_0$ so that $h(t_0, (\phi_0 - \psi_0)(0)) \geq \epsilon$ and $h_0(\tau_0, x(t_0, \phi_0) - x(t_0, \psi_0)) < \delta$.

Let\n
\[
\tilde{\phi} := x(t_0, \phi_0); \quad \tilde{\psi} := x(t_0, \psi_0).
\]

Then $\tilde{\phi}, \tilde{\psi} \in C$ and $h_0(t_0, \tilde{\phi} - \tilde{\psi}) < \delta$. From the $(h_0, h)$-uniform stability of Eq. (1), it follows that

\[
h(t, (x(t_0, \tilde{\phi}) - x(t_0, \tilde{\psi}))(\tau)) < \epsilon, \quad t \geq \tau_0.
\]

In particular, let $t = t_0$. Then

\[
h(t_0, (x(t_0, \phi_0) - x(t_0, \psi_0)) < \epsilon, \quad \text{i.e.,}
\]

\[
h(t_0, (x(t_0, \phi_0) - x(t_0, \psi_0)) < \epsilon, \quad \text{which is equivalent to}
\]

\[
h(t_0, (\phi_0 - \psi_0)(0)) < \epsilon, \quad \text{in contradiction with the above hypothesis that}
\]

\[
h(t_0, (\phi_0 - \psi_0)(0)) \geq \epsilon.
\]

Hence if $h(t, (\phi - \psi)(0)) \geq \epsilon$, then $h_0(\tau, x(t, \phi) - x(t, \psi)) \geq \delta, \quad t \geq \tau$, which implies $V(t, \phi, \psi) \geq \delta, \quad t \in \mathbb{R}_+$.

From **Lemma 1**, it follows that $V(t, \phi, \psi)$ is an $h$-normal functional.

We are going to show that $V(t, x(t_0, \phi), x(t_0, \psi))$ is nonincreasing.

For any $t_1 < t_2, \ t_0 \in \mathbb{R}_+$ and $\phi, \psi \in C$,

\[
V(t_1, x(t_0, \phi), x(t_0, \psi)) = \inf_{0 \leq \tau \leq T} h_0(\tau, x(t_1, \phi) - x(t_2, \psi))(0))
\]

\[
= \inf_{0 \leq \tau \leq T} h_0(\tau, x(t_1, \phi) - x(t_2, \psi)) \geq \inf_{0 \leq \tau \leq T} h_0(\tau, x(t_1, \phi) - x(t_2, \psi))
\]

\[
= \inf_{0 \leq \tau \leq T} h_0(\tau, x(t_2, \phi) - x(0, \psi))
\]

\[
= V(t_2, x(t_2, \phi), x(t_0, \psi)).
\]
Sufficiency. Suppose, by contradiction, that Eq. (1) is not \((h_0, h)\)-uniformly stable. Then there is an \(\epsilon_0\) such that for any \(\delta_n = \epsilon_0/n\), there are \(\phi_n, \psi_n \in C\) and \(t_n, \tau_n \in \mathbb{R}_+, t_n \geq \tau_n\), such that if \(h_0(\tau_n, \phi_n - \psi_n) < \delta_n\), then
\[
h(t_n, (x(\tau_n, \phi_n) - x(\tau_n, \psi_n))(t_n)) \geq \epsilon_0.
\]

We can choose a \(\tilde{t}_n \in [t_n, t_n]\) such that \(h(\tilde{t}_n, (x(\tau_n, \phi_n) - x(\tau_n, \psi_n))(\tilde{t}_n)) = \epsilon_0\).

Let \(D := \{\mu \in C : h(t, \mu(0)) \leq \epsilon_0\}\). Then \(x_{\tilde{t}_n}(\tau_n, \phi_n) - x_{\tilde{t}_n}(\tau_n, \psi_n) \in D\). Since \(V(t, \phi, \psi)\) is an \(h\)-normal functional, for the \(\epsilon_0\) given above, there exists a \(\delta_0 > 0\) such that \(V(t, x_{\tilde{t}_n}(\tau_n, \phi_n), x_{\tilde{t}_n}(\tau_n, \psi_n)) \geq \delta_0, t \in \mathbb{R}_+\).

By taking \(t = \tilde{t}_n\), we have that \(V(\tilde{t}_n, x_{\tilde{t}_n}(\tau_n, \phi_n), x_{\tilde{t}_n}(\tau_n, \psi_n)) \geq \delta_0\). Since \(V(t, \phi, \psi)\) is an \(h_0\)-controllable functional, from Lemma 2, for the above \(\delta_0\), there is an \(\eta_0 > 0\) such that if \(h_0(t_0, \phi - \psi) < \eta_0\), then \(V(t, \phi, \psi) < \frac{\delta_0}{2}\), for any \(t \in \mathbb{R}_+\). There is an \(n\) such that \(h_0(\tau_n, \phi_n - \psi_n) < \delta_n < \eta_0\). So, for any \(t \in \mathbb{R}_+\), we have \(V(t, \phi_n, \psi_n) < \frac{\delta_0}{2}\). By taking \(t = \tau_n\), it follows that \(V(\tau_n, \phi_n, \psi_n) < \frac{\delta_0}{2}\). Since \(V(t, x_{\tau_n}(\tau_n, \phi_n), x_{\tau_n}(\tau_n, \psi_n))\) is monotonically nonincreasing with respect to \(t\), we can write
\[
\delta_0 \leq V(\tilde{t}_n, x_{\tilde{t}_n}(\tau_n, \phi_n), x_{\tilde{t}_n}(\tau_n, \psi_n)) \leq V(\tau_n, x_{\tau_n}(\tau_n, \phi_n), x_{\tau_n}(\tau_n, \psi_n)) = V(\tau_n, \phi_n, \psi_n) < \frac{\delta_0}{2},
\]
which is a contradiction. Therefore Eq. (1) is \((h_0, h)\)-uniformly stable. \(\square\)

**Definition 5.** For a continuous functional \(V : \mathbb{R}_+ \times C \times C \rightarrow \mathbb{R}_+\), we define
\[
V'(t, \phi, \psi) := \lim_{k \rightarrow 0^+} \frac{1}{k} [V(t + k, x_{t+k}(t, \phi), x_{t+k}(t, \psi)) - V(t, \phi, \psi)].
\]

**Theorem 2.** Let \(h_0 \in \Gamma_0\) and \(h \in \Gamma\). If there is an \(h\)-normal and an \(h_0\)-controllable functional \(V(t, \phi, \psi)\) such that \(-V'(t, \phi, \psi)\) is an \(h\)-normal functional, then Eq. (1) is \((h_0, h)\)-uniformly asymptotically stable.

**Proof.** We note that since \(-V'(t, \phi, \psi)\) is an \(h\)-normal functional, there is a positive definite function \(W(x)\) so that \(V'(t, \phi, \psi) \leq -W(h(t, (\phi - \psi)(0)))\). So, \(V(t, \phi, \psi)\) is a nonincreasing functional.

By Theorem 1, Eq. (1) is \((h_0, h)\)-uniformly stable. Hence, there exists a
\[
\delta_0 > 0, \delta_0 = \delta_0(\epsilon) \text{ such that if } h_0(t, \phi - \psi) < \delta_0 \text{ then } h(t + \tau, (x(t, \phi) - x(t, \psi))(t + \tau)) < 1, \tau \geq 0.
\]

We must to show that for any \(\epsilon > 0\), there exists a \(T = T(\epsilon) > 0\) so that for any \(t \in \mathbb{R}_+\) and \(\phi, \psi \in C\), if \(\tau \geq T\) and \(h_0(t, \phi - \psi) < \delta_0\) then
\[
h(t + \tau, (x(t, \phi) - x(t, \psi))(t + \tau)) < \epsilon.
\]
We do it by contradiction. Suppose this is not true. Then there exists an \(\epsilon_0 > 0\) and for any sequence \(T_m \rightarrow \infty\) as \(m \rightarrow \infty\), there exist sequences \(\phi_m, \psi_m \in C\) and \(t_m \in \mathbb{R}_+\) so that \(h_0(t_m, \phi_m - \psi_m) < \delta_0\), but
\[
h(t_m + T_m, (x(t_m, \phi_m) - x(t_m, \psi_m))(t_m + T_m)) \geq \epsilon_0.
\]
Consider
\[
D := \{\mu \in C : h(t, \mu(0)) < 1\}.
\]
Then
\[
x_{t_m + T_m}(t_m, \phi_m) - x_{t_m + T_m}(t_m, \psi_m) \in D.
\]
Since $V(t, \phi, \psi)$ is an $h$-normal functional, Lemma 1 implies that there exists an $\alpha_0 > 0$ so that
\[ V(t_m + T_m, x_{t_m + T_m}(t_m, \phi_m), x_{t_m + T_m}(t_m, \psi_m)) \geq \alpha_0. \]

From the fact that $-V'(t, \phi, \psi)$ is an $h$-normal functional, for $\epsilon_0$ given above, Lemma 1 implies that there exists a $\gamma_0 > 0$ such that
\[ -V'(t_m + \tau, x_{t_m + \tau}(t_m, \phi_m), x_{t_m + \tau}(t_m, \psi_m)) \geq \gamma_0. \]

Integrating this expression on the interval $t_m \leq \tau \leq t_m + T_m$, we have
\[ V(t_m + T_m, x_{t_m + T_m}(t_m, \phi_m), x_{t_m + T_m}(t_m, \psi_m)) \leq -\gamma_0 T_m + V(t_m, \phi_m, \psi_m). \]

Since $h_0(t_m, \phi_m - \psi_m) < \delta_0$ and $V(t, \phi, \psi)$ is an $h_0$-controllable functional, it follows that $V(t_m, \phi_m, \psi_m)$ is bounded and therefore
\[ \lim_{m \to \infty} V(t_m + T_m, x_{t_m + T_m}(t_m, \phi_m), x_{t_m + T_m}(t_m, \psi_m)) = -\infty. \]

This contradicts the fact that $V$ is an $h$-normal functional. Then Eq. (1) is $(h_0, h)$-uniformly asymptotically stable. □

References