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Planar Ramsey numbers for cycles

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Abstract

For two given graphs G and H the planar Ramsey number $\mathrm{PR}(G, H)$ is the smallest integer n such that every planar graph F on n vertices either contains a copy of G or its complement contains a copy H. By studying the existence of subhamiltonian cycles in complements of sparse graphs, we determine all planar Ramsey numbers for pairs of cycles.

MSC: 05D10; 05C55

Keywords: Ramsey number; Planar graph; Cycles

1. Introduction

Given two graphs G and H, *the Ramsey number* $R(G, H)$ is the smallest integer n such that every graph F on n vertices contains a copy of G or its complement contains a copy of H. The determination of Ramsey numbers is, in general, an extremely difficult problem. Often in such cases graph theorists turn to specific classes of graphs in hope for some positive results, and the class of planar graphs is one of the most attractive. This was apparently the reason why Walker [12] in 1969, and, independently (sic!), Steinberg and Tovey [11] in 1993 introduced the notion of planar Ramsey number.

The planar Ramsey number $\mathrm{PR}(G, H)$ is the smallest integer n for which every *planar* graph F on n vertices contains a copy of G or its complement contains a copy of H. Note that $\mathrm{PR}(G, H) \leqslant R(G, H)$, but unlike $R(G, H)$, as a simple consequence of Turán's Theorem, the numbers $\mathrm{PR}(G, H)$ grow only linearly with $|V(H)|$. Moreover, they do not need to be symmetric with respect to G and H. A related feature is that quite often all planar graphs F with $\mathrm{PR}(G, H)$ vertices, out of the two alternatives satisfy only the latter, i.e., $F \not\supset G$ but $F^c \supset H$. This is obviously true when G is not planar, but even for planar G it may be the case (see Theorem 6).

Both papers [12,11] focus on computing the planar Ramsey numbers for complete graphs and link them with the Four Color Conjecture, and, resp. Four Color Theorem. To pinpoint the values of these numbers, the authors of [11] use Grünbaum's Theorem, which generalizes the better known Grötzsch's Theorem. They show that $\mathrm{PR}(K_3, K_l) = 3l - 3$ and $\mathrm{PR}(K_k, K_l) = 4l - 3$ for all $k \geqslant 4$ and $l \geqslant 3$.

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In this paper we determine all planar Ramsey numbers for pairs of cycles, that is, all numbers $\mathrm{PR}(C_m, C_n)$, where $m \geqslant 3$ and $n \geqslant 3$. It turns out that for n large enough these numbers do not depend on m, while, on the other hand, for small n they do not grow with m, which emphasizes the asymmetry mentioned above. In our proofs we rely on well-known sufficient conditions for the existence of hamiltonian cycles and for pancyclicity. On the other hand, for some small cycles we make use of programs generating planar graphs [13].

Planar Ramsey numbers were also considered in [1,2]. For more on Ramsey numbers see, e.g., [8]. Throughout the paper we use the standard graph theory notation (see, e.g., [5]). In particular, $\delta(G)$, $\alpha(G)$ and $\kappa(G)$ stand, respectively, for the minimum degree, the independence number and the vertex-connectivity of a graph G. By G^c we denote the complement of G.

2. Cycles in complements of sparse graphs

The study of planar Ramsey numbers for cycles reduces, in most instances, to finding long cycles in complements of sparse, but not necessarily planar, graphs. In this section we present three similar results in this direction, all leading to the determination of the planar Ramsey numbers in question.

In our proofs we will frequently use the well-known sufficient conditions for a graph to be hamiltonian and to be pancyclic. All of them can be found, for instance, in the monograph [3]. Here is the list of what we will need.

- (Dirac) If $\delta(G) \geqslant n/2$ then G is hamiltonian.
- (Chvátal–Erdős) If $\alpha(G) \leqslant \kappa(G)$ then G is hamiltonian.
- (Bondy) If G is hamiltonian and $e(G) \geqslant n^2/4$ then G is pancyclic unless $G = K_{n/2,n/2}$.

Our first result asserts that there is a subhamiltonian cycle in each graph which, in some sense, is sparse both, locally and globally. As the example of $K_{3,3} - e$ shows, Theorem 1 does not need to be true for $n \leqslant 6$. Also, it fails to be true if we instead request a copy of C_{n-1} in the complement. Here $K_{2,n-2}$ is a counterexample.

Theorem 1. *Let G be an arbitrary graph on $n \geqslant 7$ vertices, at most* $\max\{\binom{n}{2} - n^2/4, 2n-4\}$ *edges and containing neither K_3 nor $K_{3,3}$. Then G^c contains C_{n-2}.*

Proof. We have $\alpha(G^c) \leqslant 2$. If $\kappa(G^c) \geqslant 2$, then G^c is hamiltonian by the Chvátal–Erdős condition. For $n \geqslant 8$, we have $e(G^c) \geqslant n^2/4$ and $G^c \neq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Thus G^c is pancyclic and in particular contains C_{n-2}. For $n=7$, if $e(G^c) < 7^2/4$ then $e(G^c) = 11$ or 12, so there must be at least four diagonals in the C_7, and thus there is a C_5 in G^c.

If $\kappa(G^c) \leqslant 1$, then there is a vertex x which separates a nonempty set of vertices U from the rest. Note that $|U| \leqslant 2$, since G does not contain $K_{3,3}$. Set $W = V - U - \{x\}$. Because G is K_3-free, $G^c[W]$ must be complete. If $|U| = 1$ then we have C_{n-2} in G^c, so consider $|U| = 2$. Since $|W| \geqslant 4$, there are at least two edges from x to W (otherwise there would be $K_{3,3}$ in G). So, $W \cup \{x\}$ spans a C_{n-2}. □

The next result is not new. Nevertheless, it fits well with the other theorems here, and we give the proof because of its simplicity. Together with a lower bound given by the star $K_{1,n-1}$, it yields that $\mathrm{PR}(C_4, C_n) = n+1$.

Theorem 2 (*[6,9,10]*). *For $n \geqslant 6$, the Ramsey number $R(C_4, C_n) = n+1$. In other words, the complement of every C_4-free graph on $n \geqslant 7$ vertices contains C_{n-1}.*

Proof. Let G be a graph fulfilling the assumptions of the theorem. We have $\alpha(G^c) \leqslant 3$. If $\kappa(G^c) \geqslant 3$ then, by the Chvátal–Erdős condition, G^c is hamiltonian. It is easy to see that for $n \geqslant 8$, the Turán number $T(n, C_4)$ is smaller than $\binom{n}{2} - n^2/4$ (use the well-known bound $T(n, C_4) \leqslant \frac{1}{4} n(1 + 2\sqrt{n})$—see, e.g., [8, p. 144]), and thus G^c is pancyclic, containing, in particular, C_{n-1}. For $n=7$, $T(n, C_4) = 9$ (see [4]) and $\binom{n}{2} - n^2/4 = 8.75$, but it can be checked by hand that there is no C_4-free graph on seven vertices whose complement is hamiltonian.

If $\kappa(G^c) \leqslant 2$ then there are two vertices x, y which separate one vertex, a, from the rest. Set $W = V - \{a, x, y\}$. Because G is C_4-free, every vertex of W has in G^c at least $n-5$ neighbors in W, while x and y have each at least $n-4$ neighbors in W. Hence, by Dirac's condition, $G^c[W]$ has a Hamilton cycle which can be extended, by adding x and y, to a cycle C_{n-1}. □

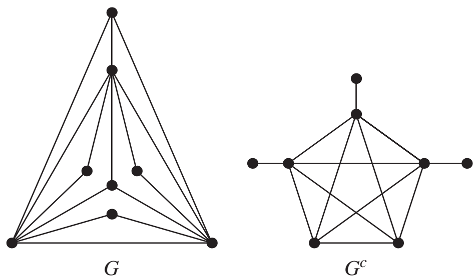

Fig. 1. A graph G and its complement.

We now present our main result. Note that the assumption $n \geqslant 12$ is only technical and Theorem 3 should remain true all the way down to $n = 9$. This is the case for planar graphs, which we prove later using graph generation programs. This is partly why we waive our hands at the end of the proof and do not show the details for $n = 12$ and 13. Theorem 3 is not true for $n = 8$—see the graph shown in Fig. 1. It is best possible in the sense that it fails if one would instead request a copy of C_{n-1}. Again, $K_{2,n-2}$ is a counterexample.

Theorem 3. *Let G be an arbitrary graph on $n \geqslant 12$ vertices, at most $\binom{n}{2} - n^2/4$ edges and containing neither K_5 nor $K_{3,3}$. Then G^c contains C_{n-2}.*

Proof. Let G be a graph fulfilling the assumptions of the theorem. Certainly $\alpha(G^c) \leqslant 4$, since otherwise we would have a K_5 in G. If $\kappa(G^c) \geqslant 4$ then by Chvátal–Erdős condition the graph G^c is hamiltonian. Moreover, $|E(G^c)| \geqslant n^2/4$, and so G^c is also pancyclic and therefore contains C_{n-2}. Thus, we may assume that $\kappa(G^c) \leqslant 3$. Observe that no vertex cut in G^c separates two vertex sets with at least three vertices each (there would be a $K_{3,3}$ in G in that case).

Assume first that $\delta(G^c) \geqslant 4$. Then a certain set S of three vertices would separate two vertices x and y from the set $W = V(G) - S - \{x, y\}$ in the complement of G. Since the subgraph $G^c[W]$ has minimum degree at least $n - 8$ (otherwise G would contain a $K_{3,3}$), by Dirac's condition it contains a Hamilton cycle. Similarly, each vertex from S has at least $n - 7$ neighbors in W, so this cycle can be extended successively to a cycle with the vertex set $W \cup S$.

Assume now that $\delta(G^c) \leqslant 3$. Let us remove one vertex v with $d(v) = \delta(G^c)$. If $\delta(G^c - v) \geqslant 4$ then $\kappa(G^c - v) \geqslant 4$, otherwise there would be a $K_{3,3}$ in G. In this case $G^c - v$ is hamiltonian and even pancyclic, since it has at least $n^2/4 - 3 > (n-1)^2/4$ edges.

Next assume that $\delta(G^c - v) \leqslant 3$. Let us remove one vertex u with $d(u) = \delta(G^c - v)$. If $\delta(G^c - v - u) \geqslant 4$ and $\kappa(G^c - v - u) \leqslant 3$, then a certain set of three vertices S separates two vertices x and y from the set $W = V(G) - S - \{x, y\}$. We have $|W| = n - 7$ and, for $n \geqslant 13$, the vertices x, y, u form in G, together with some three nonneighbors of v in W a $K_{3,3}$—a contradiction. For $n = 12$, it is straightforward to show that $S \cup W$ spans $K_{3,5}$ in G^c (or, again, a $K_{3,3}$ in G), and thus one can find a cycle through $S \cup W \cup \{u, v\}$.

Finally, consider the case $\delta(G^c - v - u) \leqslant 3$. Let us remove one vertex w with $d(w) = \delta(G^c - v - u)$. Let $W = V(G) - \{v, u, w\}$. Note that all but two vertices from W must be among the neighbors of v, u, or w. Hence, $n \leqslant 14$. For the rest of the proof, we assume that $n = 14$ and each of v, u, w has degree three in W. Similar, but more tedious argument in the cases $n = 12$ and 13 is omitted.

Let N_u, N_v and N_w be the (disjoint) neighborhoods of u, v, w and let x, y be the remaining two vertices. To avoid K_5 and $K_{3,3}$ in G, one can deduce the existence of several edges in G^c. For instance, there $xy \in E(G^c)$, moreover, there is at least one edge in $G^c[N_u]$, and, for each $a \in N_u$ at least one edge to a vertex in N_v and at least one edge to a vertex in N_w, etc. Altogether, the structure of G^c is so rich and spread out that it is easy to find there a Hamilton cycle, together with two shortcuts which yield the existence of C_{12} in G^c. $\square$

For planar graphs things seem to be a bit easier, but the proofs for small values of n remain more cumbersome. In order not to overwhelm the reader with tedious case by case analysis we turned to the existing graph generating programs to settle some questions for planar graphs with less than 10 vertices. We summarize them now.

Fact 1. (i) *The complements of all planar graphs on nine vertices contain C_5, C_6 and C_7.*
(ii) *All planar graphs on eight vertices either contain C_5 or their complements contain C_6.*

Proof. (i) It is enough to verify the statement for all edge maximal planar graphs, that is, for all triangulations on nine vertices. Using the program PLANTRI by Brinkmann and Mc Kay [13] we indeed have checked that all planar triangulations on nine vertices contain C_5, C_6 and C_7 in their complements.

(ii) Using the same package as above we have checked all planar graphs on eight vertices. The results of our computer search can be found at [14]. □

Based on Fact 1 we can now easily prove the following result.

Theorem 4. *Let G be a planar graph on $n \geqslant 9$ vertices. Then G^c contains C_{n-2}.*

Proof. We use induction on n and reduce the statement to the case $n = 9$ which is settled by Fact 1(i). If $n \geqslant 14$ and Theorem 4 is true for $n - 1$, then remove a vertex v of degree at most five in G and apply the induction assumption. There is a C_{n-3} in $G^c - v$ and v has at least $n - 1 - 5 - 2 = n - 8 > (n - 3)/2$ neighbors on that cycle, yielding an extension to a C_{n-2}.

For $n = 13$ consider two cases. If G contains a vertex v of degree at most four then its degree in the complement is at least eight. The graph $G^c - v$ contains a C_{10} by the induction assumption, so, again, v has two consecutive neighbors on that cycle, yielding a copy of C_{11}.

On the other hand, if $\delta(G) \geqslant 5$ then G has 12 vertices of degree five and one vertex v of degree six. Then $G^c - v$ has the degree sequence $6 \times 7 + 6 \times 6$ and is hamiltonian by Dirac condition. Moreover, it is pancyclic.

The case $n = 12$ can be dealt with similarly. If there is a vertex of degree at least seven in G^c, we are done. Otherwise, G^c is 6-regular and is hamiltonian by Dirac condition. Since by planarity of G, $G^c \neq K_{6,6}$, it has to be pancyclic.

If $n = 11$ then, by planarity, $\delta(G) \leqslant 4$. So there is a vertex v of degree at least six in G^c. There is a C_8 in $G^c - v$ by the induction assumption. If v has two adjacent neighbors on this cycle we have C_9 in G^c. Otherwise, v has exactly four neighbors which are nonadjacent on the cycle. In that case there is at least one edge in G^c joining some two of the remaining four vertices on the C_8, since otherwise there would be a K_5 in G. Now we are able to lead a C_9 through the vertices of that C_8 and vertex v.

In the last step of our reduction, let G be an arbitrary planar graph on 10 vertices. By planarity $\delta(G) \leqslant 4$. So there is a vertex v of degree at least five in G^c. There is a C_7 in $G^c - v$ by the induction assumption. If v has two adjacent neighbors on this cycle, we have C_8 in G^c. Otherwise, v has exactly three neighbors on it which are nonadjacent on the cycle. Let a, b, c, d be the remaining vertices of the C_7 with $cd \in E(C_7)$. If there is at least one other edge in G^c among $\{a, b, c, d\}$, then it is easy to find a C_8 in G^c on the vertices of the C_7 and vertex v. Otherwise, let x and y be the remaining two vertices lying outside the cycle. Note that x is joined in G^c to at least one of c and d, since in the opposite case there would be a topological minor of K_5 in G. Then we have a C_8 in G^c through v, x, c, and the remaining vertices of C_7 except d. □

To satisfy those readers who are sceptical about computer supported proofs, we now provide a relatively short, theoretical proof that all 8-vertex planar graphs have C_4 in their complement. Obviously, it fails to be true for C_4 replaced by C_3, C_5 (take $G = 2K_4$) or C_6 (the graph in Fig. 1).

Theorem 5. *Let G be a planar graph on eight vertices. Then G^c contains C_4.*

Proof. Note that it is enough to consider triangulations only. Let G be a planar triangulation on eight vertices, and thus, with 18 edges and minimum degree $\delta(G) \geqslant 3$. Suppose that $C_4 \not\subset G^c$. Then, for every pair of vertices there is at most one vertex which is nonadjacent in G to both of them.

Assume first that there is a vertex v of degree three in G. Let $W = N_G(v) = \{w_1, w_2, w_3\}$ and $U = V(G) - \{v\} - W = \{u_1, u_2, u_3, u_4\}$. Since every vertex from W may be nonadjacent in G to at most one vertex in U, there is at least one vertex in U which is adjacent to all vertices from W. Let u_1 be that vertex and $U' = U - \{u_1\}$. By planarity of G there is at least one edge missing between each vertex of U' and W. On the other hand, each vertex from W is adjacent to at least two vertices from U', and thus $G[W \cup U'] = K_{3,3} - 3K_2$. Without loss of generality, we may assume that

$w_i u_{i+1} \notin E(G)$, $i = 1, 2, 3$. Observe that $G[\{u_1, u_2, u_3\}]$ must contain an edge, since otherwise the vertices v, u_1, u_2, u_3 would form a C_4 in G^c. Thus, G contains a topological minor of $K_{3,3}$—a contradiction.

If $\delta(G) \geqslant 4$ then $\Delta(G^c) \leqslant 3$. The graph G^c has precisely 10 edges, so there are at least four vertices of degree three in G^c, and at least two of them must be adjacent in G^c. Let x and y be those vertices. If they have no common neighbor in G^c, then it is easy to see that there is a topological minor of $K_{3,3}$ in G.

Hence, consider the case when x and y have exactly one common neighbor z in G^c. Let x' and y' be, respectively, the remaining neighbors of x and y in G^c, and let $V(G) \setminus \{x, y, z, x', y'\} = \{a, b, c\}$. Clearly, x, y and a, b, c form a copy of $K_{2,3}$ in G. Moreover, since $\Delta(G^c) \leqslant 3$ and there is no $K_{3,3}$ in G, there is exactly one edge between z and a, b, c in G^c. Let az be that edge. Then $ax' \in E(G)$ and we have a topological minor of $K_{3,3}$ in G— a contradiction. □

3. Planar Ramsey numbers for cycles

Given two graphs, G and H, call a graph *Ramsey* if it has $R(G, H) - 1$ vertices, contains no copy of G and its complement contains no copy of H. Trivially,

$$\mathrm{PR}(G, H) \leqslant R(G, H),$$

and the two numbers coincide if at least one Ramsey graph is planar. Since this is often the case for pairs of cycles, it is crucial for us to know the values of $R(C_m, C_n)$. Fortunately, the Ramsey numbers for cycles have been determined already long time ago [6,9,10]: the numbers are provided by the following formula which depends strongly on the parity of the parameters:

$$R(C_m, C_n) = \begin{cases} 6 & \text{for } m = n \in \{3, 4\}, \\ 2n - 1 & \text{for } m \text{ odd and } (m, n) \neq (3, 3), \\ \max\left\{n + \dfrac{m}{2}, 2m\right\} - 1 & \text{for } m \text{ even and } n \text{ odd}, \\ n + \dfrac{m}{2} - 1 & \text{otherwise.} \end{cases}$$

Using the results from the previous section, the above formulas and some known facts about planar graphs, we are now in position to determine all planar Ramsey numbers for cycles.

Theorem 6. *The planar Ramsey numbers* $\mathrm{PR}(C_m, C_n)$ *are as shown in Table 1.*

Proof. The lower bounds follow by considering the Ramsey graphs listed in Table 2 (Fig. 2) and noticing that they are all planar. For the upper bounds we argue as follows. The cases we consider correspond to respective

Table 1
The planar Ramsey numbers for cycles

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th></th></tr>
<tr><td>3</td><td>6</td><td>7</td><td colspan="6">n + 2</td><td>(a)</td></tr>
<tr><td>4</td><td>7</td><td>6</td><td>7</td><td colspan="5">n + 1</td><td>(b)</td></tr>
<tr><td>5</td><td rowspan="6">9
(d)</td><td>7</td><td rowspan="6">9
(f)</td><td>8</td><td colspan="5" rowspan="6">(c)
n + 2</td></tr>
<tr><td>6</td><td>7</td><td>8</td></tr>
<tr><td>7</td><td rowspan="4">8
(e)</td><td rowspan="4">9
(g)</td></tr>
<tr><td>8</td></tr>
<tr><td>9</td></tr>
<tr><td>10</td></tr>
</table>

Table 2
Planar graphs with no C_m and no C_n in the complement

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th></tr>
<tr><td>3</td><td>C_5</td><td>$K_{3,3}-e$</td><td colspan="6">$K_{2,n-1}$</td></tr>
<tr><td>4</td><td>$2K_3+e$</td><td>C_5</td><td>$2K_3+e$</td><td colspan="5">$K_{1,n-1}$</td></tr>
<tr><td>5</td><td rowspan="6">$2K_4$</td><td>$K_{3,3}-e$</td><td rowspan="6">$2K_4$</td><td>$K_{2,5}$</td><td colspan="4" rowspan="6">$K_{2,n-1}$</td></tr>
<tr><td>6</td><td>$K_1\cup(K_5-e)$</td><td>$K_{2,5}$</td></tr>
<tr><td>7</td><td rowspan="4">$K_1\cup H$ (Fig.2)</td><td rowspan="4">G (Fig.1)</td></tr>
<tr><td>8</td></tr>
<tr><td>9</td></tr>
<tr><td>10</td></tr>
</table>

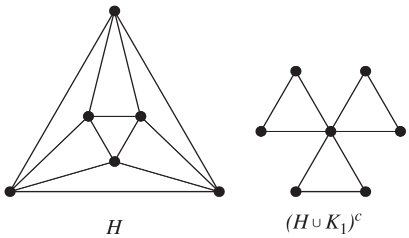

Fig. 2. A graph H and a complement of $K_1 \cup H$.

zones in Table 1.

(a) For $m = 3$ and $n \geqslant 5$ we have $\mathrm{PR}(C_3, C_n) \leqslant n + 2$ by Theorem 1.
(b) For $m = 4$ and $n \geqslant 6$ we have $\mathrm{PR}(C_4, C_n) \leqslant R(C_4, C_n) = n + 1$.
(c) For $m \geqslant 5$ and $n \geqslant 7$ we have $\mathrm{PR}(C_m, C_n) \leqslant n + 2$ by Theorem 4.
(d) For $m \geqslant 5$ and $n = 3$, by the Four Color Theorem and the Pigeon-hole Principle, the complement of every planar graph on nine vertices contains a triangle.
(e) For $m \geqslant 7$ and $n = 4$ we have $\mathrm{PR}(C_m, C_4) \leqslant 8$ by Theorem 2.
(f) For $m \geqslant 5$ and $n = 5$ we have $\mathrm{PR}(C_m, C_5) \leqslant 9$ by Fact 1(i).
(g) For $m \geqslant 5$ and $n = 6$ we have $\mathrm{PR}(C_m, C_6) \leqslant 9$ by Fact 1(i).

Now consider the nine remaining numbers not covered by the cases a–g. For $(m, n) = (3, 3), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (6, 4), (6, 6)$ the upper bounds on $\mathrm{PR}(C_m, C_n)$ follow from the inequality $\mathrm{PR}(C_m, C_n) \leqslant R(C_m, C_n)$. Finally, $\mathrm{PR}(C_5, C_6) \leqslant 8$ by Fact 1(ii). □

4. Remarks

Remark 1. Some theorems presented in Table 1 can be formulated more generally with the first cycle replaced by a graph not contained in a planar Ramsey graph for this number. For example, for all $m \geqslant 5$ and $n \geqslant 10$ (zone (c)), and for every graph G containing C_m, we also have $\mathrm{PR}(G, C_n) = n + 2$, simply because $K_{2,n-1}$ remains to serve as a Ramsey graph in this case. As another example, take an arbitrary graph G not contained in the union of two copies of K_4. Then we still have $\mathrm{PR}(G, C_3) = \mathrm{PR}(G, C_5) = 9$ for the same reason as above.

Remark 2. As we mentioned in the Introduction, there is, in general, no symmetry in the numbers $\mathrm{PR}(G, H)$. However, for several pairs of integers $m < n$ we do have $\mathrm{PR}(C_m, C_n) = \mathrm{PR}(C_n, C_m)$ (namely, for (3, 4), (3, 7), (4, 5), (4, 6), (4, 7), (5, 7), (6, 7)). Does it happen by coincidence, or all these cases are somehow related? For what other pairs of graphs G, H, we have $\mathrm{PR}(G, H) = \mathrm{PR}(H, G)$?

Remark 3. As we have seen from Theorem 6, quite often the planar Ramsey numbers coincide with the classic Ramsey numbers, including the infinite sequence of numbers $\mathrm{PR}(C_4, C_n) = R(C_4, C_n) = n + 1$ for all $n \geqslant 6$. What do these cases have in common? Can we determine all pairs of graphs G, H for which $\mathrm{PR}(G, H) = R(G, H)$?

Note Added in Proof

For more results on planar Ramsey numbers see also: A. Dudek, A. Rucinski, Planar Ramsey Numbers for Small Graphs, Congr. Numer. 176 (2005) 201–220.

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