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A bijection between 2-triangulations and pairs of non-crossing Dyck paths

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Abstract

A k-triangulation of a convex polygon is a maximal set of diagonals so that no $k+1$ of them mutually cross in their interiors. We present a bijection between 2-triangulations of a convex n-gon and pairs of non-crossing Dyck paths of length $2(n-4)$. This solves the problem of finding a bijective proof of a result of Jonsson for the case $k=2$. We obtain the bijection by constructing isomorphic generating trees for the sets of 2-triangulations and pairs of non-crossing Dyck paths.

Keywords: Generalized triangulation; Dyck path; Bijection; Crossing

1. Introduction

A triangulation of a convex n-gon can be defined as a maximal set of diagonals so that no two of them intersect in their interiors. It is well known that the number of triangulations of a convex n-gon is the Catalan number $C_{n-2} = \frac{1}{n-1}\binom{2(n-2)}{n-2}$, and that all such triangulations have $n-3$ diagonals (not counting the n sides of the polygon as diagonals).

We say that two diagonals *cross* if they intersect in their interiors. Define an m-crossing to be a set of m diagonals where any two of them mutually cross. A natural way to generalize a triangulation is to allow diagonals to cross, but to forbid m-crossings for some fixed m. For any positive integer k, define a *k-triangulation* to be a maximal set of diagonals not containing any $(k+1)$-crossing. For example, a 1-triangulation is just a triangulation in the standard sense. Generalized triangulations appear in [2,5,6,8,11]. It was shown in [5,11] that all k-triangulations

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of a convex n-gon have the same number of diagonals. Counting also the n sides of the polygon, the total number of diagonals and sides in a k-triangulation is always $k(2n-2k-1)$.

Jacob Jonsson [8] enumerated k-triangulations of a convex n-gon, proving the following remarkable result.

Theorem 1. *The number of k-triangulations of a convex n-gon is equal to the determinant*

$$\det(C_{n-i-j})_{i,j=1}^{k} = \begin{vmatrix} C_{n-2} & C_{n-3} & \dots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \dots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \dots & C_{n-2k+1} & C_{n-2k} \end{vmatrix}, \tag{1}$$

where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number.

On the other hand, it can be shown [4] using the lattice path determinant formula of Lindström [10], Gessel and Viennot [7] that this determinant counts certain fans of non-crossing lattice paths. Indeed, recall that *Dyck path* can be defined as a lattice path with north steps $N=(0,1)$ and east steps $E=(1,0)$ from the origin $(0,0)$ to a point (m,m), with the property that it never goes below the diagonal $y=x$. We say that m is the *size* or *semilength* of the path. The number of k-tuples $(P_1, P_2, \dots, P_k)$ of Dyck paths from $(0,0)$ to $(n-2k, n-2k)$ such that each P_i never goes below P_{i+1} is given by the same determinant (1).

In the case $k=1$, this determinant is just C_{n-2}, which counts Dyck paths from $(0,0)$ to $(n-2, n-2)$. There are several simple bijections between triangulations of a convex n-gon and such paths (see for example [12, Problem 6.19]). However, for $k \geqslant 2$, the problem becomes more complicated. One of the main open questions left in [8], stated also in [9, Problem 1], is to find a bijection between k-triangulations and k-tuples of non-crossing Dyck paths, for general k. In this paper we solve this problem for $k=2$, that is, we find a bijection between 2-triangulations of a convex n-gon and pairs (P, Q) of Dyck paths from $(0,0)$ to $(n-4, n-4)$ so that P never goes below Q.

In Section 2 we present the bijection explicitly. In Section 3 we describe a generating tree for 2-triangulations, and in Section 4 we give a generating tree for pairs of non-crossing Dyck paths. In Section 5 we show that these two generating trees are isomorphic, and that our bijection maps each node of one tree to the corresponding node in the other. In Section 6 we discuss possible generalizations of our results to arbitrary k.

1.1. Notation

From now on, the term n-gon will refer to a convex n-gon, which can be assumed to be regular. We label its vertices clockwise with the integers from 1 to n. For any $n > 2k > 0$, let $\mathcal{T}_n^{(k)}$ denote the set of k-triangulations of an n-gon. Let $\mathcal{D}_m^{(k)}$ denote the set of k-tuples $(P_1, P_2, \dots, P_k)$ of Dyck paths from $(0,0)$ to (m,m) such that P_i never goes below P_{i+1} for $1 \leqslant i \leqslant k-1$.

Given n points labeled $1, 2, \dots, n$, a segment connecting a and b (with $a < b$) can be associated to the square (a,b) in an $n \times n$ board with rows indexed increasingly from top to bottom and columns from left to right. A collection of segments connecting some of the points can then be represented as a subset of the squares of the triangular array $\Omega_n = \{(a,b)\colon 1 \leqslant a < b \leqslant n\}$, as it was done in [8]. If the points are the vertices of an n-gon labeled clockwise, then the squares $(a, a+1)$, for $1 \leqslant a \leqslant n-1$, and $(1,n)$ correspond to the sides of the polygon. The remaining

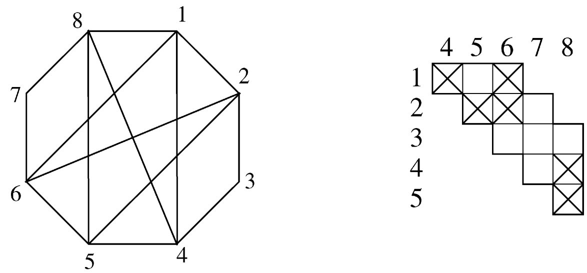

Fig. 1. A 2-triangulation of an octagon and its representation as a subset of Λ_8.

squares of Ω_n correspond to diagonals. The diagonal connecting two vertices a and b will be denoted (a,b).

It is easy to check (see for example [8]) that t diagonals $(a_1,b_1),\dots,(a_t,b_t)$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_t$ and $a_i < b_i$ for all i form a t-crossing if and only if $a_1 < a_2 < \cdots < a_t < b_1 < b_2 < \cdots < b_t$. The condition that $a_t < b_1$ can be replaced with the condition that the smallest rectangle containing the t squares (a_i,b_i), $1 \leqslant i \leqslant t$, fits inside Ω_n.

Note that the diagonals joining two vertices that have less than k vertices in between them can never be part of a k-crossing. We will call these *trivial diagonals*. They are those of the form $(a,a+j)$ (or $(a+j-n,a)$ if $a+j>n$), for $2 \leqslant j \leqslant k$, $1 \leqslant a \leqslant n$. Any k-triangulation of the polygon contains all these diagonals. For simplicity, we will ignore trivial diagonals. Deleting from Ω_n the squares corresponding to trivial diagonals and to the sides of the polygon, we get the shape $\Lambda_n^{(k)} = \{(a,b)\colon\ 1 \leqslant a < b-k \leqslant n-k,\ a > b-n+k\}$. We will represent k-triangulations as subsets of the squares of $\Lambda_n^{(k)}$. We will draw a cross in a square to indicate that the corresponding diagonal belongs to the k-triangulation. The number of crosses is then precisely $k(n-2k-1)$, since that is the number of diagonals of a k-triangulation after the superfluous ones have been omitted [5]. See Fig. 1 for an example of a 2-triangulation of an octagon, where the trivial diagonals have been omitted. To simplify notation, $\Lambda_n^{(2)}$ will be denoted Λ_n.

2. The bijection

In this section we give a bijection Ψ between 2-triangulations of an n-gon and pairs (P,Q) of Dyck paths from $(0,0)$ to $(n-4,n-4)$ so that P never goes below Q. We assume that $n \geqslant 5$.

Let $T \in \mathcal{T}_n^{(2)}$ be a 2-triangulation of an n-gon. The number of diagonals, not counting the trivial ones (which are present in any 2-triangulation) is $2n-10$. We represent T by placing $2n-10$ crosses in Λ_n. Index the columns of Λ_n from 4 to n, so that the leftmost column is called "column 4," and index the rows from 1 to $n-3$. This way, a cross in row a and column b corresponds to the diagonal (a,b).

In the first part of the bijection we will color half of these crosses blue and the other half red. Along the process, some adjacent columns of Λ_n will be merged. We use the term *block* to refer to a column or to a set of adjacent columns that have been merged. Blocks are ordered from left to right, so that "block j" refers to the one that has $j-1$ blocks to its left. At the beginning there are $n-3$ blocks, and block j contains only column $j+3$, for $j=1,\dots,n-3$ (see Fig. 2). Next we describe an iterative step that will be repeated $n-5$ times. At each iteration one cross will be colored blue, another one red, and two blocks will be merged into one. At the end, all $2n-10$ crosses will be colored, and there will be only 2 blocks.

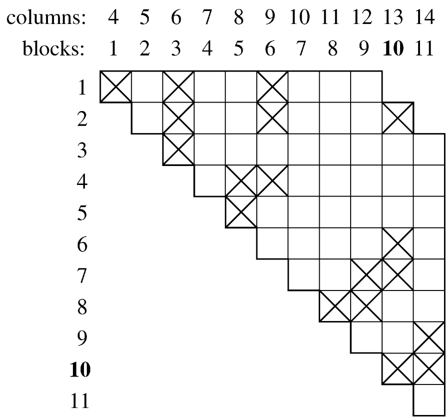

Fig. 2. A 2-triangulation of a 14-gon, with $r = 10$.

Here is the part that is iterated:

- Let r be the largest index so that row r has a cross in block r.
- Color blue the leftmost uncolored cross in block r (in case of a tie pick the lowest one).
- Merge blocks $r-2$ and $r-1$ (if $r = 2$, we consider that block 1 disappears when it is merged with "block 0").
- Color red the rightmost uncolored cross in the merged block (in case of a tie pick the highest one).

The choice of cross in case of a tie is irrelevant. The reason is that the second part of the bijection, which will construct a pair of Dyck paths from the colored diagram of crosses, only takes into account the number of red and blue crosses in each column, but not what color each particular cross has.

Let us see how crosses are colored in an example. Consider the 2-triangulation of a 14-gon shown in Fig. 2. In the following pictures, red crosses will be drawn with a circle around them, and blue crosses will be drawn as a star. At the beginning there are 11 blocks, and $r = 10$. In the first iteration, a cross in block 10 is colored blue, a cross in block 9 is colored red, and blocks 8 and 9 are merged into one block, leaving us with Fig. 3(a). In the second iteration, we have again $r = 10$. A cross in block 10 is colored blue, blocks 8 and 9 are merged, and the leftmost uncolored cross in the merged block is colored red, as shown in Fig. 3(b). In the third iteration, $r = 9$, and we get Fig. 3(c). In the fourth iteration, $r = 7$, so blocks 5 and 6 are merged, giving Fig. 3(d). Next, $r = 6$, and blocks 4 and 5 are merged. In the sixth iteration, $r = 4$, and we get Fig. 3(f). In the next step, $r = 2$, so block 1 disappears and the cross that it contained is colored red (see Fig. 3(g)). In the last two iterations, $r = 2$ again, and we end with Fig. 3(i), where all the crosses have been colored.

In the second part of the bijection we construct a pair of non-crossing Dyck paths out of the colored diagram of crosses. For $j = 4, \dots, n$, let α_j (respectively β_j) be the number of blue (respectively red) crosses in column j of Λ_n. Let

$$P = NE^{\alpha_5}NE^{\alpha_6}\cdots NE^{\alpha_{n-1}}NE^{\alpha_n}E,$$
$$Q = NE^{\beta_4}NE^{\beta_5}\cdots NE^{\beta_{n-2}}NE^{\beta_{n-1}}E,$$

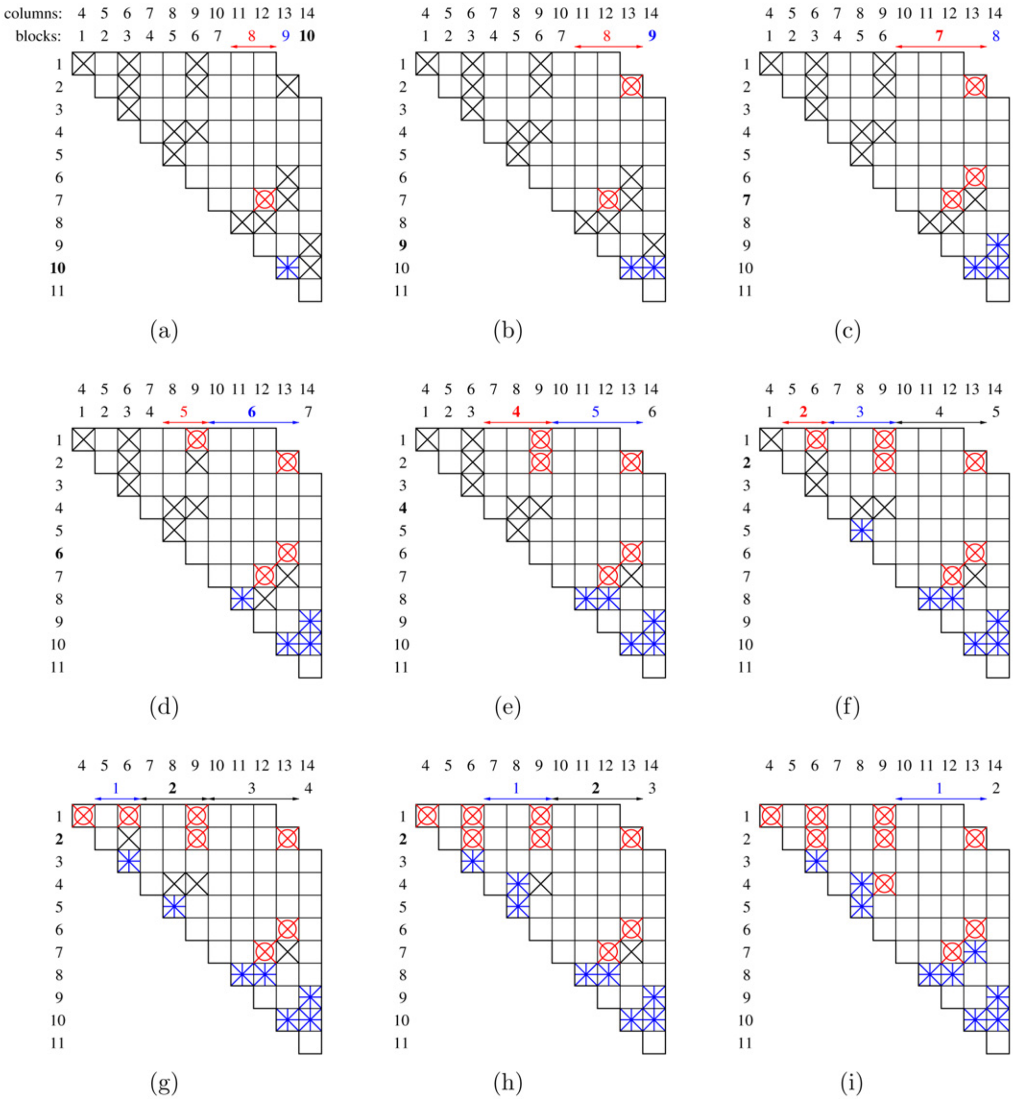

Fig. 3. An example of the coloring algorithm.

where N and E are steps north and east, and exponentiation indicates repetition of a step. We claim that P and Q are Dyck paths from $(0,0)$ to $(n-4, n-4)$, and that P never goes below Q. We define $\Psi(T) = (P, Q)$.

For example, if T is the 2-triangulation from Fig. 2, we get from Fig. 3(i) that

$$P = NNENNEENNNENENENEENEEE,$$

$$Q = NENNEENNNEEENNNENEEE.$$

These paths are drawn in Fig. 4.

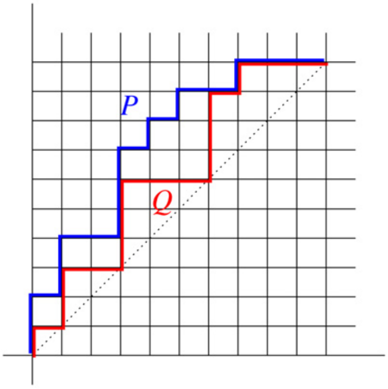

Fig. 4. The pair $\Psi(T) = (P, Q)$, where T is the 2-triangulation from Fig. 2.

We claim that at each step of the coloring algorithm there is always a cross to be colored red and a cross to be colored blue in the appropriate blocks, so all crosses get colored at the end. We have also stated that P and Q are non-crossing Dyck paths. Finally, we claim that Ψ is in fact a bijection between $\mathcal{T}_n^{(2)}$ and $\mathcal{D}_{n-4}^{(2)}$. We will justify these assertions in the next three sections, by giving more insight on the bijection. The idea is to construct isomorphic generating trees for the set of 2-triangulations and the set of pairs of non-crossing Dyck paths. The natural isomorphism between the two generating trees determines Ψ.

3. A generating tree for 2-triangulations

In this section we describe a generating tree where nodes at level ℓ correspond to 2-triangulations of an $(\ell + 5)$-gon. The root of the tree is the only 2-triangulation of a pentagon, which has no diagonals.

In the rest of this paper, when we refer to a 2-triangulation we will not consider the trivial diagonals. In particular, all 2-triangulations of an n-gon have $2n - 10$ diagonals. The *degree* of a vertex is the number of (non-trivial) diagonals that have it as an endpoint. The degree of a is denoted $\deg(a)$.

3.1. The parent of a 2-triangulation

To describe the generating tree, we specify the parent of any given 2-triangulation of an n-gon, where $n \geqslant 6$. For this purpose we need a few simple lemmas.

Lemma 2. *Let $T \in \mathcal{T}_n^{(2)}$ be a 2-triangulation containing the diagonal (a, b), with $a < b - 3$. Then T contains the diagonal $(a, b - 1)$ or a diagonal of the form (a', b) with $a < a' \leqslant b - 3$.*

Proof. Assume that $(a, b - 1)$ is not in T. Then, since T is a maximal set of diagonals with no 3-crossings, adding the diagonal $(a, b-1)$ would create a 3-crossing together with two diagonals in T. But these two diagonals together with (a, b) do not form a 3-crossing. This means that at least one of these two diagonals crosses $(a, b - 1)$ but not (a, b). This can only happen if such a diagonal is of the form (a', b) with $a < a' \leqslant b - 3$. $\square$

Lemma 3. *Let $T \in \mathcal{T}_n^{(2)}$ be a 2-triangulation containing the diagonal (a, b), with $a \leqslant b - 3$. Then there exists a vertex $i \in \{a, \dots, b-3\}$ such that T contains the diagonal $(i, i+3)$.*

Proof. If follows easily by iterating Lemma 2. □

Lemma 4. *Assume that $n \geqslant 6$, and consider the labels of the vertices to be taken modulo n (for example, vertex $n+1$ would be vertex 1). Let $T \in \mathcal{T}_n^{(2)}$ be a 2-triangulation that does not contain the diagonal $(a, a+3)$. Then the degrees of the vertices $a+1$ and $a+2$ are both non-zero.*

Proof. Since T is a maximal set of diagonals with no 3-crossing, adding the diagonal $(a, a+3)$ would create a 3-crossing. This can only happen if in T there is a diagonal with endpoint $a+1$ and another diagonal with endpoint $a+2$ that cross. □

Lemma 5. *Assume that $n \geqslant 6$, and consider the labels of the vertices to be taken modulo n. Let $T \in \mathcal{T}_n^{(2)}$ be a 2-triangulation and let a be a vertex whose degree is 0. Then T contains the diagonals $(a-2, a+1)$ and $(a-1, a+2)$.*

Proof. If $(a-2, a+1)$ was not in T, then by Lemma 4 the degree of a would be non-zero. Similarly if $(a-1, a+2)$ was not in T. □

Now we can define the parent of any given 2-triangulation. Let $n \geqslant 6$, and let T be a 2-triangulation of an n-gon. Let r be the largest number with $1 \leqslant r \leqslant n-3$ such that T contains the diagonal $(r, r+3)$. This number $r = r(T)$ will be called the *corner* of T. Diagonals of the form $(i, i+3)$ will be called *short diagonals*.

Let us note look at some useful properties of T. First, note that T does not contain any diagonals of the form (a, b) with $r < a \leqslant b - 3 \leqslant n - 3$, since otherwise, by Lemma 3, there would be a short diagonal contradicting the choice of r. In particular, T has no diagonals of the form $(r+1, b)$ or $(r+2, b)$ with $r+4 \leqslant b \leqslant n$. We also have that $r \geqslant 2$. Indeed, if $r = 1$ then all the diagonals would have to be of the form $(1, b)$, but there can only be $n-5$ such diagonals, which is half of the number needed in a 2-triangulation. There are three possibilities for the degrees of the vertices $r+1$ and $r+2$.

If the degree of $r+2$ is zero, then by Lemma 5 the diagonal $(r+1, r+4)$ belongs to T. In this case we have necessarily that $n = r+3$, in order not to contradict the choice of r, and this diagonal is in fact $(1, r+1)$.

If the degree of $r+1$ is zero, again by Lemma 5 we have that $(r-1, r+2)$ belongs to T.

If the degrees of $r+1$ and $r+2$ are both non-zero, let i be the smallest index so that the diagonal $(i, r+1)$ belongs to T, and let j be the largest index so that the diagonal $(j, r+2)$ belongs to T. By the previous reasoning, we know that $i, j < r$. It is also clear that $j \leqslant i$, since otherwise the diagonals $(i, r+1)$, $(j, r+2)$ and $(r, r+3)$ would form a 3-crossing. We claim that in fact $i = j$. Indeed, by Lemma 2 applied to the diagonal $(j, r+2)$, we have that either $(j, r+1)$ belongs to T, in which case $i \leqslant j$ by the choice of i, or there is a diagonal in T of the form $(j', r+2)$ with $j < j'$, which would contradict the choice of j.

With these properties in mind, we define the parent of T in the generating tree to be the 2-triangulation $p(T) \in \mathcal{T}_{n-1}^{(2)}$ obtained as follows:

- Delete the diagonal $(r, r+3)$ from T (recall that $r := \max\{a\colon 1 \leqslant a \leqslant n-3,\ (r, r+3) \in T\}$).
- If $\deg(r+1) = 0$, delete the diagonal $(r-1, r+2)$;

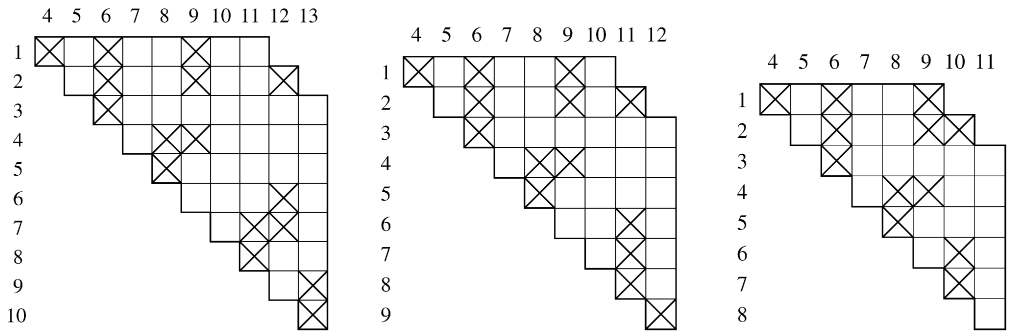

Fig. 5. From left to right, the parent, the grandparent, and the great grandparent of the 2-triangulation from Fig. 2.

if $\deg(r+2)=0$ (in which case $r=n-3$), delete the diagonal $(1,r+1)$;
if $\deg(r+1)>0$ and $\deg(r+2)>0$, delete the diagonal $(j,r+2)$, where $j:=\max\{a\colon 1\leqslant a<r,\ (a,r+2)\in T\}$ (in this case we also have $j=\min\{a\colon 1\leqslant a<r,\ (a,r+1)\in T\}$).

- Contract the side $(r+1,r+2)$ of the polygon (that is, move all the diagonals from $r+2$ to $r+1$, delete the vertex $r+2$, and decrease by one the labels of the vertices $b>r+2$).

It is clear that $p(T)$ contains no 3-crossings, because it has been obtained from T by deleting diagonals. Also, by the above reasoning, $p(T)$ has exactly 2 diagonals less than T. Therefore, $p(T)$ is a 2-triangulation of an $(n-1)$-gon.

It will be convenient to give an equivalent description of $p(T)$ in terms of diagrams of 2-triangulations. Consider the representation of T as a subset of Λ_n. Next we describe how the diagram of $p(T)$ as a subset of Λ_{n-1} is obtained from it. Observe that if r is the corner of T, then the diagram of T has no crosses below row r, because crosses in squares (a,b) with $r<a\leqslant b-3\leqslant n-3$ would contradict the choice of r, by Lemma 3. To obtain the diagram of $p(T)$, first delete all the squares $(a,a+3)$ for $a=r-1,r,\dots,n-3$. (Note that aside from $(r,r+3)$, the only square among these where there may be a cross is $(r-1,r+2)$, and if this cross is present, then column $r+1$ is empty.) Next we merge columns $r+1$ and $r+2$. We do this so that the new merged column, which will be the new column $r+1$, has a cross in those rows where either the old column $r+1$ or $r+2$ (or both) had a cross. (Note that there is at most one row where both columns had a cross.) This yields the diagram of $p(T)$ as a subset of Λ_{n-1}. For example, if T is the 2-triangulation from Fig. 2, then $p(T)$, $p(p(T))$ and $p(p(p(T)))$ are shown in Fig. 5.

Note that in the bijection Ψ defined in Section 2, the iterated step that merges blocks $r-2$ and $r-1$ corresponds to moving up one level in this generating tree of 2-triangulations. At each iteration, if $n'-3$ is the current number of blocks, this indicates that we have moved up in the tree to a 2-triangulation T' of an n'-gon. Then, for $1\leqslant a\leqslant b\leqslant n'-3$, a cross in row a and block b indicates that the diagonal $(a,b+3)$ is present in T'. The largest r such that there is a cross in row r and block r is precisely the corner of T'. Merging blocks $r-2$ and $r-1$ in the original diagram is equivalent to merging columns $r+1$ and $r+2$ in T'.

3.2. *The children of a 2-triangulation*

Even though the generating tree is already completely specified by the above subsection, it will be useful to characterize the children of a given 2-triangulation $T\in\mathcal{T}_n^{(2)}$ in the tree. By definition,

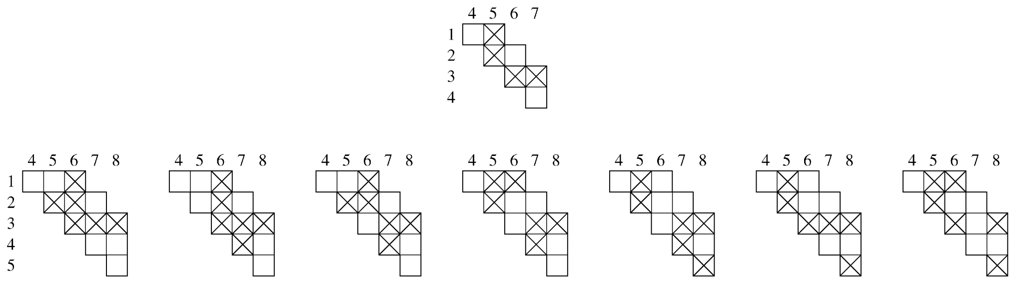

Fig. 6. A 2-triangulation of a heptagon and its 7 children in the generating tree.

the children are all those elements $\widehat{T} \in \mathcal{T}_{n+1}^{(2)}$ such that $p(\widehat{T}) = T$. Again, let $r \in \{1, 2, \ldots, n-3\}$ be the corner of T. Equivalently, r is the largest index of a non-empty row in the diagram of T. Note that for any child $\widehat{T}$ of T, if $\hat{r}$ is the corner of $\widehat{T}$, one must have $\hat{r} \geqslant r$. It is not hard to check that all the children of T are obtained in the following way:

- Choose a number $u \in \{r, \ldots, n-2\}$.
- Add one to the labels of the columns j with $u+2 \leqslant j \leqslant n$.
- Add the square $(u, u+3)$ with a cross in it, and add empty squares $(j, j+3)$ for $j = u+1, \ldots, n-2$.
- Split column $u+1$ into two columns labeled $u+1$ and $u+2$ as follows:
  (1) Let $(a_1, u+1), \ldots, (a_h, u+1)$ be the crosses in column $u+1$ (assume that $a_1 > \cdots > a_h$). Choose a number $i \in \{0, 1, \ldots, h\}$. If $u = n-2$, there is an additional available choice $i = h+1$; if this is chosen, skip to (5) below.
  (2) Leave the crosses $(a_1, u+1), \ldots, (a_i, u+1)$ in column $u+1$.
  (3) Add a cross in position $(a_i, u+2)$ if $i > 0$, or in position $(u-1, u+2)$ if $i = 0$.
  (4) Move the crosses $(a_{i+1}, u+1), \ldots, (a_h, u+1)$ to $(a_{i+1}, u+2), \ldots, (a_h, u+2)$.
  (5) In the special case that $u = n-2$ and that $i = h+1$ has been chosen, column $u+1$ is split by leaving all the crosses $(a_1, u+1), \ldots, (a_h, u+1)$ in it, adding a new cross $(1, u+1)$, and leaving column $u+2$ empty.

Each choice of u and i gives rise to a different child of T. Note that each choice of u generates those children with $\hat{r} = u$. Figure 6 shows a 2-triangulation and its seven children, of which one is obtained with $u = 3$, three with $u = 4$, and three with $u = 5$. It follows from the above characterization that the total number of children of T is

$$(h_{r+1} + 1) + (h_{r+2} + 1) + \cdots + (h_{n-1} + 1) + 1 = h_{r+1} + h_{r+2} + \cdots + h_{n-1} + n - r,$$

where, for $r < j < n$, h_j is the number of crosses in column j of the diagram of T. This observation allows us to easily describe the generating tree for 2-triangulations by labeling the nodes with the list of numbers $(h_{r+1}, \ldots, h_{n-1})$. For each chosen $u \in \{r, \ldots, n-2\}$, the h_{u+1} crosses in column $u+1$ can be split into two columns for each choice of i. We have proved the following result.

Proposition 6. *The generating tree described above for the set $\mathcal{T}^{(2)}$ is isomorphic to the tree with root labeled* $(0, 0)$ *and with generating rule*

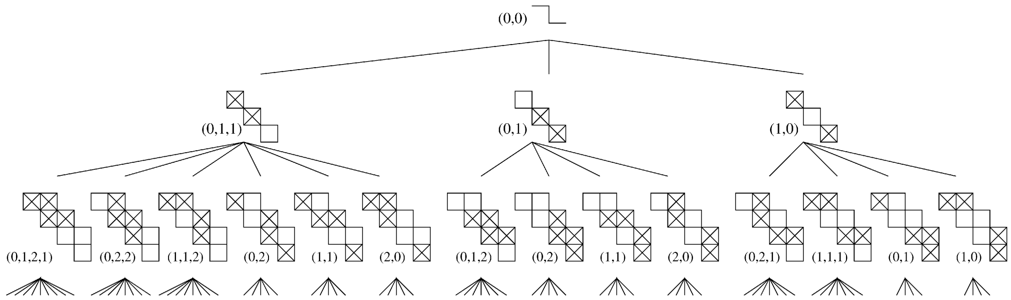

Fig. 7. The first levels of the generating tree for 2-triangulations.

$$(d_1, d_2, \ldots, d_s) \longrightarrow \bigl\{(i, d_j - i + 1, d_{j+1} + 1, d_{j+2}, \ldots, d_s)\colon 1 \leqslant j \leqslant s-1,\ 0 \leqslant i \leqslant d_j\bigr\} \\ \cup \bigl\{(i, d_s - i + 1)\colon 0 \leqslant i \leqslant d_s + 1\bigr\}.$$

For example, the children of a node labeled $(0, 1, 3, 2)$ have labels $(0, 1, 2, 3, 2)$, $(0, 2, 4, 2)$, $(1, 1, 4, 2)$, $(0, 4, 3)$, $(1, 3, 3)$, $(2, 2, 3)$, $(3, 1, 3)$, $(0, 3)$, $(1, 2)$, $(2, 1)$, and $(3, 0)$. In Fig. 6, the parent has label $(0, 2, 1)$ and the children, from left to right, are labeled $(0, 1, 3, 1)$, $(0, 3, 2)$, $(1, 2, 2)$, $(2, 1, 2)$, $(0, 2)$, $(1, 1)$, and $(2, 0)$. The first levels of the generating tree for $\mathcal{T}^{(2)}$ with their labels are drawn in Fig. 7.

4. A generating tree for pairs of non-crossing Dyck paths

In this section we define a generating tree for $\mathcal{D}^{(2)}$, where nodes at level ℓ correspond to pairs of Dyck paths of size $\ell + 1$ such that the first never goes below the second, and we show that it is isomorphic to the generating tree from Proposition 6. The root of our tree is the pair (P, Q), where $P = Q = NE$.

Every Dyck path P of size m can be expressed uniquely as

$$P = NE^{p_m} NE^{p_{m-1}} \cdots NE^{p_2} NE^{p_1} E$$

for some non-negative integers p_i. The sequence $(p_1, p_2, \ldots, p_m)$ determines the path, and it must satisfy $p_1 + p_2 + \cdots + p_t \geqslant t - 1$ for all $1 \leqslant t \leqslant m$, and $p_1 + p_2 + \cdots + p_m = m - 1$. Given a pair $(P, Q) \in \mathcal{D}_m^{(2)}$, we will write P as above, and Q as

$$Q = NE^{q_m} NE^{q_{m-1}} \cdots NE^{q_2} NE^{q_1} E.$$

We set $p_{m+2} = p_{m+1} = q_{m+1} = 0$ by convention. It will be convenient to encode the pair (P, Q) by the matrix

$$[P, Q] := \begin{bmatrix} p_{m+2} & p_{m+1} & p_m & p_{m-1} & \cdots & p_3 & p_2 & p_1 \\ q_{m+1} & q_m & q_{m-1} & q_{m-2} & \cdots & q_2 & q_1 & 0 \end{bmatrix}.$$

The leftmost column has zero entries, so it is superfluous, but it will make the notation easier later on. The condition that P never goes below Q is equivalent to the fact that for any $t \in \{1, \ldots, m\}$, $p_1 + p_2 + \cdots + p_t \geqslant q_1 + q_2 + \cdots + q_t$. We will write $p_j(P, Q)$ and $q_j(P, Q)$ when we want to emphasize that these are parameters of the pair (P, Q). We define

$$s = s(P, Q) = \min\{j \geqslant 2\colon\ p_j q_j = 0\}.$$

In terms of the paths, $s-2$ is the minimum number of N steps following the last occurrence of NN in P or Q (defining $s=m+1$ if there is no NN). In particular, $2\leqslant s\leqslant m+1$. For example, the encoding of the pair (P,Q) of paths in Fig. 4 is

$$[P,Q]=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 2 & 2\\ 0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 & 2 & 0\end{bmatrix}$$

and $s(P,Q)=3$.

The parent of (P,Q) in the generating tree is defined to be the pair $(P',Q')\in\mathcal{D}^{(2)}_{m-1}$ whose encoding is

$$[P',Q']:=\begin{bmatrix} p_{m+2} & p_{m+1} & p_m & \cdots & p_{s+2} & p_{s+1}+p_s & p_{s-1}-1 & p_{s-2} & \cdots & p_2 & p_1\\ q_{m+1} & q_m & q_{m-1} & \cdots & q_{s+1} & q_s+q_{s-1}-1 & q_{s-2} & q_{s-3} & \cdots & q_1 & 0\end{bmatrix}.$$

Note that in the case that $s=m+1$, both $[P,Q]$ and $[P',Q']$ have the form

$$\begin{bmatrix} 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & \cdots & 1 & 0 & 0\end{bmatrix}. \tag{2}$$

For example, the parent of the pair of Dyck paths drawn in Fig. 4 is

$$[P',Q']=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 2\\ 0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 2 & 0\end{bmatrix}.$$

Letting $s'=s(P',Q')$, it is clear from the definitions that $s'\geqslant s-1$. Finally, let us check that P' never goes below Q'. If we let $\mu_t=p'_1+\cdots+p'_t$, $\nu_t=q'_1+\cdots+q'_t$, where p'_j and q'_j are the entries in the encoding of (P',Q'), then we need to show that $\mu_t\geqslant\nu_t$ for $1\leqslant t\leqslant m-1$. For $t\leqslant s-2$, this is clear since $\mu_t=p_1+\cdots+p_t\geqslant q_1+\cdots+q_t=\nu_t$. For $t=s-1$, we have that $\mu_{s-1}=p_1+\cdots+p_{s-2}+p_{s-1}-1$ and $\nu_{s-1}=q_1+\cdots+q_{s-2}+q_{s-1}+q_s-1$. We know that by the choice of s, we must have $p_s=0$ or $q_s=0$. If $p_s=0$, then $\mu_{s-1}=p_1+\cdots+p_{s-2}+p_{s-1}-1+p_s\geqslant\nu_{s-1}$, and if $q_s=0$, then $\mu_{s-1}\geqslant q_1+\cdots+q_{s-2}+q_{s-1}-1=\nu_{s-1}$. For $t\geqslant s$, $\mu_t=p_1+\cdots+p_{s-1}-1+p_s+p_{s+1}+\cdots+p_{t+1}\geqslant q_1+\cdots+q_{s-1}+q_s-1+q_{s+1}+\cdots+q_{t+1}=\nu_t$.

The above description completely specifies the generating tree for $\mathcal{D}^{(2)}$. As in the case of 2-triangulations, it will be useful to characterize the children of the pair $(P,Q)\in\mathcal{D}^{(2)}_m$. Let p_j,q_j, for $j=1,\dots,m$, and s be defined as above. The children are the pairs $(\widehat{P},\widehat{Q})\in\mathcal{D}^{(2)}_{m+1}$ whose parent $((\widehat{P})',(\widehat{Q})')$ obtained using the above construction is again (P,Q). Note that if $\hat{s}=s(\widehat{P},\widehat{Q})$, then $\hat{s}\leqslant s+1$. It is easy to check that the children of (P,Q) are the pairs $(\widehat{P},\widehat{Q})$ obtained in the following way.

- Choose a number $t\in\{1,2,\dots,s\}$.
- The following are the encodings of the children of (P,Q):

$$[\widehat{P},\widehat{Q}]=\begin{bmatrix} p_{m+2} & p_{m+1} & \cdots & p_{t+2} & p_{t+1}-i & i & p_t+1 & p_{t-1} & \cdots & p_2 & p_1\\ q_{m+1} & q_m & \cdots & q_{t+1} & 0 & q_t+1 & q_{t-1} & q_{t-2} & \cdots & q_1 & 0\end{bmatrix} \tag{3}$$

for each $i\in\{1,\dots,p_{t+1}\}$,

$$[\widehat{P},\widehat{Q}]=\begin{bmatrix} p_{m+2} & p_{m+1} & \cdots & p_{t+2} & p_{t+1} & 0 & p_t+1 & p_{t-1} & \cdots & p_2 & p_1\\ q_{m+1} & q_m & \cdots & q_{t+1} & 0 & q_t+1 & q_{t-1} & q_{t-2} & \cdots & q_1 & 0\end{bmatrix}, \tag{4}$$

and

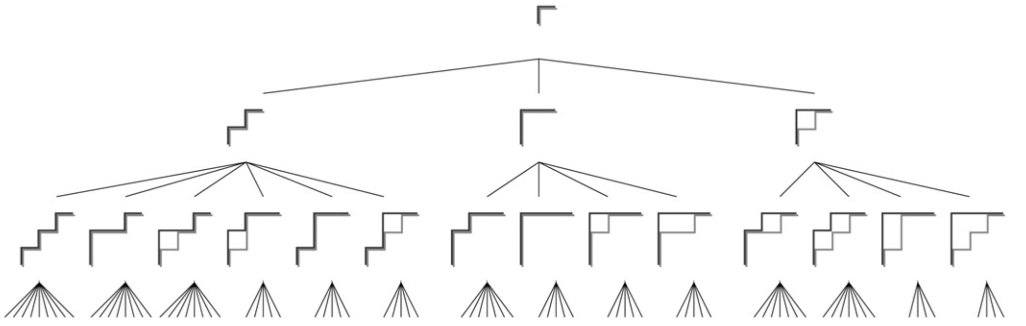

Fig. 8. The first levels of the generating tree for 2-triangulations.

$$[\widehat{P},\widehat{Q}]=\begin{bmatrix} p_{m+2} & p_{m+1} & \cdots & p_{t+2} & p_{t+1} & 0 & p_t+1 & p_{t-1} & \cdots & p_2 & p_1 \\ q_{m+1} & q_m & \cdots & q_{t+1} & j & q_t-j+1 & q_{t-1} & q_{t-2} & \cdots & q_1 & 0 \end{bmatrix} \tag{5}$$

for each $j \in \{1,\dots,q_t\}$ if $t \geqslant 2$, or $j \in \{1,\dots,q_t+1\}$ if $t=1$.

Essentially, to obtain the parent we add up two columns (determined by the parameter s) in the encoding, and subtract one to the appropriate entries, whereas to obtain a child we split a column (determined by the choice of $t \leqslant s$) into two, and add one to the appropriate entries. However, this splitting has to be done carefully, because we want each choice of t to generate precisely the children with $\hat{s}=t+1$. This is why when the column of $[P,Q]$ is split into two columns,

$$\begin{matrix} p_{t+1} \\ q_t \end{matrix} \rightsquigarrow \begin{matrix} a & b \\ c & d \end{matrix},$$

we require that either $b=0$ or $c=0$, to force $\hat{s}=t+1$ in the child. The first levels of the generating tree for $\mathcal{D}^{(2)}$ are drawn in Fig. 8.

A convenient way to represent a pair $(P,Q) \in \mathcal{D}_m^{(2)}$ is to shift the paths slightly, drawing P as a path from $(0,1)$ to $(m,m+1)$, which we call $\dot{P}$, and Q as a path from $(1,0)$ to $(m+1,m)$, which we call $\dot{Q}$ (see Fig. 9). The fact that P does not go below Q is equivalent to the fact that $\dot{P}$ and $\dot{Q}$ do not intersect. In the drawing of $\dot{P}$ and $\dot{Q}$, the number of east steps with ordinate j is then $p_{m-j+2}+q_{m-j+1}$ for $j=1,\dots,m-1$; p_2+q_1+1 for $j=m$; and p_1+1 for $j=m+1$.

Columns of $[P,Q]$ correspond to east steps in $\dot{P}$ and $\dot{Q}$ with a fixed ordinate, and the rule that describes the children of a pair (P,Q) has a simple interpretation. The choice of $t \in \{1,2,\dots,s\}$ indicates that level (ordinate) $n+1-t$ in the Dyck paths will be split into two levels by inserting an N step in each of $\dot{P}$ and $\dot{Q}$. Where these N steps can be inserted is given by the rules (3), (4) and (5). Also, an E step is inserted in $\dot{Q}$ at the same level, and in $\dot{P}$ one level higher. Figure 10 gives an example of the possible positions where N and E steps can be inserted to create a child with $t=2$. Additional restrictions apply to where the N steps are added, namely, either the N step in $\dot{Q}$ must be inserted at the leftmost possible position, or the N step in $\dot{P}$ must be inserted at the rightmost possible place.

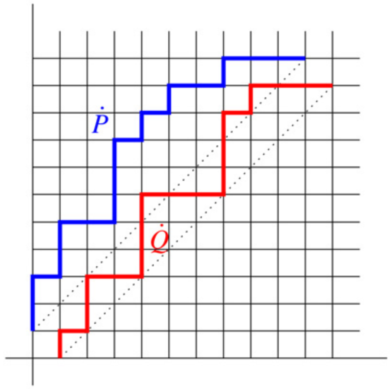

Fig. 9. The paths $\dot{P}$ and $\dot{Q}$, where (P, Q) are drawn in Fig. 4.

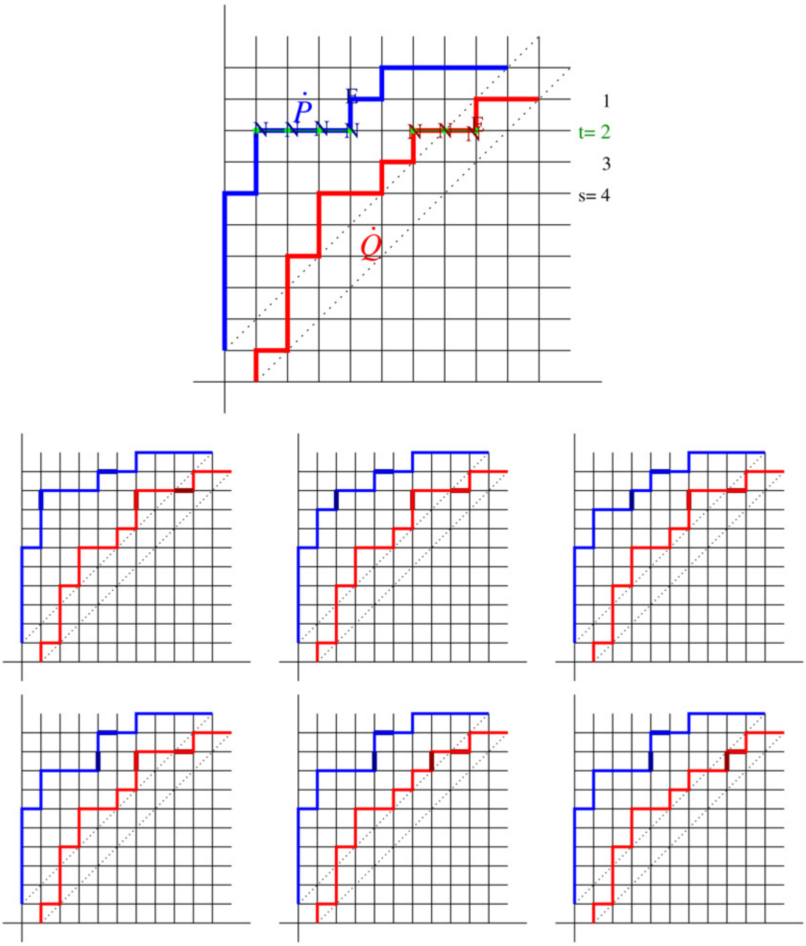

Fig. 10. A pair of paths, possible positions where N and E steps can be inserted in $\dot{P}$ and $\dot{Q}$, and all the children for the choice $t = 2$.

5. Why is Ψ a bijection?

In this section we prove that Ψ is indeed a bijection. We start by showing that the generating tree for pairs of non-crossing Dyck paths from the previous section is the same as the one we constructed for 2-triangulations.

Theorem 7. *The generating tree for $\mathcal{T}^{(2)}$ given in Section* 3 *is isomorphic to the generating tree for $\mathcal{D}^{(2)}$ given in Section* 4.

Proof. For our generating tree for 2-triangulations, Proposition 6 gives a simple description of the generating rule, with an appropriate labeling of the nodes. All we need to show is that we can assign labels to pairs of non-crossing Dyck paths so that our tree for $\mathcal{D}^{(2)}$ obeys the same generating rule.

Given a pair $(P,Q)\in\mathcal{D}^{(2)}_m$, let $p_1,p_2,\dots,p_{m+1},p_{m+2},q_1,q_2,\dots,q_m,q_{m+1}$, and $s=s(P,Q)$ be defined as in Section 4. We define the label associated to the corresponding node of the tree to be

$$(p_{s+1}+q_s,\ p_s+q_{s-1},\dots,p_2+q_1).$$

Note that the root is labeled $(0,0)$.

For each node (P,Q) in the tree for $\mathcal{D}^{(2)}$, each choice of $t\in\{1,2,\dots,s\}$ yields children $(\widehat{P},\widehat{Q})$ with $\hat{s}=s(\widehat{P},\widehat{Q})=t+1$. If $t\geqslant 2$, then the number of children generated by a particular choice of t is $p_{t+1}+q_t+1$, and their labels, according to (3), (4), (5), and the above definition, are

$$\begin{cases} (p_{t+1}-i,\ q_t+i+1,\ p_t+q_{t-1}+1,\ p_{t-1}+q_{t-2},\dots,p_2+q_1) \\ \quad \text{for each } i\in\{1,\dots,p_{t+1}\}, \\ (p_{t+1},\ q_t+1,\ p_t+q_{t-1}+1,\ p_{t-1}+q_{t-2},\dots,p_2+q_1), \quad \text{and} \\ (p_{t+1}+j,\ q_t-j+1,\ p_t+q_{t-1}+1,\ p_{t-1}+q_{t-2},\dots,p_2+q_1) \\ \quad \text{for each } j\in\{1,\dots,q_t\}, \end{cases}$$

or equivalently,

$$(l,\ p_{t+1}+q_t-l+1,\ p_t+q_{t-1}+1,\ p_{t-1}+q_{t-2},\dots,p_2+q_1)$$
$$\text{for each } l\in\{1,\dots,p_{t+1}+q_t\}.$$

Similarly, the choice $t=1$ generates p_2+q_1+2 children, whose labels are

$$(l,\ p_2+q_1-l+1) \quad \text{for each } l\in\{1,\dots,p_2+q_1+1\}.$$

This is clearly equivalent to the generating rule from Proposition 6, so the theorem is proved. □

Note that in the generating trees in the above proof, the labels of the children of any particular node are all different. This uniquely determines an isomorphism of the generating trees, which in turn naturally induces a bijection $\widetilde{\Psi}$ between 2-triangulations of an n-gon and pairs of Dyck paths of size $n-4$ so that the first never goes below the second. Let us analyze some properties of this bijection that follow immediately from the above proof. Consider a 2-triangulation $T\in\mathcal{T}^{(2)}_n$ and its corresponding pair $\widetilde{\Psi}(T)=(P,Q)\in\mathcal{D}^{(2)}_{n-4}$. By induction on n, we see that the parameter r in T and the parameter s in (P,Q) are related by $r+s=n-1$. The value of $u\in\{r,\dots,n-2\}$ chosen to generate a child of T and the value of $t\in\{1,\dots,s\}$ chosen to generate a child of

(P, Q) are related by $u + t = n - 1$. Also, if h_j, for $j = r + 1, \ldots, n - 1$, is defined to be the number of crosses in column j of the diagram of T, and p_j, q_j, for $j = 1, \ldots, n - 2$, are defined as above, then the label $(d_1, \ldots, d_s)$ of the nodes corresponding to T and (P, Q) is

$$(d_1, \ldots, d_s) = (h_{r+1}, \ldots, h_{n-1}) = (p_{s+1} + q_s, p_s + q_{s-1}, \ldots, p_2 + q_1). \tag{6}$$

These observations will be useful when proving Lemma 8 and Proposition 9.

Given a 2-triangulation $T \in \mathcal{T}_n^{(2)}$, in order to compute $\widetilde{\Psi}(T)$ we find the path in the tree from the node corresponding to T to the root, keeping track of the labels of the nodes encountered along the path. Then, starting from the root (NE, NE) in the generating tree for $\mathcal{D}^{(2)}$, these labels determine how to descend in the tree level by level, until we end with a pair (P, Q) of Dyck paths of size $n - 4$, which is $\widetilde{\Psi}(T)$ by definition. In a similar way we can compute the inverse $\widetilde{\Psi}^{-1}((P, Q))$, where $(P, Q) \in \mathcal{D}_m^{(2)}$. Both $\widetilde{\Psi}$ and $\widetilde{\Psi}^{-1}$, as well as the original description of Ψ in terms of the coloring algorithm, have been programmed in *Maple*.

For example, consider $T \in \mathcal{T}_{14}^{(2)}$ to be the 2-triangulation represented in Fig. 2. Its corner is $r = 10$, and the label of the corresponding node in the tree for 2-triangulations is $(1, 2, 4)$, since those are the numbers of crosses in columns 11, 12 and 13, respectively. Its parent, shown in the left of Fig. 5, has $r = 10$ and label $(2, 3)$. Its grandparent, drawn in the middle of Fig. 5, has $r = 9$ and label $(0, 4)$. Its great grandparent has $r = 7$ and label $(2, 3, 3)$. If we continue going up in the generating tree, the next labels that we get are $(0, 4, 2)$, $(0, 3, 3, 1)$, $(0, 1, 2, 2, 1)$, $(0, 1, 2, 1)$, $(0, 1, 1)$, and $(0, 0)$, the last one being the label of the root. To obtain $\widetilde{\Psi}(T)$, we start with the root of the tree for $\mathcal{D}^{(2)}$, whose encoding is $\left[\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right]$. Of its three children, the one with label $(0, 1, 1)$ is generated by rule (4) with $t = 2$, and its encoding is $\left[\begin{smallmatrix}0&0&1&0\\0&1&0&0\end{smallmatrix}\right]$. The next node down the tree with label $(0, 1, 2, 1)$ is encoded by $\left[\begin{smallmatrix}0&0&1&1&0\\0&1&1&0&0\end{smallmatrix}\right]$. Its child with label $(0, 1, 2, 2, 1)$ is $\left[\begin{smallmatrix}0&0&1&1&1&0\\0&1&1&1&0&0\end{smallmatrix}\right]$. Rule (3) with $t = 3$ and $i = 1$ generates the next node $\left[\begin{smallmatrix}0&0&0&1&2&1&0\\0&1&0&2&1&0&0\end{smallmatrix}\right]$, with label $(0, 3, 3, 1)$. Again, rule (3) with $t = 2$ and $i = 2$ generates its child $\left[\begin{smallmatrix}0&0&0&1&0&2&2&0\\0&1&0&2&0&2&0&0\end{smallmatrix}\right]$, with label $(0, 4, 2)$. Rule (4) with $t = 2$ generates the next node $\left[\begin{smallmatrix}0&0&0&1&0&2&0&3&0\\0&1&0&2&0&0&3&0&0\end{smallmatrix}\right]$, with label $(2, 3, 3)$. Its child with label $(0, 4)$ is generated using rule (3) with $t = 1$, and it is $\left[\begin{smallmatrix}0&0&0&1&0&2&0&0&3&1\\0&1&0&2&0&0&3&0&1&0\end{smallmatrix}\right]$. Following the path down according to the labels we got, we obtain pairs of Dyck paths whose encodings are $\left[\begin{smallmatrix}0&0&0&1&0&2&0&0&2&1&2\\0&1&0&2&0&0&3&0&0&2&0\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}0&0&0&1&0&2&0&0&1&1&2&2\\0&1&0&2&0&0&3&0&0&1&2&0\end{smallmatrix}\right]$. The last one is by definition the encoding of $\widetilde{\Psi}(T)$, which is the pair in Fig. 4.

Let us now see an example of how the inverse $\widetilde{\Psi}^{-1}$ is computed. Start with the pair of paths (P, Q) on the left of Fig. 11, whose encoding is $[P, Q] = \left[\begin{smallmatrix}0&0&0&1&0&2&1\\0&1&0&1&1&1&0\end{smallmatrix}\right]$. We have that $s(P, Q) = 3$, so the parent is $\left[\begin{smallmatrix}0&0&0&1&1&1\\0&1&0&1&1&0\end{smallmatrix}\right]$. For this pair, $s = 3$ again, so its parent is $\left[\begin{smallmatrix}0&0&1&0&1\\0&1&0&1&0\end{smallmatrix}\right]$. Now $s = 2$, and the pair one level up in the generating tree for $\mathcal{D}^{(2)}$ is $\left[\begin{smallmatrix}0&0&1&0\\0&1&0&0\end{smallmatrix}\right]$. Here $s = 3$, and its parent is the root $\left[\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right]$. To reconstruct the path in the generating tree for $\mathcal{T}^{(2)}$ that goes from the root to $\widetilde{\Psi}^{-1}(P, Q)$, we use the fact that the values of r and s at the corresponding nodes at each level satisfy $r + s = n - 1$. The child of the root that we take is the one with $r = 2$, and the next values of r are 4, 4, 5. How to split column $r + 2$ when we go down a level is determined by the encodings of the pairs of Dyck paths that we have kept track of. In this example, the nodes along the path in the generating tree are shown in Fig. 11, the last one being $\widetilde{\Psi}^{-1}(P, Q)$.

We claim that the bijection $\widetilde{\Psi}$ is precisely the map Ψ defined in Section 2. The description that we gave of Ψ is non-recursive, although implicitly it also computes the path to the root in the generating tree for $\mathcal{T}^{(2)}$. To justify this claim first we need the following lemma.

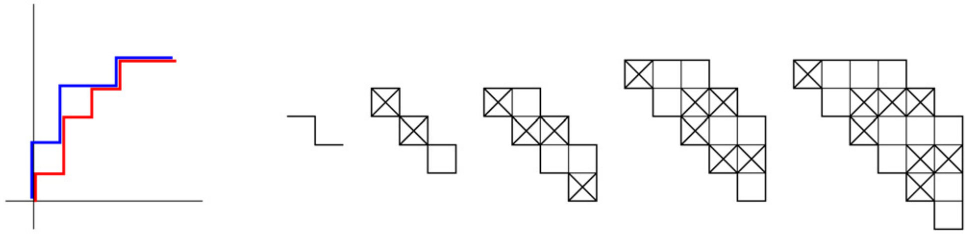

Fig. 11. The computation of $\widetilde{\Psi}^{-1}$, by reconstructing the path from the root of the generating tree.

Lemma 8. *Fix $n \geqslant 5$. Let $T \in \mathcal{T}_n^{(2)}$, and let $(P, Q) = \widetilde{\Psi}(T) \in \mathcal{D}_{n-4}^{(2)}$. For $4 \leqslant j \leqslant n$, let h_j be the number of crosses in column j of the representation of T as a subset of Λ_n. For $1 \leqslant j \leqslant n-4$, let $p_j = p_j(P, Q)$ and $q_j = q_j(P, Q)$. Then,*

$$(h_4, h_5, \ldots, h_{n-1}, h_n) = (q_{n-4}, p_{n-4} + q_{n-5}, \ldots, p_2 + q_1, p_1).$$

Proof. First notice that Eq. (6) shows that the lemma holds for the rightmost s components not including the last one, where $s = s(P, Q)$.

We prove the lemma by induction on n. For $n = 5$, the empty 2-triangulation has $h_4 = h_5 = 0$, and the pair of Dyck paths of size one has $p_1 = q_1 = 0$. Assume now that $n \geqslant 6$ and the result holds for $n-1$. Given $T \in \mathcal{T}_n^{(2)}$, let $T' = p(T) \in \mathcal{T}_{n-1}^{(2)}$ be its parent, and let $(P', Q') = \widetilde{\Psi}(T')$. For $4 \leqslant j \leqslant n-1$, let h'_j be the number of crosses in column j of the representation of T' as a subset of Λ_{n-1}. For $1 \leqslant j \leqslant n-5$, let $p'_j = p_j(P', Q')$ and $q'_j = q_j(P', Q')$, and let $p'_{n-4} = q'_0 = 0$. By the induction hypothesis, $(h'_4, h'_5, \ldots, h'_{n-2}, h'_{n-1}) = (q'_{n-5}, p'_{n-5} + q'_{n-6}, \ldots, p'_2 + q'_1, p'_1)$.

Let r be the corner of T, as usual. Let us first assume that $r \geqslant 3$. It follows from the rule that describes the children of T' (see Proposition 6) that

$$(h_4, h_5, \ldots, h_{n-1}, h_n) = \left(h'_4, \ldots, h'_r, i, h'_{r+1} - i + 1, h'_{r+2} + 1, h'_{r+3}, \ldots, h'_{n-1}\right) \tag{7}$$

for some $0 \leqslant i \leqslant h'_{r+1}$ if $r \leqslant n-4$, or $0 \leqslant i \leqslant h'_{r+1} + 1$ if $r = n-3$. Similarly, using that $s = n-1-r$, rules (3), (4) and (5) describing the children of (P', Q') imply that

$$\begin{aligned}(q_{n-4}, &\; p_{n-4} + q_{n-5}, \ldots, p_2 + q_1, p_1) \\ &= \big(p'_{n-4} + q'_{n-5}, \ldots, p'_{s+1} + q'_s, i, p'_s + q'_{s-1} - i + 1, p'_{s-1} + q'_{s-2} + 1, \\ &\quad p'_{s-2} + q'_{s-3}, \ldots, p'_1 + q'_0\big).\end{aligned} \tag{8}$$

We claim that the value of i has to be the same in (7) and (8). This is because by the definition of $\widetilde{\Psi}$, the label of T has to agree with the label of (P, Q); but these labels are given by the rightmost s components, not including the last one, of (7) and (8) respectively, and the first entry is i in both labels. It follows that (7) and (8) coincide, so the lemma holds.

In the special case $r = 2$, all the crosses in the diagram of T have to be in the first two rows, and we have that $(h_4, h_5, \ldots, h_{n-1}, h_n) = (1, 2, 2, \ldots, 2, 1, 0)$. In this case, $s = n-1-r = n-3$, and $[P, Q]$ has the form given in (2), so we have that $(q_{n-4}, p_{n-4} + q_{n-5}, \ldots, p_2 + q_1, p_1) = (1, 2, 2, \ldots, 2, 1, 0)$ as well. $\square$

As an example of the fact stated in this lemma, take T to be the 2-triangulation from Fig. 2, for which we have seen that $\widetilde{\Psi}(T)$ is then the pair (P, Q) of Dyck paths drawn in Fig. 4. In this case we have that

$$(h_3, h_4, \dots, h_{13}) = (1, 0, 3, 0, 2, 3, 0, 1, 2, 4, 2).$$

On the other hand,

$$[P, Q] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 & 2 & 0 \end{bmatrix},$$

so $(q_{10}, p_{10} + q_9, \dots, p_2 + q_1, p_1) = (0, 1, 0, 3, 0, 2, 3, 0, 1, 2, 4, 2)$ as well.

Lemma 8 states that if $T \in \mathcal{T}_n^{(2)}$ and $(P, Q) = \widetilde{\Psi}(T)$, then the number h_j of crosses in column j of Λ_n equals the number of east steps with ordinate $j - 3$ in the drawing of $(\dot{P}, \dot{Q})$ (except when j equals $n - 1$ or n, where these numbers are off by 1). This explains why in the definition of Ψ we considered the number of crosses in each column of Λ_n to determine where to put the east steps in P and Q. It remains to see how many of these h_j east steps belong to P and how many belong to Q, that is, how to split h_j into $p_{n-j+1} + q_{n-j}$. In the definition of Ψ, this is done by coloring the crosses red and blue. We show now that the paths given by this coloring coincide with the ones obtained by applying $\widetilde{\Psi}(T)$.

Proposition 9. *We have that $\widetilde{\Psi} = \Psi$. In particular, Ψ is a bijection.*

Proof. We will show by induction on n that for any $T \in \mathcal{T}_n^{(2)}$, $\widetilde{\Psi}(T) = \Psi(T)$. This statement is trivially true for $n = 5$. Assume that $n \geqslant 6$, and let $(\widetilde{P}, \widetilde{Q}) = \widetilde{\Psi}(T)$, $(P, Q) = \Psi(T)$. We will show that $(\widetilde{P}, \widetilde{Q}) = (P, Q)$. Let T' be the parent of T, and let r and r' be the corners of T and T', respectively. This means that to obtain T as a child of T', column $r + 1$ of the diagram of T' has been split into two columns, which in T are columns $r + 1$ and $r + 2$. Let h be the number of crosses in column $r + 1$ of T', and assume that they are $(a_1, r + 1), \dots, (a_h, r + 1)$ with $a_1 > \dots > a_h$.

By induction hypothesis, we know that $\widetilde{\Psi}(T') = \Psi(T')$. Let $(P', Q') = \widetilde{\Psi}(T')$, and let $s = n - 1 - r$. By Lemma 8, we have that $p'_{s+1} + q'_s = h$, where p'_{s+1} and q'_s are the number of east steps with ordinate $n - s$ in $\dot{P}'$ and $\dot{Q}'$, respectively. This means that when Ψ is applied to T', the bottom p'_{s+1} crosses in column $r + 1$ are colored blue, and the top q'_s crosses are colored red.

Let i be the number of crosses in column $r + 1$ of T. To simplify notation, we assume that $1 \leqslant i \leqslant h$ (the cases $i = 0$ and $i = h + 1$ can be analyzed in a similar way). Then, the crosses in column $r + 1$ of T are $(a_1, r + 1), \dots, (a_i, r + 1)$, and the crosses in column $r + 2$ are $(a_i, r + 2), (a_{i+1}, r + 2), \dots, (a_h, r + 2)$.

When $(\widetilde{P}, \widetilde{Q})$ is obtained as a child of (P', Q'), the column $\begin{smallmatrix} p'_{s+1} \\ q'_s \end{smallmatrix}$ of $[P', Q']$ is split into two columns

$$\begin{matrix} i & p'_{s+1} - i \\ 0 & q'_s + 1 \end{matrix} \tag{9}$$

if $i \leqslant p'_{s+1}$, or

$$\begin{matrix} p'_{s+1} & 0 \\ i - p'_{s+1} & p'_{s+1} + q'_s - i + 1 \end{matrix} \tag{10}$$

if $i \geqslant p'_{s+1}$, according to the definition of $\widetilde{\Psi}$. Besides, the upper entry p'_s of column immediately to the right of the split one becomes $p'_s + 1$.

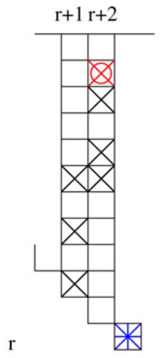

Fig. 12. The first step of the coloring algorithm.

Let us now look at how $\Psi(T)$ is computed. The first step of the coloring algorithm takes r to be the corner of T, colors blue the cross $(r, r+3)$, colors red the cross $(a_h, r+2)$ (or $(a_h, r+1)$ if column $r+2$ is empty), and merges columns $r+1$ and $r+2$ into one single block (see Fig. 12). The crucial observation is that in the successive iterations, the coloring algorithm will act on the remaining crosses (all except for the two that have just been colored) exactly in the same way that it acts on the diagram of T', if we identify the sequence of crosses $(a_1, r+1), \dots, (a_h, r+1)$ in T' with the sequence of crosses $(a_1, r+1), \dots, (a_i, r+1), (a_i, r+2), \dots, (a_{h-1}, r+2)$ in T', in this order. The reason of this is that for crosses inside the same block, the one that gets colored blue in the description of Ψ is always the leftmost one (the lowest one among them if there is a tie), and the one that gets colored red is always the rightmost one (the highest one among them if there is a tie). Thus, columns to the left of column $r+1$ will be colored identically in both T and T'; columns to the right of column $r+3$ in T will be colored like columns to the right of column $r+2$ in T', shifting one position to the right; and column $r+3$ in T will be colored like column $r+2$ in T', except that it will have an extra blue cross $(r, r+3)$ at the bottom. Finally, the bottom p'_{s+1} crosses (or all of them if $i \leqslant p'_{s+1}$) of column $r+1$ in T will be colored blue and the remaining ones red; and the top $q'_s + 1$ crosses (or all of them if $i \geqslant p'_{s+1}$) of column $r+1$ in T will be colored red and the remaining ones blue.

Therefore, when $\Psi(T) = (P, Q)$ is computed from this colored diagram of crosses, column $r+1$ of T will produce i east steps in P and 0 east steps in Q if $i \leqslant p'_{s+1}$, or p'_{s+1} east steps in P and $i - p'_{s+1}$ east steps in Q if $i \geqslant p'_{s+1}$. Similarly, column $r+2$ of T will produce $p'_{s+1} - i$ east steps in P and $q'_s + 1$ east steps in Q if $i \leqslant p'_{s+1}$, or 0 east steps in P and $h - i + 1$ east steps in Q if $i \geqslant p'_{s+1}$. Finally, the additional blue cross at the bottom of column $r+3$ of T will contribute the extra east step added to p'_s by $\widetilde{\Psi}$. Comparing with (9) and (10), it follows that $(P, Q) = (\widetilde{P}, \widetilde{Q})$. $\square$

6. Generalization to k-triangulations

The natural question at this point is whether one can give a similar bijection between k-triangulations of an n-gon and k-tuples $(P_1, P_2, \dots, P_k)$ of Dyck paths of size $n - 2k$ such that each P_i never goes below P_{i+1}, for $k \geqslant 3$. While we have not succeeded in finding such a bijection, some of the ideas in our construction for $k = 2$ generalize to arbitrary k. In this section we show that it is possible to construct an analogous generating tree for k-triangulations. It is worthwhile to mention that the number of 2-triangulations shows up in other contexts where there is no obvious extension to $k \geqslant 3$ that gives the number of k-triangulations. Examples of this are [3] and [1].

6.1. A generating tree for k-triangulations

Fix an integer $k \geqslant 2$. Next we describe a generating tree where nodes at level ℓ correspond to k-triangulations of an $(\ell + 2k + 1)$-gon. We ignore trivial diagonals, so all k-triangulations of an n-gon have $k(n - 2k - 1)$ diagonals. The root of the tree is the empty k-triangulation of a $(2k + 1)$-gon.

The lemmas in Section 3 have an immediate generalization to arbitrary k. We will only use two of them.

Lemma 10. *Let $T \in \mathcal{T}_n^{(k)}$ be a k-triangulation containing the diagonal (a, b), with $a < b-k-1$. Then T contains the diagonal $(a, b-1)$ or a diagonal of the form (a', b) with $a < a' \leqslant b-k-1$.*

Proof. Assume that $(a, b - 1)$ is not in T. Then, since T is a maximal set of diagonals with no $(k + 1)$-crossings, adding the diagonal $(a, b - 1)$ would create a $(k + 1)$-crossing together with k diagonals in T. But these k diagonals together with (a, b) do not form a $(k + 1)$-crossing. This means that at least one of these k diagonals crosses $(a, b - 1)$ but not (a, b). This can only happen if such a diagonal is of the form (a', b) with $a < a' \leqslant b - k - 1$. □

Lemma 11. *Let $T \in \mathcal{T}_n^{(k)}$ be a k-triangulation containing the diagonal (a, b), with $a \leqslant b-k-1$. Then there exists a vertex $i \in \{a, \dots, b-k-1\}$ such that T contains the diagonal $(i, i+k+1)$.*

Lemma 11 follows easily by iteration of Lemma 10.

Diagonals of the form $(a, a + k + 1)$ are called *short diagonals*. Let $n \geqslant 2k + 2$, and let T be a k-triangulation of an n-gon. To define the parent of T we will need some definitions. Let r be the largest number with $1 \leqslant r \leqslant n - k - 1$ such that T contains the short diagonal $(r, r + k + 1)$. We call r the *corner* of T. Note that T does not contain any diagonals of the form (a, b) with $r < a \leqslant b - k - 1 \leqslant n - k - 1$, since otherwise, by Lemma 11, there would be a short diagonal contradicting the choice of r. So, the diagram of T has no crosses below row r. Note that in particular we have $r \geqslant k$, since each $a \leqslant r$ can be an endpoint of at most $n - 2k - 1$ diagonals, compared to the $k(n - 2k - 1)$ needed in a k-triangulation.

For $i = 1, 2, \dots, k - 1$, let

$$A_i := \big\{a\colon\ (a, r + i) \in T\big\} \cup \{r + i - k\}.$$

Let $a_1 := \min A_1$, and for $i = 2, \dots, k - 1$, let

$$a_i := \min\{a \in A_i\colon\ a > a_{i-1}\}.$$

For example, in the 3-triangulation from Fig. 1, $r = 7$, $a_1 = 3$, and $a_2 = 6$. The following property of T will be crucial to define its parent.

Lemma 12. *Let $i \in \{1, 2, \dots, k-1\}$, and let a_i be defined as above. Then, either $(a_i, r+i+1) \in T$ or $(a_i, r + i + 1)$ is a trivial diagonal.*

Proof. First notice that if $a \in A_i$, then $a \leqslant r + i - k$. This is because the diagram of T has no crosses below row r, so all diagonals incident to $r + i$ are represented by crosses in column $r + i$, whose lowest square is in row $r + i - k - 1$.

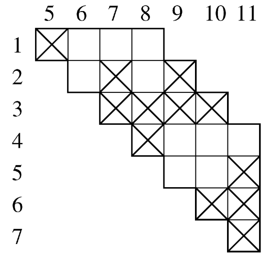

Fig. 13. The diagram of a 3-triangulation of an 11-gon.

We start with the case $i = 1$. If the square $(a_1, r+2)$ falls outside of $\Lambda_n^{(k)}$, then $(a_1, r+2)$ is a trivial diagonal and we are done. Otherwise, let us assume for contradiction that $(a_1, r+2) \notin T$. Since T is a maximal set of diagonals with no $(k+1)$-crossings, this means that if we added $(a_1, r+2)$ to T, it would form a $(k+1)$-crossing together with k diagonals in T, none of which corresponds in the diagram to a cross below row r (since there are no such crosses). By the definition of a_1, none of these diagonals can correspond to a cross in column $r+1$. Therefore, if in this $(k+1)$-crossing we replace $(a_1, r+2)$ with $(a_1, r+1)$, we obtain a $(k+1)$-crossing containing $(a_1, r+1)$, which contradicts the fact that T is a k-triangulation.

For $i > 1$ the reasoning is very similar. In this case, we assume for contradiction that $(a_i, r+i+1) \notin T$ and that it is not a trivial diagonal. Then, maximality of the set T implies that adding $(a_i, r+i+1)$ would create a $(k+1)$-crossing C, together with k diagonals in T. By the definition of $a_1, a_2, \ldots, a_i$, there must be at least one among the columns $r+1, r+2, \ldots, r+i$ which has no diagonals belonging to C. Let $r+j$ be the rightmost such column. Then, if for each $l = j, j+1, \ldots, i$ we replace the element in C in column $r+l+1$ with $(a_l, r+l)$, we still obtain a $(k+1)$-crossing. But the fact that the diagonal $(a_i, r+i)$ is part of a $(k+1)$-crossing is a contradiction, since either $(a_i, r+i) \in T$ or $(a_i, r+i)$ is a trivial diagonal. $\square$

An additional property of T is that column $r+k$ of its diagram has no crosses below row a_{k-1}. This is because if there was such a cross, then it would form a $(k+1)$-crossing together with diagonals $(a_1, r+1), \ldots, (a_{k-1}, r+k-1)$, and $(r, r+k+1)$, all of which belong to T or are trivial diagonals.

Consider now the representation of T as a subset of $\Lambda_n^{(k)}$. We define the parent of T in the generating tree to be the k-triangulation $p(T) \in \mathcal{T}_{n-1}^{(k)}$ whose diagram, as a subset of $\Lambda_{n-1}^{(k)}$, is obtained from the diagram of T as follows.

- Delete the squares $(a, a+k+1)$ for $a = r, r+1, \ldots, n-k-1$. (Note that only the first one of such squares contains a cross.)
- For each $i = 1, 2, \ldots k-1$:
  - Keep all the crosses of the form $(a, r+i)$ with $a \geqslant a_i$ in column $r+i$.
  - Move all the crosses of the form $(a, r+i+1)$ with $a < a_i$ from column $r+i+1$ to column $r+i$, and delete the cross $(a_i, r+i+1)$ if it is in T.
- Delete column $r+k$ (which at this point is empty, by the observation following Lemma 12), and move all the columns to the right of it one position to the left. If $r > n-2k$, delete also the squares $(a, n-k-1+a)$ for $a = 1, 2, \ldots, r+2k-n$.

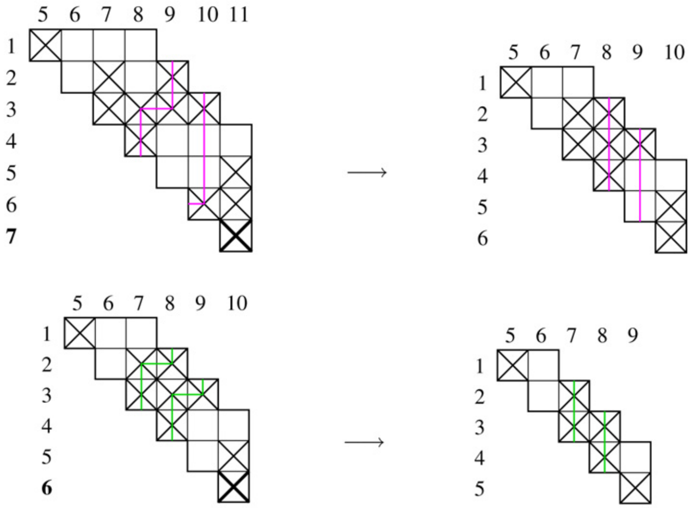

Fig. 14. The parent (top) and the grandparent (bottom) of the 3-triangulation from Fig. 13.

This yields the diagram of $p(T)$ as a subset of $\Lambda_{n-1}^{(k)}$. For example, if T is the 3-triangulation from Fig. 13, then $p(T)$ and $p(p(T))$ are shown in Fig. 14. Note that in $p(T)$, $r = 6$, $a_1 = 2$, and $a_2 = 3$.

We next characterize the children of a given k-triangulation $T \in \mathcal{T}_n^{(k)}$ in the generating tree. By definition, the children are all those elements $\widehat{T} \in \mathcal{T}_{n+1}^{(k)}$ such that $p(\widehat{T}) = T$. Again, let $r \in \{1, 2, \dots, n-k-1\}$ be the corner of T. Note that for any child $\widehat{T}$, if $\hat{r}$ is the corner of $\widehat{T}$, then $\hat{r} \geqslant r$. All the children of T are obtained in the following way:

- Choose a number $u \in \{r, \dots, n-k\}$.
- Add one to the labels of the columns j with $u+k \leqslant j \leqslant n$, and add an empty column labeled $u+k$.
- Add the square $(u, u+k+1)$ with a cross in it, and add empty squares $(j, j+k+1)$ for $j = u+1, \dots, n-k$. If $u > n-2k$, add also empty squares $(j, n-k+j)$ for $j = 1, \dots, u+2k-n$.
- For $i = 1, \dots, k-1$, let $B_i := \{b\colon (b, u+i) \in T\} \cup \{u+i-k\}$. If $u = n-k$, add also the element i to B_i, for each i.
- For each $i = 1, \dots, k-1$, choose a number $b_i \in B_i$, so that $b_1 < b_2 < \cdots < b_{k-1}$.
- For each $i = k-1, k-2, \dots, 1$, add a cross $(b_i, u+i+1)$ (except if $b_i = i$, in which case we add the cross $(b_i, u+i)$ instead), and move all the crosses of the form $(b, u+i)$ with $b < b_i$ from column $u+i$ to column $u+i+1$.

Each choice of u and $b_1, b_2, \dots, b_{k-1}$ gives rise to a different child of T. Note that each choice of u generates those children with $\hat{r} = u$. Figure 15 shows a 3-triangulation and its twelve children, of which two are obtained with $u = 4$, three with $u = 5$, and seven with $u = 6$. An important difference between the case $k = 2$ and the case $k \geqslant 3$ is that, in the latter, the number of children of a k-triangulation depends not only on the number of crosses in the columns of its diagram but also on the relative position of the crosses in different columns (this is caused by the condition $b_1 < b_2 < \cdots < b_{k-1}$). As a consequence, there is no obvious way to associate simple

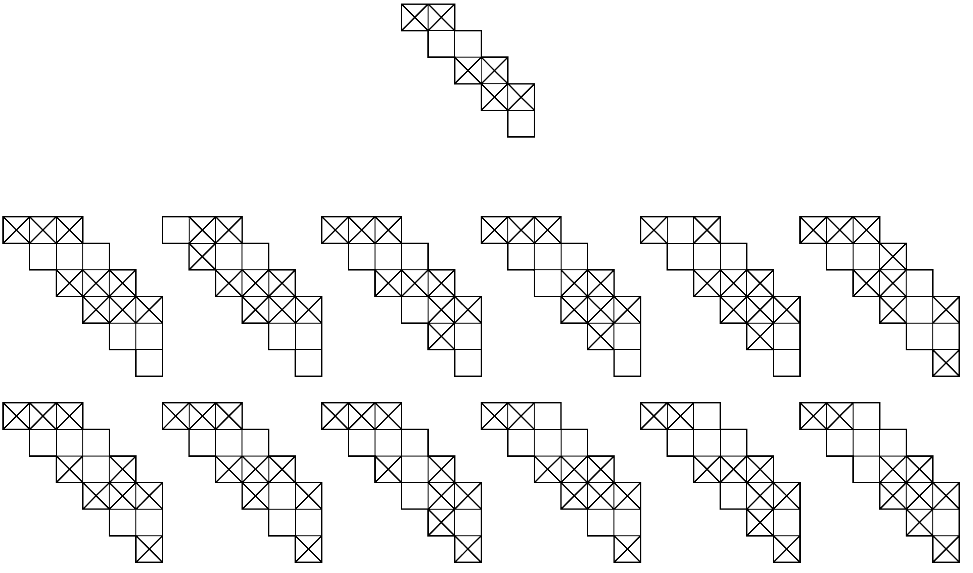

Fig. 15. A 3-triangulation of a 9-gon and its 12 children in the generating tree.

labels to each node of the generating tree, as we did for $k = 2$. This is an obstacle when trying to construct a generating tree for k-tuples of non-crossing Dyck paths isomorphic to the one that we have given for $\mathcal{T}^{(k)}$.

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