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J. Math. Anal. Appl. 318 (2006) 253–270

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On semilinear elliptic equations involving concave–convex nonlinearities and sign-changing weight function

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Received 16 March 2005

Available online 20 June 2005

Submitted by A. Cellina

Abstract

In this paper, we study the combined effect of concave and convex nonlinearities on the number of positive solutions for semilinear elliptic equations with a sign-changing weight function. With the help of the Nehari manifold, we prove that there are at least two positive solutions for Eq. $(E_{\lambda,f})$ in bounded domains.

Keywords: Semilinear elliptic equations; Nehari manifold; Concave–convex nonlinearities

1. Introduction

In this paper, we consider the multiplicity results of positive solutions of the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = u^p + \lambda f(x) u^q & \text{in } \Omega, \\ 0 \leqslant u \in H_0^1(\Omega), \end{cases} \qquad (E_{\lambda,f})$$

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[1] Partially supported by the National Science Council of Republic of China.

doi:10.1016/j.jmaa.2005.05.057

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < q < 1 < p < 2^*$ ($2^* = \frac{N+2}{N-2}$ if $N \geqslant 3$, $2^* = \infty$ if $N = 2$), $\lambda > 0$ and $f : \overline{\Omega} \to \mathbb{R}$ is a continuous function which change sign in $\overline{\Omega}$. Associated with Eq. $(E_{\lambda,f})$, we consider the energy functional J_λ, for each $u \in H_0^1(\Omega)$,

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx - \frac{\lambda}{q+1}\int_\Omega f(x)|u|^{q+1}\,dx.$$

It is well known that the solutions of Eq. $(E_{\lambda,f})$ are the critical points of the energy functional J_λ (see Rabinowitz [12]).

The fact that the number of positive solutions of Eq. $(E_{\lambda,f})$ is affected by the concave and convex nonlinearities has been the focus of a great deal of research in recent years. If the weight function $f(x) \equiv 1$, the authors Ambrosetti et al. [2] have investigated Eq. $(E_{\lambda,1})$. They found that there exists $\lambda_0 > 0$ such that Eq. $(E_{\lambda,1})$ admits at least two positive solution for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Actually, Adimurthy et al. [1], Damascelli et al. [7], Ouyang and Shi [11], and Tang [16] proved that there exists $\lambda_0 > 0$ such that Eq. $(E_{\lambda,1})$ in the unit ball $B^N(0;1)$ has exactly two positive solution for $\lambda \in (0, \lambda_0)$, has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$.

The purpose of this paper is to consider the multiplicity of positive solution of Eq. $(E_{\lambda,f})$ for a changing sign potential function $f(x)$. We prove that Eq. $(E_{\lambda,f})$ has at least two positive solutions for λ is sufficiently small.

Theorem 1. *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, Eq. $(E_{\lambda,f})$ has at least two positive solutions.*

Among the other interesting problems which are similar of Eq. $(E_{\lambda,f})$ for $q = 0$, Bahri [3], Bahri and Berestycki [4], and Struwe [13] have investigated the following equation:

$$\begin{cases} -\Delta u = |u|^{p-1}u + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{E_f}$$

where $f \in L^2(\Omega)$ and Ω is a bounded domain in $\mathbb{R}^N$. They found that Eq. (E_f) possesses infinitely many solutions. Furthermore, Cîrstea and Rădulescu [5], Cao and Zhou [6], and Ghergu and Rădulescu [10] have been investigated the analogue Eq. (E_f) in $\mathbb{R}^N$.

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we prove that Eq. $(E_{\lambda,f})$ has at least two positive solutions for λ sufficiently small.

2. Notations and preliminaries

Throughout this section, we denote by S the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^{p+1}(\Omega)$. Now, we consider the Nehari minimization problem: for $\lambda > 0$,

$$\alpha_\lambda(\Omega) = \inf\{J_\lambda(u) \mid u \in \mathbf{M}_\lambda(\Omega)\},$$

where $\mathbf{M}_\lambda(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} \mid \langle J'_\lambda(u), u\rangle = 0\}$. Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle = \|u\|_{H^1}^2 - \int_\Omega |u|^{p+1}\,dx - \lambda \int_\Omega f(x)|u|^{q+1}\,dx.$$

Then for $u \in \mathbf{M}_\lambda(\Omega)$,

$$\langle \psi'_\lambda(u), u\rangle = 2\|u\|_{H^1}^2 - (p+1)\int_\Omega |u|^{p+1}\,dx - (q+1)\lambda \int_\Omega f(x)|u|^{q+1}\,dx.$$

Similarly to the method used in Tarantello [14], we split $\mathbf{M}_\lambda(\Omega)$ into three parts:

$$\mathbf{M}_\lambda^+(\Omega) = \{u \in \mathbf{M}_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u\rangle > 0\},$$
$$\mathbf{M}_\lambda^0(\Omega) = \{u \in \mathbf{M}_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u\rangle = 0\},$$
$$\mathbf{M}_\lambda^-(\Omega) = \{u \in \mathbf{M}_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u\rangle < 0\}.$$

Then, we have the following results.

Lemma 2. *There exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$ we have $\mathbf{M}_\lambda^0(\Omega) = \emptyset$.*

Proof. We consider the following two cases.

Case (I). $u \in \mathbf{M}_\lambda(\Omega)$ and $\int_\Omega f(x)|u|^{q+1}\,dx = 0$. We have

$$\|u\|_{H^1}^2 - \int_\Omega |u|^{p+1}\,dx = 0.$$

Thus,

$$\langle \psi'_\lambda(u), u\rangle = 2\|u\|_{H^1}^2 - (p+1)\int_\Omega |u|^{p+1}\,dx = (1-p)\|u\|_{H^1}^2 < 0$$

and so $u \notin \mathbf{M}_\lambda^0(\Omega)$.

Case (II). $u \in \mathbf{M}_\lambda(\Omega)$ and $\int_\Omega f(x)|u|^{q+1}\,dx \neq 0$.

Suppose that $\mathbf{M}_\lambda^0(\Omega) \neq \emptyset$ for all $\lambda > 0$. If $u \in \mathbf{M}_\lambda^0(\Omega)$, then we have

$$\begin{aligned} 0 = \langle \psi'_\lambda(u), u\rangle &= 2\|u\|_{H^1}^2 - (p+1)\int_\Omega |u|^{p+1}\,dx - (q+1)\lambda \int_\Omega f(x)|u|^{q+1}\,dx \\ &= (1-q)\|u\|_{H^1}^2 - (p-q)\int_\Omega |u|^{p+1}\,dx. \end{aligned}$$

Thus,

$$\|u\|_{H^1}^2 = \frac{p-q}{1-q}\int_\Omega |u|^{p+1}\,dx \tag{1}$$

and

$$\lambda\int_\Omega f(x)|u|^{q+1}\,dx=\|u\|_{H^1}^2-\int_\Omega |u|^{p+1}\,dx=\frac{p-1}{1-q}\int_\Omega |u|^{p+1}\,dx. \tag{2}$$

Moreover,

$$\begin{aligned}\left(\frac{p-1}{p-q}\right)\|u\|_{H^1}^2&=\|u\|_{H^1}^2-\int_\Omega |u|^{p+1}\,dx=\lambda\int_\Omega f(x)|u|^{q+1}\,dx\\&\leqslant \lambda\|f\|_{L^{p^*}}\|u\|_{L^{p+1}}^{q+1}\leqslant \lambda\|f\|_{L^{p^*}}S^{q+1}\|u\|_{H^1}^{q+1},\end{aligned}$$

where $p^*=\frac{p+1}{p-q}$. This implies

$$\|u\|_{H^1}\leqslant\left[\lambda\left(\frac{p-q}{p-1}\right)\|f\|_{L^{p^*}}S^{q+1}\right]^{\frac{1}{1-q}}. \tag{3}$$

Let $I_\lambda:\mathbf{M}_\lambda(\Omega)\to\mathbb{R}$ be given by

$$I_\lambda(u)=K(p,q)\left(\frac{|u|_{H^1}^{2p}}{\int_\Omega |u|^{p+1}\,dx}\right)^{\frac{1}{p-1}}-\lambda\int_\Omega f(x)|u|^{q+1}\,dx,$$

where $K(p,q)=\left(\frac{1-q}{p-q}\right)^{\frac{p}{p-1}}\left(\frac{p-1}{1-q}\right)$. Then $I_\lambda(u)=0$ for all $u\in\mathbf{M}_\lambda^0(\Omega)$. Indeed, from (1) and (2) it follows that for $u\in\mathbf{M}_\lambda^0(\Omega)$ we have

$$\begin{aligned}I_\lambda(u)&=K(p,q)\left(\frac{\|u\|_{H^1}^{2p}}{\int_\Omega |u|^{p+1}\,dx}\right)^{\frac{1}{p-1}}-\lambda\int_\Omega f(x)|u|^{q+1}\,dx\\&=\left(\frac{1-q}{p-q}\right)^{\frac{p}{p-1}}\left(\frac{p-1}{1-q}\right)\left(\frac{\left(\frac{p-q}{1-q}\right)^p\left(\int_\Omega |u|^{p+1}\,dx\right)^p}{\int_\Omega |u|^{p+1}\,dx}\right)^{\frac{1}{p-1}}\\&\quad-\frac{p-1}{1-q}\int_\Omega |u|^{p+1}\,dx\\&=0.\end{aligned} \tag{4}$$

However, by (3), the Hölder and Sobolev inequality, for $u\in\mathbf{M}_\lambda^0(\Omega)$,

$$\begin{aligned}I_\lambda(u)&\geqslant K(p,q)\left(\frac{|u|_{H^1}^{2p}}{\int_\Omega |u|^{p+1}\,dx}\right)^{\frac{1}{p-1}}-\lambda\|f\|_{L^{p^*}}\|u\|_{L^{p+1}}^{q+1}\\&\geqslant \|u\|_{L^{p+1}}^{q+1}\left(K(p,q)\left(\frac{\|u\|_{H^1}^{2p}}{S^{q(p-1)+2p}\|u\|_{H^1}^{q(p-1)+2p}}\right)^{\frac{1}{p-1}}-\lambda\|f\|_{L^{p^*}}\right)\\&=\|u\|_{L^{p+1}}^{q+1}\left(K(p,q)\left(\frac{1}{S^{q(p-1)+2p}}\right)^{\frac{1}{p-1}}\frac{1}{\|u\|_{H^1}^q}-\lambda\|f\|_{L^{p^*}}\right)\end{aligned}$$

$$\geqslant \|u\|_{L^{p+1}}^{q+1}\left\{K(p,q)\left(\frac{1}{S^{q(p-1)+2p}}\right)^{\frac{1}{p-1}}\lambda^{\frac{-q}{1-q}}\left[\left(\frac{p-q}{p-1}\right)|f|_{L^{p^*}}S^{q+1}\right]^{\frac{-q}{1-q}} - \lambda\|f\|_{L^{p^*}}\right\}.$$

This implies that for λ sufficiently small we have $I_\lambda(u) > 0$ for all $u \in \mathbf{M}_\lambda^0(\Omega)$, this contradicts (4). Thus, we can conclude that there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, we have $\mathbf{M}_\lambda^0(\Omega) = \emptyset$. □

Lemma 3. *If $u \in \mathbf{M}_\lambda^+(\Omega)$, then $\int_\Omega f(x)|u|^{q+1}\,dx > 0$.*

Proof. We have

$$\|u\|_{H^1}^2 - \int_\Omega |u|^{p+1}\,dx - \lambda\int_\Omega f(x)|u|^{q+1}\,dx = 0$$

and

$$\|u\|_{H^1}^2 > \frac{p-q}{1-q}\int_\Omega |u|^{p+1}\,dx.$$

Thus,

$$\lambda\int_\Omega f(x)|u|^{q+1}\,dx = \|u\|_{H^1}^2 - \int_\Omega |u|^{p+1}\,dx > \frac{p-1}{1-q}\int_\Omega |u|^{p+1}\,dx > 0.$$

This completes the proof. □

By Lemma 2, for $\lambda \in (0, \lambda_1)$ we write $\mathbf{M}_\lambda(\Omega) = \mathbf{M}_\lambda^+(\Omega) \cup \mathbf{M}_\lambda^-(\Omega)$ and define

$$\alpha_\lambda^+(\Omega) = \inf_{u\in\mathbf{M}_\lambda^+(\Omega)} J_\lambda(u), \qquad \alpha_\lambda^-(\Omega) = \inf_{u\in\mathbf{M}_\lambda^-(\Omega)} J_\lambda(u).$$

The following lemma shows that the minimizers on $\mathbf{M}_\lambda(\Omega)$ are "usually" critical points for J_λ.

Lemma 4. *For $\lambda \in (0, \lambda_1)$. If u_0 is a local minimizer for J_λ on $\mathbf{M}_\lambda(\Omega)$, then $J_\lambda'(u_0) = 0$ in $H^{-1}(\Omega)$.*

Proof. If u_0 is a local minimizer for J_λ on $\mathbf{M}_\lambda(\Omega)$, then u_0 is a solution of the optimization problem

$$\text{minimize} \quad J_\lambda(u) \quad \text{subject to} \quad \psi_\lambda(u) = 0.$$

Hence, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that

$$J_\lambda'(u_0) = \theta\psi_\lambda'(u_0) \quad \text{in } H^{-1}(\Omega).$$

Thus,

$$\langle J_\lambda'(u_0), u_0\rangle_{H^1} = \theta\langle \psi_\lambda'(u_0), u_0\rangle_{H^1}. \tag{5}$$

Since $u_0 \in \mathbf{M}_\lambda(\Omega)$, we have $\|u_0\|_{H^1}^2 - \int_\Omega |u_0|^{p+1}\,dx - \lambda \int_\Omega f(x)|u_0|^{q+1}\,dx = 0$. Hence,

$$\langle \psi_\lambda'(u_0), u_0 \rangle_{H^1} = (1-q)\|u_0\|_{H^1}^2 - (p-q)\int_\Omega |u_0|^{p+1}\,dx.$$

Moreover, $\langle \psi_\lambda'(u_0), u_0 \rangle_{H^1} \neq 0$ and so by (5) $\theta = 0$. This completes the proof. □

For each $u \in H_0^1(\Omega)\backslash\{0\}$, we write

$$t_{\max} = \left(\frac{(1-q)\|u\|_{H^1}^2}{(p-q)\int_\Omega |u|^{p+1}\,dx} \right)^{\frac{1}{p-1}} > 0.$$

Then, we have the following lemma.

Lemma 5. *Let $p^* = \frac{p+1}{p-q}$ and $\lambda_2 = \left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}} S^{\frac{2(q-p)}{p-1}} \|f\|_{L^{p^*}}^{-1}$. Then for each $u \in H_0^1(\Omega)\backslash\{0\}$ and $\lambda \in (0, \lambda_2)$, we have*

- (i) *there is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^- u \in \mathbf{M}_\lambda^-(\Omega)$ and $J_\lambda(t^- u) = \max_{t \geqslant t_{\max}} J_\lambda(tu)$;*
- (ii) *$t^-(u)$ is a continuous function for nonzero u;*
- (iii) $\mathbf{M}_\lambda^-(\Omega) = \left\{ u \in H_0^1(\Omega)\backslash\{0\} \;\middle|\; \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) = 1 \right\}$;
- (iv) *if $\int_\Omega f(x)|u|^{q+1}\,dx > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+ u \in \mathbf{M}_\lambda^+(\Omega)$ and $J_\lambda(t^+ u) = \min_{0 \leqslant t \leqslant t^-} J_\lambda(tu)$.*

Proof. (i) Fix $u \in H_0^1(\Omega)\backslash\{0\}$. Let

$$s(t) = t^{1-q}\|u\|_{H^1}^2 - t^{p-q}\int_\Omega |u|^{p+1}\,dx \quad \text{for } t \geqslant 0.$$

We have $s(0) = 0$, $s(t) \to -\infty$ as $t \to \infty$, $s(t)$ is concave and achieves its maximum at $t_{\max}$. Moreover,

$$\begin{aligned}
&s(t_{\max}) \\
&\quad = \left(\frac{(1-q)\|u\|_{H^1}^2}{(p-q)\int_\Omega |u|^{p+1}\,dx} \right)^{\frac{1-q}{p-1}} \|u\|_{H^1}^2 \\
&\qquad - \left(\frac{(1-q)\|u\|_{H^1}^2}{(p-q)\int_\Omega |u|^{p+1}\,dx} \right)^{\frac{p-q}{p-1}} \int_\Omega |u|^{p+1}\,dx \\
&\quad = \|u\|_{H^1}^{q+1}\left[\left(\frac{(1-q)\|u\|_{H^1}^{p+1}}{(p-q)\int_\Omega |u|^{p+1}\,dx} \right)^{\frac{1-q}{p-1}} - \left(\frac{(1-q)\|u\|_{H^1}^{\frac{(p+1)(1-q)}{p-q}}}{(p-q)(\int_\Omega |u|^{p+1}\,dx)^{\frac{1-q}{p-q}}} \right)^{\frac{p-q}{p-1}} \right]
\end{aligned}$$

$$= \|u\|_{H^1}^{q+1}\left[\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}} - \left(\frac{1-q}{p-q}\right)^{\frac{p-q}{p-1}}\right]\left(\frac{\|u\|_{H^1}^{p+1}}{\int_\Omega |u|^{p+1}\,dx}\right)^{\frac{1-q}{p-1}}$$
$$\geqslant \|u\|_{H^1}^{q+1}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}}\left(\frac{1}{S^{p+1}}\right)^{\frac{1-q}{p-1}},$$

or

$$s(t_{\max}) \geqslant \|u\|_{H^1}^{q+1}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}}\left(\frac{1}{S^{p+1}}\right)^{\frac{1-q}{p-1}}. \tag{6}$$

Case (I). $\int_\Omega f(x)|u|^{q+1}\,dx \leqslant 0$.

There is a unique $t^- > t_{\max}$ such that $s(t^-) = \int_\Omega f(x)|u|^{q+1}\,dx$ and $s'(t^-) < 0$. Now,

$$(1-q)\|t^-u\|_{H^1}^2 - (p-q)\int_\Omega |t^-u|^{p+1}\,dx$$
$$= (t^-)^{2+q}\left[(1-q)(t^-)^{-q}\|u\|_{H^1}^2 - (p-q)(t^-)^{p-q-1}\int_\Omega |t^-u|^{p+1}\,dx\right]$$
$$= (t^-)^{2+q}s'(t^-) < 0,$$

and

$$\left\langle J_\lambda'(t^-u), t^-u\right\rangle$$
$$= (t^-)^2\|u\|_{H^1}^2 - (t^-)^{p+1}\int_\Omega |u|^{p+1}\,dx - (t^-)^{q+1}\lambda\int_\Omega f(x)|u|^{q+1}\,dx$$
$$= (t^-)^{q+1}\left[s(t^-) - \lambda\int_\Omega f(x)|u|^{q+1}\,dx\right] = 0.$$

Thus, $t^-u \in \mathbf{M}_\lambda^-(\Omega)$. Since for $t > t_{\max}$, we have

$$(1-q)\|tu\|_{H^1}^2 - (p-q)\int_\Omega |tu|^{p+1}\,dx < 0, \qquad \frac{d^2}{dt^2}J_\lambda(tu) < 0$$

and

$$\frac{d}{dt}J_\lambda(tu) = t\|u\|_{H^1}^2 - t^p\int_\Omega |u|^{p+1}\,dx - t^q\lambda\int_\Omega f(x)|u|^{q+1}\,dx = 0 \quad \text{for } t = t^-.$$

Therefore, $J_\lambda(t^-u) = \max_{t\geqslant t_{\max}} J_\lambda(tu)$.

Case (II). $\int_\Omega f(x)|u|^{q+1}\,dx > 0$.

By (6) and

$$s(0)=0<\lambda\int_{\Omega} f(x)|u|^{q+1}\,dx \leqslant \lambda\|f\|_{L^{p^*}}S^{q+1}\|u\|_{H^1}^{q+1}$$
$$<\|u\|_{H^1}^{q+1}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}}\left(\frac{1}{S^{p+1}}\right)^{\frac{1-q}{p-1}}\leqslant s(t_{\max})\quad \text{for } \lambda\in(0,\lambda_2),$$

there are unique t^+ and t^- such that $0<t^+<t_{\max}<t^-$,

$$s\big(t^+\big)=\lambda\int_{\Omega} f(x)|u|^{q+1}\,dx=s(t^-)$$

and

$$s'\big(t^+\big)>0>s'(t^-).$$

We have $t^+u\in \mathbf{M}_\lambda^+(\Omega)$, $t^-u\in \mathbf{M}_\lambda^-(\Omega)$, and $J_\lambda(t^-u)\geqslant J_\lambda(tu)\geqslant J_\lambda(t^+u)$ for each $t\in[t^+,t^-]$ and $J_\lambda(t^+u)\leqslant J_\lambda(tu)$ for each $t\in[0,t^+]$. Thus,

$$J_\lambda(t^-u)=\max_{t\geqslant t_{\max}} J_\lambda(tu),\qquad J_\lambda\big(t^+u\big)=\min_{0\leqslant t\leqslant t^-} J_\lambda(tu).$$

(ii) By the uniqueness of $t^-(u)$ and the external property of $t^-(u)$, we have that $t^-(u)$ is a continuous function of $u\neq 0$.

(iii) For $u\in \mathbf{M}_\lambda^-(\Omega)$, let $v=\frac{u}{\|u\|_{H^1}}$. By part (i), there is a unique $t^-(v)>0$ such that $t^-(v)v\in \mathbf{M}_\lambda^-(\Omega)$, that is $t^-\big(\frac{u}{\|u\|_{H^1}}\big)\frac{1}{\|u\|_{H^1}}u\in \mathbf{M}_\lambda^-(\Omega)$. Since $u\in \mathbf{M}_\lambda^-(\Omega)$, we have $t^-\big(\frac{u}{\|u\|_{H^1}}\big)\frac{1}{\|u\|_{H^1}}=1$, which implies

$$\mathbf{M}_\lambda^-(\Omega)\subset\left\{u\in H_0^1(\Omega)\backslash\{0\}\;\middle|\;\frac{1}{\|u\|_{H^1}}t^-\left(\frac{u}{\|u\|_{H^1}}\right)=1\right\}.$$

Conversely, let $u\in H_0^1(\Omega)\backslash\{0\}$ such that $\frac{1}{\|u\|_{H^1}}t^-\big(\frac{u}{\|u\|_{H^1}}\big)=1$. Then

$$t^-\left(\frac{u}{\|u\|_{H^1}}\right)\frac{u}{\|u\|_{H^1}}\in \mathbf{M}_\lambda^-(\Omega).$$

Thus,

$$\mathbf{M}_\lambda^-(\Omega)=\left\{u\in H_0^1(\Omega)\backslash\{0\}\;\middle|\;\frac{1}{\|u\|_{H^1}}t^-\left(\frac{u}{\|u\|_{H^1}}\right)=1\right\}.$$

(iv) By Case (II) of part (i). □

By $f:\Omega\to\mathbb{R}$ is a continuous function which change sign in Ω, we have $\Theta=\{x\in\Omega\mid f(x)>0\}$ is a open set in $\mathbb{R}^N$. Without loss of generality, we may assume that Θ is a domain in $\mathbb{R}^N$. Consider the following elliptic equation:

$$\begin{cases}-\Delta u=u^p & \text{in } \Theta,\\ 0\leqslant u\in H_0^1(\Theta).\end{cases}\tag{7}$$

Associated with Eq. (7), we consider the energy functional

$$K(u)=\frac{1}{2}\int_{\Theta}|\nabla u|^2\,dx-\frac{1}{p+1}\int_{\Theta}|u|^{p+1}\,dx$$

and the minimization problem

$$\beta_\lambda(\Theta) = \inf\{K_\lambda(u) \mid u \in \mathbf{N}_\lambda(\Theta)\},$$

where $\mathbf{N}_\lambda(\Theta) = \{u \in H_0^1(\Theta)\backslash\{0\} \mid \langle K'(u), u\rangle = 0\}$. It is known that Eq. (7) has a positive solution w_0 such that $K(w_0) = \beta_\lambda(\Theta) > 0$. Then we have the following results.

Lemma 6.

(i) *There exists* $t_\lambda > 0$ *such that*

$$\alpha_\lambda(\Omega) \leqslant \alpha_\lambda^+(\Omega) < -\frac{1-q}{q+1} t_\lambda^2 \beta_\lambda(\Theta) < 0;$$

(ii) J_λ *is coercive and bounded below on* $\mathbf{M}_\lambda(\Omega)$ *for all* $\lambda \in \left(0, \frac{p-1}{p-q}\right]$.

Proof. (i) Let w_0 be a positive solution of Eq. (7) such that $K(w_0) = \beta_\lambda(\Theta)$. Then

$$\int_\Omega f(x) w_0^{q+1}\,dx = \int_\Theta f(x) w_0^{q+1}\,dx > 0.$$

Set $t_\lambda = t^+(w_0)$ as defined by Lemma 5(iv). Hence $t_\lambda w_0 \in \mathbf{M}_\lambda^+(\Omega)$ and

$$\begin{aligned} J_\lambda(t_\lambda w_0) &= \frac{t_\lambda^2}{2}\|w_0\|_{H^1}^2 - \frac{t_\lambda^{p+1}}{p+1}\int_\Omega |w_0|^{p+1}\,dx - \frac{\lambda t_\lambda^{q+1}}{q+1}\int_\Omega f(x)|w_0|^{q+1}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{q+1}\right)\frac{t_\lambda^2}{2}\|w_0\|_{H^1}^2 + \left(\frac{1}{q+1} - \frac{1}{p+1}\right) t_\lambda^{p+1}\int_\Omega |w_0|^{p+1}\,dx \\ &< -\frac{1-q}{q+1} t_\lambda^2 \beta_\lambda(\Theta) < 0. \end{aligned}$$

This yields

$$\alpha_\lambda(\Omega) \leqslant \alpha_\lambda^+(\Omega) < -\frac{1-q}{q+1} t_\lambda^2 \beta_\lambda(\Theta) < 0.$$

(ii) For $u \in \mathbf{M}_\lambda(\Omega)$, we have $\|u\|_{H^1}^2 = \int_\Omega |u|^{p+1}\,dx + \lambda\int_\Omega f(x)|u|^{q+1}\,dx$. Then by the Hölder and Young inequality,

$$\begin{aligned} J_\lambda(u) &= \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{(p+1)(q+1)}\right)\int_\Omega f(x)|u|^{q+1}\,dx \\ &\geqslant \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{(p+1)(q+1)}\right)\|f\|_{L^{p^*}} S^{q+1}\|u\|_{H^1}^{q+1} \\ &\geqslant \left[\frac{p-1}{2(p+1)} - \lambda\left(\frac{p-q}{2(p+1)}\right)\right]\|u\|_{H^1}^2 \\ &\quad - \lambda\left(\frac{(p-q)(1-q)}{2(p+1)(q+1)}\right)\left(\|f\|_{L^{p^*}} S^{q+1}\right)^{\frac{2}{1-q}} \end{aligned}$$

$$= \frac{1}{2(p+1)}\big[(p-1)-\lambda(p-q)\big]\|u\|_{H^1}^2 - \lambda\left(\frac{(p-q)(1-q)}{2(p+1)(q+1)}\right)\left(\|f\|_{L^{p^*}}S^{q+1}\right)^{\frac{2}{1-q}}.$$

Thus, J_λ is coercive on $\mathbf{M}_\lambda(\Omega)$ and

$$J_\lambda(u) \geqslant -\lambda\left(\frac{(p-q)(1-q)}{2(p+1)(q+1)}\right)\left(\|f\|_{L^{p^*}}S^{q+1}\right)^{\frac{2}{1-q}}$$

for all $\lambda \in \left(0, \frac{p-1}{p-q}\right]$. □

3. Proof of Theorem 1

First, we will use the idea of Tarantello [14] to get the following results.

Lemma 7. *For each $u \in \mathbf{M}_\lambda(\Omega)$, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset H_0^1(\Omega) \to \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u-v) \in \mathbf{M}_\lambda(\Omega)$ and*

$$\left\langle \xi'(0), v\right\rangle = \frac{2\int_\Omega \nabla u \nabla v\,dx - (p+1)\int_\Omega |u|^{p-1}uv\,dx - (q+1)\lambda\int_\Omega f|u|^{q-1}uv\,dx}{(1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int_\Omega |u|^{p+1}\,dx} \tag{8}$$

for all $v \in H_0^1(\Omega)$.

Proof. For $u \in \mathbf{M}_\lambda(\Omega)$, define a function $F : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} F_u(\xi, w) &= \left\langle J_\lambda'\big(\xi(u-w)\big), \xi(u-w)\right\rangle \\ &= \xi^2 \int_\Omega \left|\nabla(u-w)\right|^2 dx - \xi^{p+1}\int_\Omega |u-w|^{p+1}\,dx \\ &\quad - \xi^{q+1}\lambda\int_\Omega f(x)|u-w|^{q+1}\,dx. \end{aligned}$$

Then $F_u(1,0) = \langle J_\lambda'(u), u\rangle = 0$ and

$$\begin{aligned} \frac{d}{dt}F_u(1,0) &= 2\int_\Omega |\nabla u|^2\,dx - (p+1)\int_\Omega |u|^{p+1}\,dx - (q+1)\lambda\int_\Omega f(x)|u|^{q+1}\,dx \\ &= (1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int_\Omega |u|^{p+1}\,dx \neq 0. \end{aligned}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset H^1(\mathbb{R}^N) \to \mathbb{R}$ such that $\xi(0) = 1$,

$$\left\langle \xi'(0), v\right\rangle = \frac{2\int_\Omega \nabla u \nabla v\,dx - (p+1)\int_\Omega |u|^{p-1}uv\,dx - (q+1)\lambda\int_\Omega f|u|^{q-1}uv\,dx}{(1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int_\Omega |u|^{p+1}\,dx}$$

and

$$F_u\big(\xi(v), v\big) = 0 \quad \text{for all } v \in B(0;\epsilon),$$

which is equivalent to

$$\big\langle J'_\lambda\big(\xi(v)(u-v)\big), \xi(v)(u-v)\big\rangle = 0 \quad \text{for all } v \in B(0;\epsilon),$$

that is $\xi(v)(u-v) \in \mathbf{M}_\lambda(\Omega)$. □

Lemma 8. *For each $u \in \mathbf{M}_\lambda^-(\Omega)$, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset H_0^1(\Omega) \to \mathbb{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u-v) \in \mathbf{M}_\lambda^-(\Omega)$ and*

$$\big\langle (\xi^-)'(0), v\big\rangle = \frac{2\int_\Omega \nabla u \nabla v\,dx - (p+1)\int_\Omega |u|^{p-1}uv\,dx - (q+1)\lambda\int_\Omega f|u|^{q-1}uv\,dx}{(1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int_\Omega |u|^{p+1}\,dx} \tag{9}$$

for all $v \in H_0^1(\Omega)$.

Proof. Similar to the argument in Lemma 7, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset H^1(\mathbb{R}^N) \to \mathbb{R}$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in \mathbf{M}_\lambda(\Omega)$ for all $v \in B(0;\epsilon)$. Since

$$\big\langle \psi'_\lambda(u), u\big\rangle = (1-q)\|u\|^2_{H^1} - (p-q)\int_\Omega |u|^{p+1}\,dx < 0.$$

Thus, by the continuity of the functions ψ'_λ and ξ^-, we have

$$\begin{aligned}&\big\langle \psi'_\lambda\big(\xi^-(v)(u-v)\big), \xi^-(v)(u-v)\big\rangle\\ &\quad = (1-q)\big\|\xi^-(v)(u-v)\big\|^2_{H^1} - (p-q)\int_\Omega \big|\xi^-(v)(u-v)\big|^{p+1}\,dx < 0\end{aligned}$$

if ϵ sufficiently small, this implies that $\xi^-(v)(u-v) \in \mathbf{M}_\lambda^-(\Omega)$. □

Proposition 9. *Let $\lambda_0 = \min\{\lambda_1, \lambda_2, \frac{p-1}{p-q}\}$, then for $\lambda \in (0, \lambda_0)$,*

(i) *there exists a minimizing sequence $\{u_n\} \subset \mathbf{M}_\lambda(\Omega)$ such that*

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1),$$
$$J'_\lambda(u_n) = o(1) \quad \text{in } H^{-1}(\Omega);$$

(ii) *there exists a minimizing sequence $\{u_n\} \subset \mathbf{M}_\lambda^-(\Omega)$ such that*

$$J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1),$$
$$J'_\lambda(u_n) = o(1) \quad \text{in } H^{-1}(\Omega).$$

Proof. (i) By Lemma 6(ii) and the Ekeland variational principle [9], there exists a minimizing sequence $\{u_n\} \subset \mathbf{M}_\lambda(\Omega)$ such that

$$J_\lambda(u_n) < \alpha_\lambda(\Omega) + \frac{1}{n} \tag{10}$$

and

$$J_\lambda(u_n) < J_\lambda(w) + \frac{1}{n}\|w - u_n\|_{H^1} \quad \text{for each } w \in \mathbf{M}_\lambda(\Omega). \tag{11}$$

By taking n large, from Lemma 6(i), we have

$$\begin{aligned} J_\lambda(u_n) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|_{H^1}^2 - \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\lambda \int_\Omega f(x)|u_n|^{q+1}\,dx \\ &< \alpha_\lambda(\Omega) + \frac{1}{n} < -\frac{1-q}{q+1} t_\lambda^2 \beta_\lambda(\Theta). \end{aligned} \tag{12}$$

This implies

$$\|f\|_{L^{p^*}} S^{q+1} \|u_n\|_{H^1}^{q+1} \geqslant \int_\Omega f(x)|u_n|^{q+1}\,dx > \frac{(p+1)(1-q)}{\lambda(p-q)} t_\lambda^2 \beta_\lambda(\Omega) > 0. \tag{13}$$

Consequently $u_n \neq 0$ and putting together (12), (13) and the Hölder inequality, we obtain

$$\|u_n\|_{H^1} > \left[\frac{(p+1)(1-q)}{\lambda(p-q)} t_\lambda^2 \beta_\lambda(\Theta) S^{-(q+1)} \|f\|_{L^{p^*}}^{-1}\right]^{\frac{1}{q+1}} \tag{14}$$

and

$$\|u_n\|_{H^1} < \left[\frac{2(p-q)}{(p-1)(q+1)} \|f\|_{L^{p^*}} S^{q+1}\right]^{\frac{1}{1-q}}. \tag{15}$$

Now, we will show that

$$\left\|J_\lambda'(u_n)\right\|_{H^{-1}} \to 0 \quad \text{as } n \to \infty.$$

Applying Lemma 7 with u_n to obtain the functions $\xi_n : B(0; \epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathbf{M}_\lambda(\Omega)$. Choose $0 < \rho < \epsilon_n$. Let $u \in H_0^1(\Omega)$ with $u \not\equiv 0$ and let $w_\rho = \frac{\rho u}{\|u\|_{H^1}}$. We set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$. Since $\eta_\rho \in \mathbf{M}_\lambda(\Omega)$, we deduce from (11) that

$$J_\lambda(\eta_\rho) - J_\lambda(u_n) \geqslant -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1}$$

and by the mean value theorem, we have

$$\left\langle J_\lambda'(u_n), \eta_\rho - u_n\right\rangle + o\left(\|\eta_\rho - u_n\|_{H^1}\right) \geqslant -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1}.$$

Thus,

$$\begin{aligned} &\left\langle J_\lambda'(u_n), -w_\rho\right\rangle + \left(\xi_n(w_\rho) - 1\right)\left\langle J_\lambda'(u_n), (u_n - w_\rho)\right\rangle \\ &\quad \geqslant -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1} + o\left(\|\eta_\rho - u_n\|_{H^1}\right). \end{aligned} \tag{16}$$

From $\xi_n(w_\rho)(u_n - w_\rho) \in \mathbf{M}_\lambda(\Omega)$ and (16) it follows that

$$-\rho\left\langle J_\lambda'(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle + \big(\xi_n(w_\rho) - 1\big)\big\langle J_\lambda'(u_n) - J_\lambda'(\eta_\rho), (u_n - w_\rho)\big\rangle \geqslant -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1} + o\big(\|\eta_\rho - u_n\|_{H^1}\big).$$

Thus,

$$\left\langle J_\lambda'(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle \leqslant \frac{\|\eta_\rho - u_n\|_{H^1}}{n\rho} + \frac{o(\|\eta_\rho - u_n\|_{H^1})}{\rho} + \frac{(\xi_n(w_\rho) - 1)}{\rho}\big\langle J_\lambda'(u_n) - J_\lambda'(\eta_\rho), (u_n - w_\rho)\big\rangle. \tag{17}$$

Since

$$\|\eta_\rho - u_n\|_{H^1} \leqslant \rho\big|\xi_n(w_\rho)\big| + \big|\xi_n(w_\rho) - 1\big|\|u_n\|_{H^1}$$

and

$$\lim_{\rho\to 0}\frac{|\xi_n(w_\rho) - 1|}{\rho} \leqslant \big\|\xi_n'(0)\big\|.$$

If we let $\rho \to 0$ in (17) for a fixed n, then by (15) we can find a constant $C > 0$, independent of ρ, such that

$$\left\langle J_\lambda'(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle \leqslant \frac{C}{n}\big(1 + \big\|\xi_n'(0)\big\|\big).$$

We are done once we show that $\|\xi_n'(0)\|$ is uniformly bounded in n. By (8), (15) and the Hölder inequality, we have

$$\big\langle \xi_n'(0), v\big\rangle \leqslant \frac{b\|v\|_{H^1}}{|(1-q)\int_\Omega |\nabla u_n|^2\,dx - (p-q)\int_\Omega |u_n|^{p+1}\,dx|} \quad \text{for some } b > 0.$$

We only need to show that

$$\left|(1-q)\int_\Omega |\nabla u_n|^2\,dx - (p-q)\int_\Omega |u_n|^{p+1}\,dx\right| > c \tag{18}$$

for some $c > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$ such that

$$(1-q)\int_\Omega |\nabla u_n|^2\,dx - (p-q)\int_\Omega |u_n|^{p+1}\,dx = o(1). \tag{19}$$

Combining (19) with (14), we can find a suitable constant $d > 0$ such that

$$\int_\Omega |u_n|^{p+1}\,dx \geqslant d \quad \text{for } n \text{ sufficiently large.} \tag{20}$$

In addition (19), and the fact that $u_n \in \mathbf{M}_\lambda(\Omega)$ also give

$$\lambda \int_\Omega f(x)|u_n|^{q+1}\,dx = \|u_n\|_{H^1}^2 - \int_\Omega |u_n|^{p+1}\,dx = \frac{p-1}{1-q}\int_\Omega |u_n|^{p+1}\,dx + o(1)$$

and

$$\|u_n\|_{H^1} \leqslant \left[\lambda\left(\frac{p-q}{p-1}\right)\|f\|_{L^{p^*}} S^{q+1}\right]^{\frac{1}{1-q}} + o(1). \tag{21}$$

This implies

$$\begin{aligned} I_\lambda(u) &= K(p,q)\left(\frac{\|u_n\|_{H^1}^{2p}}{\int_\Omega |u_n|^{p+1}\,dx}\right)^{\frac{1}{p-1}} - \lambda\int_\Omega f(x)|u_n|^{q+1}\,dx \\ &= \left(\frac{1-q}{p-q}\right)^{\frac{p}{p-1}}\left(\frac{p-1}{1-q}\right)\left(\frac{\left(\frac{p-q}{1-q}\right)^p\left(\int_\Omega |u_n|^{p+1}\,dx\right)^p}{\int_\Omega |u_n|^{p+1}\,dx}\right)^{\frac{1}{p-1}} \\ &\quad - \frac{p-1}{1-q}\int_\Omega |u_n|^{p+1}\,dx \\ &= o(1). \end{aligned} \tag{22}$$

However, by (20), (21) and $\lambda \in (0, \lambda_0)$,

$$\begin{aligned} I_\lambda(u) &\geqslant K(p,q)\left(\frac{\|u_n\|_{H^1}^{2p}}{\int_\Omega |u_n|^{p+1}\,dx}\right)^{\frac{1}{p-1}} - \lambda\|f\|_{L^{p^*}}\|u_n\|_{L^{p+1}}^{q+1} \\ &\geqslant \|u_n\|_{L^{p+1}}^{q+1}\left(K(p,q)\left(\frac{\|u_n\|_{H^1}^{2p}}{S^{q(p-1)+2p}\|u_n\|_{H^1}^{q(p-1)+2p}}\right)^{\frac{1}{p-1}} - \lambda\|f\|_{L^{p^*}}\right) \\ &\geqslant \|u_n\|_{L^{p+1}}^{q+1}\left\{K(p,q)\left(\frac{1}{S^{q(p-1)+2p}}\right)^{\frac{1}{p-1}}\lambda^{\frac{-q}{1-q}}\left[\left(\frac{p-q}{p-1}\right)\|f\|_{L^{p^*}}S^{q+1}\right]^{\frac{-q}{1-q}} \right. \\ &\quad \left. - \lambda\|f\|_{L^{p^*}}\right\}, \end{aligned}$$

this contradicts (22). We get

$$\left\langle J_\lambda'(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle \leqslant \frac{C}{n}.$$

This completes the proof of (i).

(ii) Similarly, by using Lemma 8, we can prove (ii). We will omit the details here. □

Now, we establish the existence of a local minimum for J_λ on $\mathbf{M}_\lambda^+(\Omega)$.

Theorem 10. *Let $\lambda_0 > 0$ as in Proposition 9, then for $\lambda \in (0, \lambda_0)$ the functional J_λ has a minimizer u_0^+ in $\mathbf{M}_\lambda^+(\Omega)$ and it satisfies*

(i) $J_\lambda(u_0^+) = \alpha_\lambda(\Omega) = \alpha_\lambda^+(\Omega)$;
(ii) u_0^+ *is a positive solution of Eq.* $(E_{\lambda,f})$;
(iii) $J_\lambda(u_0^+) \to 0$ *as* $\lambda \to 0$.

Proof. Let $\{u_n\} \subset \mathbf{M}_\lambda(\Omega)$ be a minimizing sequence for J_λ on $\mathbf{M}_\lambda(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1) \quad \text{and} \quad J'_\lambda(u_n) = o(1) \quad \text{in } H^{-1}(\Omega).$$

Then by Lemma 6 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^+ \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u_0^+ \quad \text{weakly in } H_0^1(\Omega),$$
$$u_n \to u_0^+ \quad \text{strongly in } L^{p+1}(\Omega)$$

and

$$u_n \to u_0^+ \quad \text{strongly in } L^{q+1}(\Omega). \tag{23}$$

First, we claim that $\int_\Omega f(x)|u_0^+|^{q+1}\,dx \neq 0$. If not, by (23) we can conclude that

$$\int_\Omega f(x)\left|u_0^+\right|^{q+1} dx = 0$$

and

$$\int_\Omega f(x)|u_n|^{q+1}\,dx \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega |u_n|^{p+1}\,dx + o(1)$$

and

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - \frac{1}{p+1}\int_\Omega |u_n|^{p+1}\,dx - \frac{\lambda}{q+1}\int_\Omega f(x)|u_n|^{q+1}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |u_n|^{p+1}\,dx + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \left|u_0^+\right|^{p+1} dx \quad \text{as } n \to \infty, \end{aligned}$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda(\Omega) < 0$ as $n \to \infty$. In particular, $u_0^+ \in \mathbf{M}_\lambda(\Omega)$ is a nonzero solution of Eq. $(E_{\lambda,f})$ and $J_\lambda(u_0^+) \geqslant \alpha_\lambda(\Omega)$. We now prove that $u_n \to u_0^+$ strongly in $H_0^1(\Omega)$. Supposing the contrary, then $\|u_0^+\|_{H^1} < \liminf_{n\to\infty} \|u_n\|_{H^1}$ and so

$$\|u_0^+\|_{H^1}^2 - \int_\Omega |u_0^+|^{p+1}\,dx - \lambda \int_\Omega f(x)|u_0^+|^{q+1}\,dx$$
$$< \liminf_{n\to\infty}\left(\|u_n\|_{H^1}^2 - \int_\Omega |u_n|^{p+1}\,dx - \lambda \int_\Omega f(x)|u_n|^{q+1}\,dx\right) = 0,$$

this contradicts $u_0^+ \in \mathbf{M}_\lambda(\Omega)$. Hence $u_n \to u_0^+$ strongly in $H_0^1(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^+) = \alpha_\lambda(\Omega) \quad \text{as } n \to \infty.$$

Moreover, we have $u_0^+ \in \mathbf{M}_\lambda^+(\Omega)$. In fact, if $u_0^+ \in \mathbf{M}_\lambda^-(\Omega)$, by Lemma 5, there are unique t_0^+ and t_0^- such that $t_0^+ u_0^+ \in \mathbf{M}_\lambda^+(\Omega)$ and $t_0^- u_0^+ \in \mathbf{M}_\lambda^-(\Omega)$, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J_\lambda(t_0^+ u_0^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J_\lambda(t_0^+ u_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \leqslant t_0^-$ such that $J_\lambda(t_0^+ u_0^+) < J_\lambda(\bar{t} u_0^+)$. By Lemma 5,

$$J_\lambda(t_0^+ u_0^+) < J_\lambda(\bar{t} u_0^+) \leqslant J_\lambda(t_0^- u_0^+) = J_\lambda(u_0^+),$$

which is a contradiction. Since $J_\lambda(u_0^+) = J_\lambda(|u_0^+|)$ and $|u_0^+| \in \mathbf{M}_\lambda^+(\Omega)$, by Lemma 4 we may assume that u_0^+ is nonnegative solution. By Drábek et al. [8, Lemma 2.1], we have $u_0^+ \in L^\infty(\Omega)$. Then we can apply the Harnack inequality due to Trudinger [15] in order to get that u_0^+ is positive in Ω. Moreover, by Lemma 6,

$$0 > J_\lambda(u_0^+) \geqslant -\lambda\left(\frac{(p-q)(1-q)}{2(p+1)(q+1)}\right)\left(\|f\|_{L^{p^*}} S^{q+1}\right)^{\frac{2}{1-q}}.$$

We obtain $J_\lambda(u_0^+) \to 0$ as $\lambda \to 0$. □

Next, we establish the existence of a local minimum for J_λ on $\mathbf{M}_\lambda^-(\Omega)$.

Theorem 11. *Let $\lambda_0 > 0$ as in Proposition 9, then for $\lambda \in (0, \lambda_0)$ the functional J_λ has a minimizer u_0^- in $\mathbf{M}_\lambda^-(\Omega)$ and it satisfies*

(i) $J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega)$;
(ii) *u_0^- is a positive solution of Eq. $(E_{\lambda,f})$.*

Proof. By Proposition 9(ii), there exists a minimizing sequence $\{u_n\}$ for J_λ on $\mathbf{M}_\lambda^-(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1) \quad \text{and} \quad J_\lambda'(u_n) = o(1) \quad \text{in } H^{-1}(\Omega).$$

By Lemma 6 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^- \in \mathbf{M}_\lambda^-(\Omega)$ is a nonzero solution of Eq. $(E_{\lambda,f})$ such that

$$u_n \rightharpoonup u_0^- \quad \text{weakly in } H_0^1(\Omega),$$
$$u_n \to u_0^- \quad \text{strongly in } L^{p+1}(\Omega)$$

and

$$u_n \to u_0^- \quad \text{strongly in } L^{q+1}(\Omega).$$

We now prove that $u_n \to u_0^-$ strongly in $H_0^1(\Omega)$. Suppose otherwise, then $\|u_0^-\|_{H^1} < \liminf_{n\to\infty} \|u_n\|_{H^1}$ and so

$$\begin{aligned}
&\|u_0^-\|_{H^1}^2 - \int_\Omega |u_0^-|^{p+1}\,dx - \lambda \int_\Omega f(x)|u_0^-|^{q+1}\,dx \\
&\quad < \liminf_{n\to\infty}\left(\|u_n\|_{H^1}^2 - \int_\Omega |u_n|^{p+1}\,dx - \lambda \int_\Omega f(x)|u_n|^{q+1}\,dx \right) = 0.
\end{aligned}$$

This contradicts $u_0^- \in \mathbf{M}_\lambda^-(\Omega)$. Hence $u_n \to u_0^-$ strongly in $H_0^1(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega) \quad \text{as } n \to \infty.$$

Since $J_\lambda(u_0^-) = J_\lambda(|u_0^-|)$ and $|u_0^-| \in \mathbf{M}_\lambda^-(\Omega)$ by Lemma 4 we may assume that u_0^- is nonnegative solution. By Drábek et al. [8, Lemma 2.1], we have $u_0^- \in L^\infty(\Omega)$. Then we can apply the Harnack inequality due to Trudinger [15] in order to get that u_0^- is positive in Ω. □

Now, we complete the proof of Theorem 1.

By Theorems 10, 11, for Eq. $(E_{\lambda,f})$ there exist two positive solutions u_0^+ and u_0^- such that $u_0^+ \in \mathbf{M}_\lambda^+(\Omega)$, $u_0^- \in \mathbf{M}_\lambda^-(\Omega)$. Since $\mathbf{M}_\lambda^+(\Omega) \cap \mathbf{M}_\lambda^-(\Omega) = \emptyset$, this implies that u_0^+ and u_0^- are different.

Acknowledgment

The author is grateful for the referee's valuable suggestions and helps.

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