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In Memoriam

The Work of Lothar Collatz in Approximation Theory

G. MEINARDUS, G. NÜRNBERGER, TH. RIESSINGER, AND G. WALZ

Fakultät für Mathematik und Informatik, Universität Mannheim, W-6800 Mannheim 1, Germany

Professor Collatz was one of the pioneers in the development of numerical analysis in this century and did fundamental work in all branches of this field, such as differential equations, integral equations, eigenvalue problems, bifurcation problems, functional analytic methods, optimization, and approximation theory. His great talent coupled with an enormous energy made him an outstanding figure in applied mathematics with a far-reaching and lasting influence.

The aim of this article is to describe some of his basic contributions and ideas in approximation theory including applications to partial differential equations. While the total number of his papers is about 230, a selection of 82 publications on approximation can be found in the references.

There are two striking characteristics in the work of Collatz. He was convinced that mathematics should be motivated by the study of real world phenomena, and that it is an important aim to develop methods which can be used to explain such phenomena. Moreover, efficient algorithms should be developed for computing approximate solutions of the problems that arise. All his life he never wearied of fighting for this conviction. He fulfilled this task by attacking a wide range of problems from various domains and by describing basic principles and ideas which have strongly stimulated research in numerical analysis.

Collatz published his first paper on classical approximation theory in 1938 jointly with W. Quade [1]. The authors investigate the problem of computing the Fourier coefficients of a given periodic function f on $[0, 2\pi]$, where the asymptotic behavior of these coefficients, depending on smoothness properties of f, is included in the approximation process. It turns out that under some simple assumptions on the process there exist so-called attenuation factors, independent of f, which may improve the accuracy of the approximation considerably. These factors are constructed

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a priori by interpolation at the knots $x_\nu = 2\pi\nu/k$, $\nu = 0, ..., k$, with functions from

$$S = \{s \in C^{m-1}[0, 2\pi] : s|_{[x_\mu, x_{\mu+1}]} \in \Pi_m, \mu = 0, ..., k-1, \\ s^{(\nu)}(0) = s^{(\nu)}(2\pi), \nu = 0, ..., m-1\},$$

where Π_m denotes the space of polynomials of degree m. Therefore, this paper is the first place in which periodic polynomial spline functions belonging to an equidistant set of knots are used. The full procedure for interpolation of the knots is developed, provided m is an odd integer, including the discussion of the Euler–Frobenius polynomials. In fact, Collatz and Quade are the inventors of the periodic splines.

Later in 1956, still before the enormous development of approximation theory in the sixties, Collatz began to study problems of best approximation [4]. Again, his aim was of a constructive nature, namely to develop criteria which in practice allow the estimation of the error of a given approximation. He examines best approximation of functions from $C(T)$ —the space of continuous real-valued functions on a compact subset T of $\mathbb{R}^n$—by elements of a finite-dimensional subspace V of $C(T)$ in the uniform norm $\|f\| = \sup_{t \in T} |f(t)|$. In his general approach, he considers two disjoint subsets T_1 and T_2 of T which satisfy the assumption that there does not exist a function $v \in V$ with $v(t) > 0$, $t \in T_1$, and $v(t) < 0$, $t \in T_2$. For a function $f \in C(T)$ and an approximation $v_0 \in V$ such that $f(t) - v_0(t) > 0$, $t \in T_1$, and $f(t) - v_0(t) < 0$, $t \in T_2$, he obtains the error bounds

$$I = \inf_{t \in T_1 \cup T_2} |f(t) - v_0(t)| \leqslant d(f, V) \leqslant \|f - v_0\|$$

for the minimal deviation $d(f, V) = \inf_{v \in V} \|f - v\|$. This shows that the approximation v_0 is nearly optimal if the computable values I and $\|f - v_0\|$ are close.

In addition, Collatz describes a reduction method for verifying the above assumption in the case of finite sets T_1 and T_2 which is similar to the elimination method of C. F. Gauss. He applies this algorithm to multivariate approximation by linear and quadratic polynomials, i.e., to the cases

$$V = \Pi_1(x_1, ..., x_n) = \text{span}\{1, x_1, ..., x_n\}$$

and

$$V = \Pi_2(x_1, x_2) = \text{span}\{x_1^i x_2^j : 0 \leqslant i, j, i + j \leqslant 2\}.$$

Without knowing A. N. Kolmogorov's paper on approximation, he also derives a characterization of best approximations $v_f \in V$ of f (i.e., $\|f - v_f\| = d(f, V)$) by the condition

$$\min_{t \in E(f - v_f)} (f(t) - v_f(t))\, v(t) \leqslant 0, \qquad v \in V,$$

where $E(f-v_f)$ denotes the set of extremal points of $f-v_f$. By using this result, Collatz shows that differentiable functions in two variables have a unique best approximation from $V=\Pi_1(x_1, x_2)$ for strictly convex domains T, although by Haar's theorem global unicity fails.

Even in this early paper, he goes a step further by applying his methods to partial differential equations and obtains concrete error bounds for approximate solutions with the aid of the maximum principle.

Later in 1965—approximation theory was rapidly growing—Collatz adapted his concept to nonlinear approximation [12]. For this reason, he introduced the notation of H-sets for an arbitrary class of approximating functions V in $C(T)$.

A set $\tilde{T}$ in T is called an H-set if it is the union of nonempty disjoint subsets T_1 and T_2 such that there do not exist $v_1, v_2 \in V$ with $v_1(t)-v_2(t)>0$, $t\in T_1$, and $v_1(t)-v_2(t)<0$, $t \in T_2$. Similarly to the linear case, for $f\in C(T)$ and $v_0 \in V$ with the property $f(t)-v_0(t)>0$, $t\in T_1$, and $f(t)-v_0(t)<0$, $t\in T_2$, he obtains the inclusions

$$\inf_{t\in T_1 \cup T_2} |f(t)-v_0(t)| \leqslant d(f, V) \leqslant \|f-v_0\|.$$

Moreover, Collatz develops a procedure for verifying the H-set property and constructs this type of set for bivariate approximations of the form

$$v(x_1, x_2)=\frac{\sum_{\mu=1}^{p} a_\mu e^{\alpha_\mu x_1+\beta_\mu x_2}}{\sum_{\nu=1}^{q} b_\nu e^{\gamma_\nu x_1+\delta_\nu x_2}}$$

with variable exponents, and for multivariate rational approximations which are quotients of polynomials from

$$\Pi_m(x_1, ..., x_n)=\operatorname{span}\left\{x_1^{j_1}\cdots x_n^{j_n}: 0\leqslant j_1, ..., j_n, \sum_{i=1}^{n} j_i \leqslant m\right\}.$$

He also applies this approach to multivariate segment approximation [11].

The importance of multivariate approximation as well as the difficulties in solving the corresponding numerical problems was often emphasized by Collatz (see e.g., [58]). In his opinion, the development of a general theory which can be applied to concrete problems seems not to be possible and, therefore, special function classes should be investigated.

In most of this papers on approximation problems, Collatz goes beyond the classical theory by applying his methods to partial differential equations.

The first article on this subject was published in 1952, when he introduced the concept of monotonicity [2]. In the literature, most of the results on error analysis are of a qualitative nature. These estimates guarantee the existence of a constant $K > 0$ such that the error is less than $K \cdot h^q$, where the step size h and the exponent q are known. However, in general, no information about the size of K is available.

The aim of Collatz is to obtain concrete error bounds. One method is based on operators $L: U \to W$ (U, W being real function spaces) of *monotonic type*; i.e., $L(u_1) \leqslant L(u_2)$ implies $u_1 \leqslant u_2$ for $u_1, u_2 \in U$ [2]. (More general, "$\leqslant$" denotes a certain semi-order in the space U or W.) For solving associated operator equations $L(u) = f$ numerically, he defines a suitable set V of approximating functions in U and computes approximate solutions v_1 and v_2 of the one-sided approximation problems

$$\min_{\substack{v \in V \\ L(v) \leqslant f}} \|f - L(v)\| \qquad \text{and} \qquad \min_{\substack{v \in V \\ f \leqslant L(v)}} \|f - L(v)\|.$$

Then the inclusion $v_1 \leqslant u \leqslant v_2$ yields a quantitative error estimate

$$|u - v| \leqslant \tfrac{1}{2} |v_1 - v_2|$$

for the approximation $v = \frac{1}{2}(v_1 + v_2)$. In order to minimize the right side of the above inequality, the approximation problems can be replaced by the following optimization problem:

$$\text{Minimize } d, \text{ subject to}$$
$$0 \leqslant v - \tilde{v} \leqslant d, \qquad L(v) \leqslant f \leqslant L(\tilde{v}), \qquad v, \tilde{v} \in V.$$

In particular, Collatz suggests the use of functions which depend on finitely many parameters, the choice of a discrete subset of the considered domain, and that the resulting finite optimization problem be solved [52].

A large class of operators $L = (R, S)$ which correspond to elliptic differential equations

$$R(u) = -\sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} - \sum_{i=1}^{m} b_i \frac{\partial u}{\partial x_i} + cu = f \qquad \text{on } T,$$
$$S(u) = g \qquad \text{on bd } T$$

is of monotonic type if $c \geqslant 0$ and the matrix (a_{ij}) is uniformly positive definite, where a_{ij}, b_i, and c are bounded continuous functions on a bounded region T of $\mathbb{R}^m$ [52, 76, 77].

Actually, Collatz was mainly interested in methods for solving boundary value problems for partial differential equations

$$R(u) = f \text{ on } T \qquad \text{and} \qquad S(u) = g \text{ on bd } T.$$

For certain classes of problems he considers a set V which satisfies $R(v) = f$

for all $v \in V$. He describes methods for computing approximations $v \in V$ such that $\|u - v\| \leqslant \varepsilon$ holds for the error on the boundary of the considered region. Under the assumption of the maximum principle, he obtains the same estimate on the whole domain.

A standard example is given by the heat equation

$$R(u) = k \cdot \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{on } T = (0, 2\pi) \times (0, \infty)$$

and the space

$$V = \operatorname{span}\{e^{-(\lambda_\mu^2/k) \cdot t} \cdot \cos \lambda_\mu x \colon \mu = 0, ..., n\}.$$

The method is to determine for $t = 0$ an approximation $\tilde{v}$ of $g(x, 0)$ from the space

$$\tilde{V} = \operatorname{span}\{\cos \lambda_\mu x \colon \mu = 0, ..., n\}$$

of generalized trigonometric sums. Computation of the basis coefficients of $\tilde{v}$ simultaneously yields a function $v \in V$ with the same coefficients, and for this approximation an error estimate on $T \cup \operatorname{bd} T$ via the maximum principle [82].

A different approach—the concept of monotonicity—will be illustrated by an example associated to a torsion problem:

$$T = \{(x, y) \in \mathbb{R}^2 \colon |x| < 1, |y| < 1\},$$

$$R(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad S(u) = u = g.$$

Collatz uses the space

$$V = \operatorname{span}\{\operatorname{Re}(x + iy)^{4\mu} \colon \mu = 0, ..., n\}$$

which satisfies $R(v) = 0$, $v \in V$, and is led to a univariate approximation problem on the boundary ($x = 1$, $-1 \leqslant y \leqslant 1$). Since the operator $L = (R, S)$ is of monotonic type, he obtains concrete error bounds on $T \cup \operatorname{bd} T$ [52].

In a further approach, Collatz uses sets V satisfying the boundary condition $S(v) = g$, $v \in V$. For example, plate problems can be expressed in terms of a biharmonic differential equation

$$T = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\},$$

$$R(u) = \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2\, \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f$$

and the boundary condition $u = \partial u/\partial n = 0$. A suitable choice is the polynomial space

$$V = \text{span}\{(1 - x^2 - y^2)\, x^i y^j \colon 0 \leqslant i, j;\, i + j \leqslant 2\}.$$

Since $v = \partial v/\partial n = 0$ for all $v \in V$ and the operator R is of monotonic type, one-sided approximation of f by functions from $R(V)$ yields the desired error bounds [82].

It is not always possible to choose approximating functions which satisfy either the partial differential equation or the boundary condition. Therefore, Collatz studies the more complex problem of simultaneous approximation [38, 82].

In addition, Collatz documents the efficiency of his methods by applying them to nonlinear equations which are of particular relevance for applications such as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(u) = 1,$$

or

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - c \frac{\partial^2 u}{\partial x^2} = 0,$$

the so-called Burger's equation.

He also applies his concept to partial differential equations with (hidden) singularities by choosing appropriate functions which depend on the singularities [51, 64, 66–69] to free boundary value problems [47–50] and to delay equations [61, 62].

A further principle of monotonicity is used for solving fixed point problems. Differential equations can be transformed into fixed point equations $L(u) = u$ for integral operators, with the aid of Green's functions.

Iteration methods of the type

$$v_{n+1} = L_1(v_n) + L_2(w_n)$$

$$w_{n+1} = L_1(w_n) + L_2(v_n)$$

are applied for computing an approximate solution of fixed point equations $L(u) = u$ if L is compact and monotonically decomposable, i.e., $L = L_1 + L_2$ and $u \leqslant \tilde{u}$ implies $L_1(u) \leqslant L_1(\tilde{u})$, $L_2(u) \geqslant L_2(\tilde{u})$.

Since in this situation

$$v_n < v_{n+1} < u < w_{n+1} < w_n,$$

the question arises of how to determine starting elements v_0 and w_0 such that the difference $w_1 - v_1$ is as small as possible. Therefore, Collatz is led to field approximation problems which can be formulated as optimization problems:

$$\text{Minimize } d, \text{ subject to}$$

$$0 \leqslant w_1 - v_1 \leqslant d, \qquad v_0 \leqslant v_1, w_1 \leqslant w_0$$

[29, 34, 37].

An important class of nonlinear operators that are monotonically decomposable is given by Hammerstein's integral operators:

$$L(u)(x) = \int_T K(x, t)\, \phi(u(t))\, dt.$$

Finally, the following statement of Collatz [45] summarizes the conviction which had a forming influence on his work:

> Idee und Erfahrung sollen im allgemeinen in ständiger Wechselwirkung miteinander stehen. [In general, ideas and experience shall interact permanently.]

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