Sobolev embeddings involving symmetry

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Abstract

Given a bounded regular domain with cylindrical symmetry, then functions having such symmetry and belonging to the first Sobolev space can be embedded compactly into some weighted $L^p$ spaces, with $p$ superior to the critical Sobolev exponent. A simple application to elliptic boundary value problem is also considered.

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1. Introduction

In his paper, Ni [8] has considered an elliptic BVP as follows

$$\begin{cases} -\Delta u = h(x)u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \quad (1.1)$$

and proved, among other things, that if $\Omega$ is an $n$-dimensional ball and $h(x)$ behaves like $|x|^l$, with $l > 0$, then problem (1.1) has at least a solution for every $p \in (1, (n + 2 + l)/(n - 2))$.

Generally, it is unusual for problem (1.1) to have solutions if $p \geq (n + 2)/(n - 2)$. A famous variational identity, Pohozaev’s identity, implies that problem (1.1) admits no so-
olution if \( \Omega \) is a star shaped domain about the origin and \( h(x) \) is a strictly positive function, verifying \( x \cdot \nabla h(x) \leq 0 \) on \( \Omega \). Such obstacles to the existence of a solution for problem (1.1) stem from the fact that \( q = 2n/(n - 2) \) is the critical exponent for the Sobolev embedding: \( H^1_0(\Omega) \to L^q(\Omega) \).

Although there are other examples showing that problem (1.1) may have solutions both in critical and super critical cases (see [3] and [4]), these results usually involve the “defects” of the domain, so that its topology plays a key rôle in the existence of a solution. Here, the question is: does there exist a solution to the problem if the domain is contractible? Ni’s result is a good instance to answer the question. The motive of this paper is to show that Ni’s results are not lonely phenomenons. Similar conclusions hold for a class of domains where only partial symmetry is needed.

To solve problem (1.1), an approach is to investigate sub-Sobolev spaces. We note that symmetry can improve Sobolev embeddings (see Hebey and Vaugon [6], Lions [7] and the references therein). However, symmetry alone does not provide the desired improvements if we restrict ourselves on the usual \( L^p \) spaces.

In Section 2, we establish embedding results on a cylindrically symmetric domain. We prove that functions belonging to the first Sobolev space and having the same symmetry can be embedded compactly into a higher weighted \( L^p \) space, denoting by \( L^p_h(\Omega) \), than the one given by Sobolev’s embedding theorems.

In Section 3, we give some simple applications of the embedding results to nonlinear elliptic boundary value problems with critical or super critical nonlinearities. We believe that our embedding results may have other applications.

2. Compact embeddings

Let \( \Omega = \Omega_1 \times \Omega_2 \), with \( \Omega_1 \subset \mathbb{R}^m \) being a bounded regular domain, and \( \Omega_2 \) being a \( k \geq 2 \) dimensional ball of radius \( R \), centered at the origin. Denote

\[
\tilde{H}^1_s(\Omega) = \{ u \in H^1_0(\Omega) \mid u(\cdot, x_2) = u(\cdot, |x_2|), \forall x \in \Omega_2 \}.
\]

Let \( h \) be a nonnegative Hölder continuous function on \( \Omega \), denote by \( L^p_h(\Omega) \) the Banach space for the norm

\[
\|u\|_{p,h} = \left( \int h|u|^p \right)^{1/p}.
\]

For a positive integer \( m \), we write

\[
2^*_m = \begin{cases} 
2m/(m - 2), & \text{if } m > 2; \\
\infty, & \text{if } m \leq 2.
\end{cases}
\]

Throughout the whole paper, \( m, k \) and \( n \) are specified to denote the dimensions of \( \Omega_1, \Omega_2 \) and \( \Omega \). And \( \nabla_1 \) and \( \nabla_2 \) are the partial gradient operators on \( \Omega_1 \) and on \( \Omega_2 \) respectively.

For succinctness, we will use without explanation, \( A, B, C \) etc. to denote various constants independent of functions \( u \)'s under discussion or of its variables. For the same reason, we often omit limits or volume elements in a integral if not causing confusions.
2.1. Basic lemmas

**Lemma 2.1.** Let \( \dim \Omega_2 = k \geq 3 \), then for \( u \in \tilde{H}^1_s(\Omega) \), we have

\[
|u(x)| \leq \frac{C_k}{|x_2|^{(k-2)/2}} \left( \int_{\Omega_2} \left| \nabla^2 u \right|^2 \right)^{1/2}, \quad \forall x \in \Omega,
\]

where \( C_k \) is a constant depending only on \( k \).

**Proof.** Let \( u \in \tilde{H}^1_s(\Omega) \cap C^\infty_0(\Omega) \), then

\[
u(x_1, x_2) - u(x_1, R) = \int_{|x_2|}^R u_t(x_1, t) \, dt,
\]

thus, we have by Hölder’s inequality

\[
|u(x)| \leq \int_{|x_2|}^R \left| u_t(x_1, t) \right| \, dt = \int_{|x_2|}^R \left| u_t(x_1, t) \right|^{\frac{k-1}{2}} \cdot t^{\frac{1-k}{2}} \, dt \leq \left( \int_{|x_2|}^R \left| u_t(x_1, t) \right|^{2} t^{k-1} \, dt \right)^{1/2} \left( \int_{|x_2|}^R t^{1-k} \, dt \right)^{1/2}.
\]

Denote by \( \omega_{k-1} \) the area of the unit \( k-1 \) sphere, then

\[
\int_{|x_2|}^R \left| u_t(x_1, t) \right|^{2} t^{k-1} \, dt = \frac{1}{\omega_{k-1}} \int_{|x_2|}^R \int_{|x_2|}^R \left| u_t(x_1, t) \right|^{2} t^{k-1} \, dt \, d\omega 
\leq \frac{1}{\omega_{k-1}} \int_{\Omega_2} \left| \nabla u \right|^2 \, dx_2 \leq \frac{1}{\omega_{k-1}} \int_{\Omega_2} \left| \nabla^2 u \right|^2 \, dx_2.
\]

Since

\[
\int_{|x_2|}^R \frac{1}{t^{k-1}} \, dt \leq \frac{1}{(k-2)|x_2|^{k-2}},
\]

the result therefore follows. \( \square \)

In the case of \( k = 2 \), we need a substitute for Lemma 2.1, that is

**Lemma 2.2.** Let \( \dim \Omega_2 = k \), and let \( b > 1 \) be a real number, then for \( u \in H^1_0(\Omega) \), and all \( x \in \Omega \) we have

\[
|u| \leq \frac{A_k}{|x_2|^{k-1}} \int_{\Omega_2} \left| \nabla u \right|, \quad |u|^b \leq \frac{bA_k}{|x_2|^{k-1}} \int_{\Omega_2} |u|^{b-1} \left| \nabla u \right|.
\]
Proof. In the proof of Lemma 2.1, before using Hölder’s inequality, replace \( t^{(k-1)/2} \) with \( t^{k-1} \), then by a simple estimate, we obtain the first inequality. To achieve the second, write \(|u|^b\) in place of \( u \) in the first. \( \square \)

To simplify our later proof of theorems, we will employ Sobolev’s operator and its properties. Given \( f \in L^1(\Omega_1) \), define

\[
V(f)(x_1) = \int_{\Omega_1} \frac{|f(y_1)|}{|x_1 - y_1|^{m-1}} \, dy_1.
\]

A function \( u \in H^1_0(\Omega) \) can be regarded as functions \( u(\cdot, x_2) \) in \( H^1_0(\Omega_1) \), parameterized by \( x_2 \). For these functions, we have the following (see Aubin [1], Adams [2] or Gilbarg and Trudinger [5]).

**Lemma 2.3.** Let \( u \in C^1_0(\Omega) \), then we have the following conclusions:

(i) \(|u(x)| \leq C_m V(\nabla_1 u), x \in \Omega; \)

(ii) for \( 1 < q \leq m/(m-1) \),

\[
\|V(\nabla_1 u)\|_{L^q(\Omega_1)} \leq A_m \|\nabla_1 u\|_{L^1(\Omega_1)}, \quad x_2 \in \Omega_2;
\]

(iii) moreover, for \( 1 < q \leq 2^*_m \),

\[
\|V(\nabla_1 u)\|_{L^q(\Omega_1)} \leq A_m \|\nabla_1 u\|_{L^2(\Omega_1)}, \quad x_2 \in \Omega_2.
\]

2.2. The higher dimensional cases

Now, we are in a position to announce

**Theorem 2.4.** Let \( \dim \Omega_1 = m \geq 2 \) and \( \dim \Omega_2 = k \geq 3 \). Suppose that \( h = |x_2|^l \), \( l > 0 \) being a real number. Then the embedding \( \tilde{H}^1_s(\Omega) \to L^q_h(\Omega) \) is compact for \( q \in (1, 2^*_m + \tau) \), where \( \tau = \frac{2}{n-2} \min\{\frac{2(k-2)}{m}, l\} \).

**Proof.** Our main objective in proving the theorem and the rest theorems in the section is to establish the inequality

\[
\int_{\Omega_2} |x_2|^l |u|^q \leq A \left( \int_{\Omega} |\nabla u|^2 \right)^{q/2}, \quad \forall u \in \tilde{H}^1_s(\Omega) \tag{2.1}
\]

for \( 1 < q < 2^*_m + \tau \). This means that the embedding \( \tilde{H}^1_s(\Omega) \to L^q_h(\Omega) \) is continuous. Then the compact conclusion follows from a standard procedure which can be found, for example, in [2] for the proof of Kondrachov theorem. Let \( 0 < a < 2 \) and \( 0 < b < 2 \) be real numbers. For \( u \in \tilde{H}^1_s(\Omega) \), from Lemmas 2.1 and 2.3, we draw easily the following

\[
|x_2|^l |u|^{a+b} \leq B \frac{|V(\nabla_1 u)|^a}{|x_2|^\gamma} \left( \int_{\Omega_2} |\nabla u|^2 \right)^{b/2}, \tag{2.2}
\]
where \( \gamma = b(k - 2)/2 - l \). Integrate (2.2) on \( \Omega_1 \) then application of Hölder’s inequality yields

\[
\int_{\Omega_1} |x_2|^l |u|^{a+b} \leq B \left( \int_{\Omega_1} (V(|\nabla_1 u|))^{2a/2-b} \right)^{2-b} \left( \int_{\Omega} |\nabla_2 u|^2 \right)^{b/2}. \tag{2.3}
\]

However, according to Lemma 2.3,

\[
\left( \int_{\Omega_1} (V(\nabla_1 u))^{2a/(2-b)} \right)^{(2-b)/2} \leq A \left( \int_{\Omega_1} |\nabla_1 u|^2 \, dx_1 \right)^{a/2}
\]

if \( 2a/(2-b) \leq 2^*_m \), that is

\[
(m - 2)a + mb \leq 2m. \tag{2.4}
\]

Under this condition, (2.3) becomes

\[
\int_{\Omega_1} |x_2|^l |u|^{a+b} \leq B \left( \int_{\Omega_1} |\nabla u|^2 \right)^{a/2} \left( \int_{\Omega} |\nabla_2 u|^2 \right)^{b/2}. \tag{2.5}
\]

Integrate (2.5) on \( \Omega_2 \), then apply again Hölder’s inequality, we get

\[
\int_{\Omega} |x_2|^l |u|^{a+b} \leq B \left( \int_{\Omega_2} |x_2|^{2\gamma/(2-a)} \right)^{2-a} \left( \int_{\Omega} |\nabla u|^2 \right)^{a+b/2}, \tag{2.6}
\]

if \( 2\gamma/(2-a) < k \), or equivalently

\[
ka + (k - 2)b < 2(k + l), \tag{2.7}
\]

then (2.6) is reduced to

\[
\int_{\Omega} |x_2|^l |u|^{a+b} \leq A \left( \int_{\Omega} |\nabla u|^2 \right)^{(a+b)/2}. \tag{2.8}
\]

To sum up, we have (2.8) if conditions (2.4) and (2.7) are satisfied. These conditions are equivalent to require for \( 0 < a < 2 \) and \( 0 < b < 2 \) to satisfy the following

\[
\begin{align*}
(m - 2)a + mb & \leq 2m, \\
ka + (k - 2)b & < 2(k + l).
\end{align*} \tag{2.9}
\]

What remains for us is to choose suitable \( a \) and \( b \) verifying condition (2.9) such that \( (a+b) \) approaches \( 2^*_n + \tau \) arbitrarily. To do this, let

\[
l^* = \min\{l, (2k - 4)/m\}, \quad \tau = l^*/(m + k - 2).
\]

If \( m \geq 3 \), for \( l' \in (0, l^*) \) arbitrarily given, set

\[
a = \frac{m(2 + l')}{m + k - 2}, \quad b = \frac{2(k + l') - l'm}{m + k - 2},
\]
then \(a\) and \(b\) lie in \((1, 2)\) and verify condition (2.9). Moreover, as \(l'\) goes to \(l^*\),
\[
a + b = \frac{2(m + k) + 2l'}{m + k - 2} \to 2_n^* + \tau.
\]
If \(m = 2\), then \((2 + l')/k < 1\) for \(l' \in (0, l^*)\), and
\[
a = b = 1 + (2 + l')/k
\]
answer the needs of condition (2.9). Besides, \(a + b \to 2_n^* + \tau\) as \(l'\) tends to \(l^*\). This ends the proof of the theorem. \(\Box\)

2.3. The low dimensional cases

It should be remarked that the method used in proving Theorem 2.4 does not work for low dimensional cases. The next two theorems are intended to handle the remaining cases.

**Theorem 2.5.** Assume that \(\text{dim} \, \Omega_1 = m \geq 2\) and that \(\text{dim} \, \Omega_2 = k = 2\). Let \(h(x) = |x_2|^l\), with \(l > 0\), then there exists a positive number \(\tau\), such that the embedding \(\hat{H}^1_s(\Omega) \to L^q_h(\Omega)\) is compact for \(q \in \left(1, 2_n^* + \tau\right)\). To be precise, \(\tau = (2/m)\min\{l, 1/m\}\).

**Proof.** Assume \(\alpha, \beta \in (0, 1)\) and \(b > 1\) to be positive real numbers. Let \(u \in \hat{H}^1_s(\Omega) \cap C^1_0(\Omega)\). From Lemmas 2.2 and 2.3, we draw easily the following
\[
|\langle x_2 \rangle^l |u|^{b(\alpha+\beta)} \leq \frac{B \left[ V(|u|^{b-1} \nabla u) \right]^{b(l-\gamma)}}{|\langle x_2 \rangle|^{\beta-l}} \left( \int_{\Omega_2} |u|^{b-1} |\nabla u| \right)^{\beta}. 
\]
Integrate the inequality above on \(\Omega_1\), then we have by employing Hölder’s inequality
\[
\int_{\Omega_1} |\langle x_2 \rangle^l |u|^{b(\alpha+\beta)} \leq \frac{B |\langle x_2 \rangle|^{\gamma}}{|\langle x_2 \rangle|^{\gamma}} \left( \int_{\Omega_1} [V(|u|^{b-1} \nabla u)]^{\alpha/(1-\beta)} \right)^{1-\beta} \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)^{\beta}. 
\]
Here \(\gamma = \beta - l\). The inequality is then, according to Lemma 2.3, reduced to
\[
\int_{\Omega_1} |\langle x_2 \rangle^l |u|^{b(\alpha+\beta)} \leq \frac{B |\langle x_2 \rangle|^{\gamma}}{|\langle x_2 \rangle|^{\gamma}} \left( \int_{\Omega_1} |u|^{b-1} |\nabla u| \right)^{\alpha} \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)^{\beta} \tag{2.10} 
\]
if \(\alpha/(1-\beta) \leq m/(m-1)\), or more explicitly
\[
(m-1)\alpha + m\beta \leq m. \tag{2.11} 
\]
Integrate (2.10) on \(\Omega_2\) and apply Hölder’s inequality,
\[
\int_{\Omega} |\langle x_2 \rangle^l |u|^{b(\alpha+\beta)} \leq B \left( \int_{\Omega_2} |\langle x_2 \rangle|^{\gamma/\alpha} \right)^{1-\alpha} \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)^{\alpha+\beta}. 
\]
Thus, we have the following estimate
\[
\int_{\Omega} |\langle x_2 \rangle^l |u|^{b(\alpha+\beta)} \leq B \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)^{\alpha+\beta} \tag{2.12} 
\]
if \( \gamma / (1 - \alpha) < 2 \), that is
\[
2\alpha + \beta < 2 + l.
\] (2.13)

We make still again use of Hölder’s inequality to estimate
\[
\int_{\Omega} |u|^{b-1} |\nabla u| \leq A \left( \int_{\Omega} |u|^{2(b-1)} \right)^{1/2} \left( \int_{\Omega} |\nabla u|^2 \right)^{1/2}.
\] (2.14)

Now, let \( 2(b - 1) \leq 2^*_n \), i.e.
\[
b \leq 2(m + 1)/m,
\] (2.15)

then the Sobolev embedding theorem gives
\[
\int_{\Omega} |u|^{2(b-1)} \leq A \left( \int_{\Omega} |\nabla u|^2 \right)^{b-1}.
\] (2.16)

It follows from (2.12), (2.14) and (2.16) that
\[
\int_{\Omega} |x_2|^l |u|^{b(\alpha + \beta)} \leq B \left( \int_{\Omega} |\nabla u|^2 \right)^{b(\alpha + \beta)/2}
\] (2.17)

if conditions (2.11), (2.13) and (2.15) are satisfied. To sum up, the conditions for (2.17) hold are as follows
\[
\begin{aligned}
&(m - 1)\alpha + m\beta \leq m, \\
&2\alpha + \beta < 2 + l, \\
&b \leq 2(m + 1)/m.
\end{aligned}
\] (2.18)

On analyzing these conditions, let \( \ell = \min\{l, 1/m\} \), and for \( \eta > 0 \) arbitrarily small, let
\[
\alpha = \frac{m(1 + \ell) - \eta}{m + 1}, \quad \beta = \frac{2 - (m - 1)\ell}{m + 1}, \quad b = \frac{2(m + 1)}{m},
\]
then, \( a \) and \( b \) stay in the interval \((0, 1)\) and satisfy the conditions of (2.18). Besides,
\[
b(\alpha + \beta) = \frac{2(m + 2 + \ell)}{m} + o(1),
\]
as \( \eta \to 0 \). That is the result. \( \Box \)

**Theorem 2.6.** Let \( \dim \Omega_1 = 1 \) and let \( \dim \Omega_2 = k \geq 2 \). Suppose that \( h = |x_2|^l \), \( l > 0 \) being a real number, then there exists a positive number \( \tau \) such that the embedding \( \tilde{H}^1_\Theta(\Omega) \to L^q_{\nu}(\Omega) \) is compact for \( q \in (1, 2^*_n + \tau) \). Moreover, \( \tau = \min\{2l/(k - 1), 2\} \).

**Proof.** Let \( b > 1 \) and \( 0 < c < 1 \) be real numbers. For \( u \in \tilde{H}^1_\Theta(\Omega) \), we draw from Lemma 2.2 the following inequality
\[
|x_2|^l |u(x)|^{b(1+c)} \leq \frac{B}{|x_2|^{k-1-l}} \left( \int_{\Omega_1} |u|^{b-1} |\nabla u| \right)^c \int_{\Omega_2} |u|^{b-1} |\nabla u|.
\]
Integrate the above inequality first on $\Omega_1$, we get
\[
\int_{\Omega_1} |x_2|^l |u|^{b(1+c)} \leq B \left( \int_{\Omega_1} |u|^{b-1} |\nabla u| \right)^c \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)
\]
where $\gamma = k - 1 - l$. Integrate then on $\Omega_2$ while applying Hölder’s inequality,
\[
\int_{\Omega} |x_2|^l |u|^{b(1+c)} \leq B \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)^{1+c} \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)^{1+c}.
\]  
(2.19)
Evidently, if
\[
\gamma/(1-c) < k \iff kc < 1 + l,
\]  
(2.20)
then we have
\[
\int_{\Omega} |x_2|^l |u|^{b(1+c)} \leq B \left( \int_{\Omega} |u|^{b-1} |\nabla u| \right)^{1+c}.
\]  
(2.21)
To go on estimating, now, let us suppose that $2(b - 1) \leq 2^*$, namely
\[
b \leq 2k/(k - 1),
\]  
(2.22)
we have by using Hölder’s inequality,
\[
\int_{\Omega} |u|^{b-1} |\nabla u| \leq \left( \int_{\Omega} |u|^{2(b-1)} \right)^{1/2} \left( \int_{\Omega} |\nabla u|^2 \right)^{1/2}.
\]  
(2.23)
We estimate the first integral on the right side by appealing Sobolev’s embedding theorem
\[
\int_{\Omega} |u|^{2(b-1)} \leq C \left( \int_{\Omega} |\nabla u|^2 \right)^{1/2}.
\]  
(2.24)
It follows from (2.21), (2.23) and (2.24) that
\[
\left( \int_{\Omega} |x_2|^l |u|^{b(1+c)} \right)^{1/b(1+c)} \leq A \left( \int_{\Omega} |\nabla u|^2 \right)^{1/2}
\]  
(2.25)
if conditions (2.20) and (2.22) are satisfied.

To finish our proof, set $\ell = \min\{l, k - 1\}$, and take
\[
c = (1 + \ell - \eta)/k, \quad b = 2k/(k - 1),
\]
where $\eta$ is an arbitrarily small positive number. Such $c$ and $b$ are fully in accordance with conditions (2.20) and (2.22). Moreover,
\[
b(1+c) = \frac{2(k+1+\ell)}{k-1} + o(1),
\]
as $\eta \to 0^+$. Our proof is thus complete.  \(\square\)
We end this section with a more general result. First, let us assume that $h$ is a non-negative Hölder continuous function on $\overline{\Omega}$, radially symmetric with respect to $x_2 \in \Omega_2$, satisfying $h(x_1, 0) = 0$. Define $l_h$ to be

$$ l_h = \sup\{\lambda > 0 : |h(x)|/|x_2|^\lambda < \infty, \ x \in \Omega\}. $$

Then, from the previous discussions, we draw easily the following

**Theorem 2.7.** Let $h$ be a nonnegative and non-identically null function, Hölder continuous on $\overline{\Omega}$, radially symmetric on $\Omega_2$, satisfying $l_h > 0$. Then, there exists a positive number $\tau = \tau(h, m, k)$ such that the embedding $\dot{H}^1_s(\Omega) \rightarrow L^q_h(\Omega)$ is compact for all $q \in (1, 2^*_n + \tau)$.

3. Applications

As in the previous section, let $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subset \mathbb{R}^m$ is a bounded regular domain, and $\Omega_2$ is a $k$ dimensional ball centered at the origin. Now return to problem

$$
\begin{cases}
-\Delta u = h(x)u^p, & \text{in } \Omega, \\
u > 0, & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(3.1)

where $p > 1$ is a real number, and $h(x)$ is a nonnegative Hölder continuous function on $\overline{\Omega}$.

As a simple application of Section 2, we have

**Theorem 3.1.** Let $h$ be a nonnegative and non-identically null function, Hölder continuous on $\overline{\Omega}$, radially symmetric on $\Omega_2$ satisfying $l_h > 0$, then there exists a positive number $\tau$ such that problem (3.1) has a solution $u \in \dot{H}^1_s(\Omega)$ for all $p \in (1, 2^*_n + \tau)$.

**Proof.** Let $\Sigma = \{u \in \dot{H}^1_s(\Omega) \mid \|u\|_{h, p+1} = 1\}$, and let

$$
\mu = \inf_{u \in \Sigma} \int |\nabla u|^2.
$$

(3.2)

We know by Theorem 2.7 that there is some $\tau > 0$ such that the embedding $\dot{H}^1_s(\Omega) \rightarrow L^p_{h+1}(\Omega)$ is compact for $(p + 1) \in (2, 2^*_n + \tau)$. Hence, the infimum in (3.2) is achieved by some nonnegative function $v$. Thus, we obtain a Lagrange multiplier $\mu$ such that

$$
-\Delta v = \mu h(x)v^p.
$$

After stretching $\mu$, we get a solution of (3.1). The solution $u$ is positive on $\Omega$ by the strong maximum principle.

It should be remarked that the solution so obtained lies in $H^1_0(\Omega)$. In fact, it belongs to $L_\infty(\Omega)$ (see [4] and the references therein). By a bothtrap method, $u$ is as smooth as $h$ and $\Omega$ permit. \(\square\)
References


Further reading