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# Further Reduction of the Takens–Bogdanov Normal Form

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## 1. INTRODUCTION

The purpose of this paper is to present a complete formal classification of germs of analytic vector fields in $\mathbf{R}^2$ with nilpotent linear part.[1] The topological study of this bifurcation was initiated by Takens [Ta] and Bogdanov [Bo1, Bo2], and was taken up more recently in [DRS1, DRS2]. Here we concentrate on uniqueness aspects of the formal normal forms for such singularities.

The (formal) classification for vector fields whose linear part is the harmonic oscillator was obtained in [Ba–Ch]. In that case the classical normal form (i.e., what we call in this paper the *first order normal form*) is well known to commute with the linear part, which makes it $S^1$-symmetric. More important yet, this insures that the linear part plays no role when refining the first order normal form to higher order ones, a process that is required if one is to obtain uniqueness. By contrast, in the nilpotent case the linear and nonlinear parts of the first order normal form have different symmetries (see [Cu–Sa1]). This makes the problem considerably more difficult, as one has to contend with the presence of the linear term whose effects persist through all higher order normal forms.

First results concerning uniqueness for nilpotent vector fields were

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[1] Certain exceptional subcases to be specified in the appropriate places have been left out, however.



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derived in [Ba–Sa], where in addition to the Hamiltonian version of the present problem a classification for the non-semisimple $(1:-1)$-resonance was obtained. (See [vMe] for an extended treatment of the Hamiltonian–Hopf bifurcation.) Another recent contribution to the developing theory (this time concerning the $(1:2)$-resonance) can be found in [Sa–vMe].

Our strategy is an adaptation to the particular situation of the abstract method proposed in [Ba]. The reader interested in an informal description should consult the introduction to [Ba–Sa], where also a more thorough list of references can be found. For a formal treatment he or she is also referred to the first three sections of [Ba–Sa]. Here, besides describing the results of this paper, we shall restrict ourselves to only a few general comments.

The first component of our work is a tight description of the Lie algebra $\mathrm{Vect}(\mathbf{R}^2; 0)$ of formal planar vector fields in terms of a "good" basis for it. We use the Clebsch–Gordan decomposition to break up $\mathrm{Vect}(\mathbf{R}^2; 0)$ into a direct sum of two Lie subalgebras, Eul and Ham, whose homogeneous subspaces correspond to the irreducible representations of $\mathrm{sl}(2; \mathbf{R})$. This leads to a remarkably simple description of the structure constants for this algebra. The reader will find them in Theorem 3.7. Needless to say, this result was instrumental for the successful application of the abstract method to the nilpotent problem.

From the combined work of several authors (cf. [Cu–Sa1] and also the references therein,) it has become clear in recent years that the classification problem for nilpotent vector fields is intimately related (via the Jacobson–Morozov Lemma) to various representations of $\mathrm{sl}(2; \mathbf{R})$ on $\mathrm{Vect}(\mathbf{R}^n; 0)$. It is our feeling that in order to make progress on uniqueness questions in higher dimensions along the lines of this paper, a detailed description of the Lie-theoretic properties of $\mathrm{Vect}(\mathbf{R}^n; 0)$ generalizing those of $\mathrm{Vect}(\mathbf{R}^2; 0)$ will have to be found.

A second ingredient (which pervades most aspects of our work and is indeed central to it) is the notion of "(multi-)graded Lie algebra." The way this comes about and why it is relevant is roughly as follows. When proceeding from a first order normal form to higher order ones, one is basically trying to use up all the symmetry left to produce further normalization. "Normalization," in turn, means removal of as many additional terms as possible. This process, which has to be performed without changing lower order terms already in "satisfactory" form, involves solving long sequences of equations whose purpose is to insure that certain commutators vanish, so that undesired disturbances will indeed not occur. (To alert the reader, we point out that such sequences of equations are hidden in the innocuous looking Definition 2.1.) Here is where the gradings make their appearance. The more gradings you have the better off you are because of the added control. Of course, in order for this *motto* to make sense, the operators involved must be homogeneous relative to at least one of these degrees. In our work we proceeded backwards, and *constructed* good gradings, tailoring them to our needs in such a manner that the relevant operators did possess the required homogeneity properties.

We hasten to point out that these constructions were made possible by the presence of two standard gradings on Vect($\mathbf{R}^2$; 0), namely, those given by the eigenspaces of the "Euler" operators $\mathrm{ad}(x(\partial/\partial x))$ and $\mathrm{ad}(y(\partial/\partial y))$. In generalizing to higher dimensions, one should expect the $n$ Euler operators on $\mathbf{R}^n$ to play a similar role relative to Vect($\mathbf{R}^n$; 0).

As for specifics, we found out that the typical formal nilpotent vector field falls within three categories (labeled Cases I–III) that we treat in separate sections. Cases I and II are essentially complete, whereas the results on Case III are preliminary at this time. In all instances we found what we believe to be a phenomenon not yet reported in the literature,[2] namely, that of highly periodic number-theoretical patterns for the unique normal forms, which depend on the nature of the subdominant terms that occur in first order normalization. The main results of this paper are encapsulated in Theorems 5.2.6 and 6.2.7 for Cases I and II, respectively.

The organization of the paper is as follows: in Section 2 we introduce notation as well as the abstract framework for "infinite order" normal form theory on graded Lie algebras. Section 3 contains the Clebsch–Gordan decomposition and the Structure Theorem for *Vect*($\mathbf{R}^2$; 0). This leads in Section 4 to the classical (first order) normal forms and to the trichotomy alluded-to above. The ensuing Cases I–III are studied separately in Section 5–7.

## 2. Preliminaries

Here we set notation and recall a number of basic concepts to be used throughout the paper. For background as well as for a proof of the theorem below, the reader is referred to [Ba] or [Ba–Sa].

In this paper a *graded Lie algebra* will be a Lie algebra $\mathscr{L}=\prod_{j=-\infty}^{\infty}\mathscr{L}_j$ such that $\mathscr{L}_j=0$ for $j<\gamma_0$ and $[\mathscr{L}_j,\mathscr{L}_k]\subseteq\mathscr{L}_{j+k}$ for all $j,k\in\mathbf{Z}$. We write elements $x\in\mathscr{L}$ as formal series $x=x_{\gamma_0}+\cdots+x_j+\cdots$, where $x_j\in\mathscr{L}_j$ is *homogeneous of degree j*. To any such $\mathscr{L}$ we associate the following objects:

(1) $J_k(x):=\sum_{j=\gamma_0}^{k}x_j$, the $k$th jet of $x$.

(2) The non-linear projection $\pi_0\colon\mathscr{L}\to\bigcup_{j=\gamma_0}^{\infty}\mathscr{L}_j$, defined by

$$\pi_0(x)=\begin{cases}x_\gamma & \text{if}\quad x=x_\gamma+\cdots\text{with } x_\gamma\neq 0\\ 0 & \text{if}\quad x=0.\end{cases}$$

[2] With the exception of [Ba–Sa], essentially a subcase of the present work.

(3) The degree function $\delta$ defined by

$$\delta(x)=\begin{cases}\gamma & \text{if } \pi_0(x)\in\mathscr{L}_\gamma\backslash\{0\}\\ \infty & \text{if } x=0.\end{cases}$$

(4) The decreasing filtration $\mathscr{F}_k=\mathscr{F}_k(\mathscr{L}):=\prod_{j\geqslant k}^{\infty}\mathscr{L}_j$.

The $\mathscr{F}_j$ satisfy $[\mathscr{F}_j, \mathscr{F}_k]\subseteq\mathscr{F}_{j+k}$ and form a basis of neighborhoods of 0 for a topology for which the Lie bracket is a continuous operation and all series of the form $\sum x_j$ with $x_j\in\mathscr{F}_j$ converge. A sequence $x^n\to 0$ iff $\delta(x^n)\to\infty$.

In what follows and for technical reasons that do no limit the scope of our applications, we make the blanket assumption that, unless statement to the contrary is made, the homogeneous subspaces $\mathscr{L}_j$ of all graded Lie algebras are *finite dimensional.*

The following additional conventions and notation will be in use:

(5) $\mathbf{R}[[x]]=\prod_{j=0}^{\infty}\mathscr{P}_j$, the graded ring of formal power series, where $\mathscr{P}_j=\mathscr{P}_j(\mathbf{R}^n)$ is the space of homogeneous polynomials of degree $j$.

(6) $\mathrm{Vect}_j(\mathbf{R}^n)$ is the space of homogeneous vector fields $X=\sum p_i\partial/\partial x_i$ with $p_i\in\mathscr{P}_{j+1}(\mathbf{R}^n)$. (Note the shift $j\mapsto j+1$.)

(7) $\mathrm{Vect}(\mathbf{R}^n)=\prod_{j=-1}^{\infty}\mathrm{Vect}_j(\mathbf{R}^n)$ is a graded Lie algebra. $\mathrm{Vect}(\mathbf{R}^n;0):=\mathscr{F}_0(\mathrm{Vect}(\mathbf{R}^n))$ represents the subalgebra of fields with a rest point at the origin.

(8) For $X\in\mathrm{Vect}(\mathbf{R}^n)$ as above and $f\in\mathbf{R}[[x]]$, $X(f)$ denotes the Lie derivative $\sum p_i\partial f/\partial x_i$. In particular $p_i=X(x_i)$.

(9) $\mathrm{Ham}(\mathbf{R}^n):=\prod_{j=-2}^{\infty}\mathrm{Ham}_j(\mathbf{R}^n)$ is the graded Poisson algebra of Hamiltonians on $(T^*\mathbf{R}^n\cong\mathbf{R}^n\oplus\mathbf{R}^n;\sum dx_i\wedge dy_i)$, where $\mathrm{Ham}_j(\mathbf{R}^n):=\mathscr{P}_{j+2}(\mathbf{R}^{2n})$. The required index shift is now $j\mapsto j+2$, and the Poisson bracket is given by

$$\{h,g\}=\frac{\partial h}{\partial y_i}\frac{\partial g}{\partial x_i}-\frac{\partial g}{\partial y_i}\frac{\partial h}{\partial x_i}.$$

(10) For $h\in\mathrm{Ham}(\mathbf{R}^n)$, $X_h$ denotes the Hamiltonian field defined by the derivation $\mathrm{ad}(h)$ on $\mathbf{R}[[x,y]]$. Thus $X_h(f)=\{h,f\}$. With these sign conventions the map $h\mapsto X_h$ is a graded Lie algebra homomorphism $\mathrm{Ham}(\mathbf{R}^n)\to\mathrm{Vect}(\mathbf{R}^{2n})$, which is injective as a map $\mathrm{Ham}(\mathbf{R}^n;0)\to\mathrm{Vect}(\mathbf{R}^{2n};0)$.

(11) When dealing with an algebra $\mathscr{L}$ endowed with several degree functions $\delta^{(n)}$, the superscriupt $(n)$ will be in use to signify objects associated to the appropriate degree, such as $\mathscr{L}_j^{(n)}$, $\mathscr{F}_j^{(n)}$, and the like.

(12) The notation

$$u \overset{\mathcal{S}}{\equiv} v$$

will be used to signify "$u$ is congruent to $v$ (mod $\mathcal{S}$)."

We now recall certain results from the general theory of formal normal forms on a filtered Lie algebra $\mathcal{L}$. This theory concerns itself with classification of conjugacy classes modulo the group $\exp \operatorname{ad}(\mathcal{F}_1)$ of filtration preserving automorphisms of $\mathcal{L}$. More generally, we will need to study automorphism groups of the form $\mathcal{G} = \exp \operatorname{ad}(\mathcal{J})$, where $\mathcal{J}$ is a closed homogeneous subalgebra of $\mathcal{F}_1$. (Recall that a subalgebra $\mathcal{T}$ is said to be homogeneous iff the following condition holds for arbitrary $x = \sum x_j \in \mathcal{L}$: $x \in \mathcal{J} \Leftrightarrow \forall j \; x_j \in \mathcal{J}$). For a fixed element $x \in \mathcal{L}$, we introduce the sequence $\mathcal{R}_k = \mathcal{R}_k(x, \mathcal{J}) \subseteq \mathcal{L}_k$ of maximally *removable subspaces* by:

$$\mathcal{R}_k = \pi_0([x, \mathcal{J}]) \cap \mathcal{L}_k. \tag{2.1}$$

*Remark.* These spaces are $\mathcal{G}$-invariant and depend only on the $\mathcal{G}$-orbit of $x$ (cf. [Ba–Sa, Prop. 2.7]).

We will say that two subspaces $\mathcal{U}$, $\mathcal{V}$ of a vector space $\mathcal{W}$ are *transversal* (resp. *strictly transversal*) if $\mathcal{U} + \mathcal{V} = \mathcal{W}$ (resp. $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$). The following is the basic result on existence and uniqueness of normal forms that we will use in the sequel.

THEOREM 2.2. *Let $x \in \mathcal{L}$ be a fixed element with $\delta(x) = \gamma < \infty$ and $\pi_0(x) = x_\gamma$. Suppose that a sequence of subspaces $\{\mathcal{N}_k\}_{k > \gamma}$ transverse to $\mathcal{R}_k$ in $\mathcal{L}_k$ has been chosen. Then*

(1) *There exists $z \in \mathcal{J}$ such that if $y = \exp \operatorname{ad}(z)\, x$, then $y_k \in \mathcal{N}_k$ for $k > \gamma$.*

(2) *If the $\mathcal{R}_k$, $\mathcal{N}_k$ are strictly transversal $\forall k > \gamma$, then the above normal form is unique (relative to the group $\mathcal{G}$).*

*Remark.* The proof for $\mathcal{J} = \mathcal{F}_1$ given in references [Ba, Ba–Sa] applies word for word to this (slightly more general) situation.

A classical way of constructing normal forms is to choose the $\mathcal{N}_k$ strictly transverse to $[x_\gamma, \mathcal{L}_{k-\gamma} \cap \mathcal{J}]$. Since the latter is clearly a subspace of $\mathcal{R}_k$, it follows that the $\mathcal{N}_k$ are transverse to the $\mathcal{R}_k$, but not necessarily strictly so, which is the reason why the forms found in the literature are not necessarily unique. In any case when the choice of the $\mathcal{N}_k$ has been made in this fashion, we will refer to the corresponding normal forms as "*first order*," whereas the terminology "*second order*" will refer to normal forms which are first order relative to a newly defined ad hoc grading (on the algebra $\mathscr{L}$). Here we think of $n$th-order normal forms as successive approximations to the unique normal form of the above theorem , which will be accordingly referred to as "*infinite order*." This language is only meant to be suggestive, and although a rigorous formal definition of such successive approximations can be given to all orders, we will refrain from doing so in this paper.

The following result is standard but it is included here because we have no readily available reference.

PROPOSITION 2.4. *Let $x, y \in \mathscr{L}$ such that $y = \exp \operatorname{ad}(z)\, x$ for some $z \in \mathscr{F}_1$. If $[z, x] \overset{\mathscr{F}_{j+1}}{\equiv} [z, x]_j$, then $y_k = x_k$ for $k < j$ and*

$$y_j = x_j + [z, x]_j. \tag{2.5}$$

*Proof.* Since $z \in \mathscr{F}_1$, $[z, x] \in \mathscr{F}_j \Rightarrow \operatorname{ad}(z)^k (x) \in \mathscr{F}_{j+k-1} \subseteq \mathscr{F}_{j+1}$ when $k \geqslant 2$. The result follows from $y = x + [z, x] + \sum_{k \geqslant 2} \operatorname{ad}(z)^k (x)/k! \overset{\mathscr{F}_{j+1}}{\equiv} J_j(x) + [z, x]_j$. ∎

PROPOSITION 2.6. *Let $x = x_\gamma + x_\lambda + h.o.t.$ Let $\mathscr{N}_j$ be a sequence of subspaces of $\mathscr{L}_j$ such that $[x_\gamma, \mathscr{L}_{j-\gamma} \cap \mathscr{J}] \cap \mathscr{N}_j = 0$; and let $z = z_{j_0} + h.o.t.$ be such that $[z, x] = [z, x]_j + h.o.t.$ with $[z, x]_j \in \mathscr{N}_j$. Then*

(1) $[x, z]_j = 0$ *if* $j < \lambda + j_0$.

(2) $z_k \in \ker \operatorname{ad}(x_\gamma)$ *if* $j_0 \leqslant k < j_0 + \lambda - \gamma$.

(3) $[z, x]_{\lambda + j_0} = [z_{\lambda + j_0 - \gamma}, x_\gamma] + [z_{j_0}, x_\lambda]$.

*Proof.* Clearly $[z, x]_j = 0$ when $j < j_0 + \gamma$, so let $j \geqslant j_0 + \gamma$. We have

$$[z, x]_j = \sum_{n=\gamma}^{\infty} [z_{j-n}, x_n] = \sum_{n=\gamma}^{j-j_0} [z_{j-n}, x_n]$$

Moreover $[z_{j-n}, x_n] = 0$ if $\gamma < n < \lambda$, so that

$$[z, x]_j = \begin{cases} [z_{j-\gamma}, x_\gamma] & \text{if } j - j_0 < \lambda \\ [z_{j_0+\lambda-\gamma}, x_\gamma] + [z_{j_0}, x_\lambda] & \text{if } j - j_0 = \lambda \end{cases}$$

We see that $z_{j-\gamma} \in [x_\gamma, \mathscr{L}_{j-\gamma}] \cap \mathscr{N}_j$ when $j - j_0 < \lambda$ so that $z_k \in \ker \operatorname{ad}(x_\gamma)$ when $j_0 \leqslant k < j_0 + \lambda - \gamma$. ∎

COROLLARY 2.7. *Let $x \overset{\mathscr{F}_{\lambda+1}}{\equiv} x_\gamma + x_\lambda$, $y = \exp \operatorname{ad}(z)\, x$ for some $z \in \mathscr{F}_{j_0} \cap \mathscr{J}$ and suppose that $x$ and $y$ are both in first order normal form relative to some sequence $\mathscr{N}_j$ strictly transverse to $[x_\gamma, \mathscr{L}_{j-\gamma} \cap \mathscr{J}]$. Then*

(1) $y_{\lambda + j_0} = x_{\lambda + j_0} + [z_{j_0 + \lambda - \gamma}, x_\gamma] + [z_{j_0}, x_\lambda]$.

(2) $y_j = x_j$ *for* $j < \lambda + j_0$.

(3) *The* first non-zero term beyond $x_\gamma$ *is a first order normal form invariant of $x$ for the group* $\mathscr{G}$.

*Proof.* Apply the proposition in conjunction with (2.5). ∎

## 3. Clebsch–Gordan for Planar Vector Fields

Consider the Euler field

$$E := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \in \mathrm{Vect}_0(\mathbf{R}^2).$$

We denote by $\mathrm{Eul}(\mathbf{R}^2; 0)$ the graded algebra obtained from $\mathbf{R}[[x, y]]$ with its natural grading and the Lie bracket (see (8) of Section 2 for notation):

$$[f, g] = fE(g) - gE(f).$$

The reader can easily verify that this is indeed a Lie algebra satisfying the required properties.

In this section we will show that $\mathrm{Vect}(\mathbf{R}^2; 0)$ is bigraded and decomposes into a direct sum of two subalgebras isomorphic to $\mathrm{Ham}(\mathbf{R}; 0)$ and $\mathrm{Eul}(\mathbf{R}^2; 0)$, respectively. This fact will permit an easy calculation of the structure constants relative to a conveniently chosen basis, and at the same time will give us tight control over various subrepresentations of interest in this paper.

Definition 3.1.

(3.1a) $\iota_1\colon \mathrm{Ham}(\mathbf{R}; 0) \to \mathrm{Vect}(\mathbf{R}^2; 0), \qquad h \mapsto X_h;$

(3.1b) $\iota_2\colon \mathrm{Eul}(\mathbf{R}^2; 0) \to \mathrm{Vect}(\mathbf{R}^2; 0), \qquad g \mapsto gE;$

(3.1c) $p_1\colon \mathrm{Vect}(\mathbf{R}^2) \to \mathrm{Ham}_k(\mathbf{R}), \qquad X \mapsto \dfrac{1}{k+2}\,(yX(x) - xX(y));$

(3.1d) $p_2\colon \mathrm{Vect}(\mathbf{R}^2) \to \mathrm{Eul}_k(\mathbf{R}^2), \qquad X \mapsto \dfrac{1}{k+2}\left(\dfrac{\partial X(x)}{\partial x} + \dfrac{\partial X(y)}{\partial y}\right).$

We view the $p_j$ as defined on all of $\mathrm{Vect}(\mathbf{R}^2; 0)$ by linear extension of (3.1c) and (3.1d).

Two properties of $E$ worth noting here are

$$\text{(i)}\quad E(f) = kf, \qquad f \in \mathscr{P}_k(\mathbf{R}^2)$$

$$\text{(ii)}\quad [E, X] = kX, \qquad X \in \mathrm{Vect}_k(\mathbf{R}^2).$$

PROPOSITION 3.2. *The following hold*:

(1) $p_1 \circ \iota_1 = \mathrm{id}_{\mathrm{Ham}(\mathbf{R};0)}$. *Consequently* $\pi_1 := \iota_1 \circ p_1$ *satisfies* $\pi_1^2 = \pi_1$.

(2) $p_2 \circ \iota_2 = \mathrm{id}_{\mathrm{Eul}(\mathbf{R}^2;0)}$. *Consequently* $\pi_2 := \iota_2 \circ p_2$ *satisfies* $\pi_2^2 = \pi_2$.

(3) *The* $\iota_j$ *are graded Lie algebra monomorphisms and their images are subalgebras of* $\mathrm{Vect}(\mathbf{R}^2; 0)$ *isomorphic to* $\mathrm{Ham}(\mathbf{R}; 0)$ *when* $j=1$, *and to* $\mathrm{Eul}(\mathbf{R}^2; 0)$ *when* $j=2$.

(4) $\pi_1 + \pi_2 = \mathrm{id}_{\mathrm{Vect}(\mathbf{R}^2;0)}$. *This gives rise to the Clebsch–Gordan decomposition*

$$\mathrm{Vect}(\mathbf{R}^2; 0) = \iota_1(\mathrm{Ham}(\mathbf{R}; 0)) \oplus \iota_2(\mathrm{Eul}(\mathbf{R}^2; 0)) \cong \mathbf{R}[[x, y]] \otimes \mathbf{R}^2.$$

*Proof.* (1) Let $h \in \mathrm{Ham}_k(\mathbf{R}) = \mathscr{P}_{k+2}(\mathbf{R}^2)$ so that $E(h) = (k+2)\, h$. Thus

$$\begin{aligned} p_1(\iota_1(h)) &= p_1\left(\frac{\partial h}{\partial y}\frac{\partial}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial y}\right) \\ &= \frac{1}{k+2}\left(y\frac{\partial h}{\partial y} + x\frac{\partial h}{\partial x}\right) \\ &= \frac{1}{k+2} E(h) = h. \end{aligned}$$

(2) Let $g \in \mathrm{Eul}_k(\mathbf{R}^2) = \mathscr{P}_k(\mathbf{R}^2)$ so that $E(g) = kg$. Note that $p_2 = (1/k+2)\,\mathrm{div}$ on $\mathrm{Vect}_k(\mathbf{R}^2)$, and that for any function $f$ and vector field $X$, $\mathrm{div}(fX) = X(f) + f\,\mathrm{div}(X)$. Thus

$$p_2(\iota_2(g)) = \frac{1}{k+2}\mathrm{div}(gE) = \frac{1}{k+2}(E(g) + g\,\mathrm{div}(E)) = g.$$

(3) Follows from (1) and (2).

(4) Let $X = p\partial/\partial x + q\partial/\partial y \in \mathrm{Vect}_k(\mathbf{R}^2)$ so that both $p = X(x)$ and $q = X(y)$ are in $\mathscr{P}_{k+1}$. We have

$$\begin{aligned} (\pi_1 + \pi_2)\, X &= \frac{1}{k+2}\left(\iota_1(yp - xq) + \iota_2\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)\right) \\ &= \frac{1}{k+2}\left(\frac{\partial(yp-xq)}{\partial y}\frac{\partial}{\partial x} - \frac{\partial(yp-xq)}{\partial x}\frac{\partial}{\partial y} + \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)E\right) \\ &= \frac{1}{k+2}\left((p + E(p))\frac{\partial}{\partial x} + (q + E(q))\frac{\partial}{\partial y}\right) \\ &= p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} = X. \quad \blacksquare \end{aligned}$$

To calculate the structure constants of $\mathrm{Vect}(\mathbf{R}^2; 0)$ we shall make use of the above decomposition. Since the $\iota_j$ are monomorphisms, brackets of the form $[\iota_j(h), \iota_j(g)]$ can be readily computed from the brackets on Ham and Eul. The mixed brackets are somewhat more involved and will be calculated next.

PROPOSITION 3.3. *Let* $h \in \mathrm{Ham}_k(\mathbf{R})$ *and* $g \in \mathrm{Eul}_m(\mathbf{R}^2)$. *Then*

$$[\iota_1(h), \iota_2(g)] = \frac{1}{m+k+2}(-k(k+2)\, X_{hg} \oplus (m+2)\{h, g\}\, E). \quad (3.4)$$

*Proof.* Note that $p_1(gX_h) = ((k+2)/(m+k+2))\, gp_1(X_h) = ((k+2)/(k+m+2))\, gh$ and that since $\mathrm{div}(X_h) = 0$, $\mathrm{div}(gX_h) = X_h(g) = \{h, g\}$. Thus

$$\begin{aligned}
[\iota_1(h), \iota_2(g)] &= [X_h, gE] \\
&= \{h, g\}\, E + g[X_h, E] \\
&= \{h, g\}\, E - kgX_h \\
&= -k(\pi_1 + \pi_2)(gX_h) + \{h, g\}\, E \\
&= -k\iota_1(p_1(gX_h)) - k\iota_2(p_2(gX_h)) + \{h, g\}\, E \\
&= -k\frac{k+2}{k+m+2}\iota_1(gh) - \frac{k}{k+m+2}\iota_2(\mathrm{div}(gX_h)) + \{h, g\}\, E \\
&= \frac{1}{k+m+2}(-k(k+2)\, X_{gh} + (m+2)\{h, g\}\, E). \quad \blacksquare
\end{aligned}$$

To introduce a second degree observe that $E$ acts semisimply both on $\mathbf{R}[[x, y]]$ and on $\mathrm{Vect}(\mathbf{R}^2; 0)$ with spectrum $\mathbf{Z}_+$. Since in either instance this action is a derivation, one could *define* the homogeneous components of these algebras as being the eigenspaces of $E$. Then the derivation property would automatically take care of the correct behavior of such gradings relative to multiplication (associative or Lie as the case may be). We refer to the Euler grading as *standard* or *total*, and denote the corresponding degree function $\delta^{(1)}$. Following the procedure just outlined, we introduce a new grading on the four algebras $\mathbf{R}[[x, y]]$, Vect, Ham, and Eul, the "$x$-degree," given by the Euler field in the $x$-direction $x\partial/\partial x$, and we denote the corresponding degree function $\delta^{(x)}$. Here we view Ham and Eul as subalgebras of Vect via the inclusions $\iota_j$.

*Remark.* Contrary to an earlier assumption the $\delta^{(x)}$-homogeneous components in any of these algebras are infinite dimensional. Observe, however, that any linear combination of $\delta^{(1)}$ and $\delta^{(x)}$ with positive (integral) coefficients will produce a new degree for which the finite dimensionality assumption holds. We will make use of this fact later on.

Writing $u_l^k$ to indicate typical elements satisfying $\delta^{(1)}(u_l^k) = l$ and $\delta^{(x)}(u_l^k) = k$, we define bases $\{a_k^l\}$ of Ham($\mathbf{R}$; 0) and $\{b_m^n\}$ of Eul($\mathbf{R}^2$; 0) as follows:

$$a_k^l = x^{l+1}y^{k+1-l}, \qquad -1 \leqslant l \leqslant k+1, \qquad k \geqslant 0$$

$$b_m^n = x^n y^{m-n}, \qquad 0 \leqslant n \leqslant m, \qquad m \geqslant 0.$$

Noting that $b_m^n = a_{m-2}^{n-1}$, a straightforward calculation using the definitions yields

$$\{a_k^l, a_{k'}^{l'}\} = ((l'+1)(k+2) - (l+1)(k'+2))\, a_{k+k'}^{l+l'} \tag{3.5a}$$

$$[b_m^n, b_{m'}^{n'}] = (m'-m)\, b_{m+m'}^{n+n'} \tag{3.5b}$$

$$a_k^l b_m^n = a_{k+m}^{l+n} \tag{3.5c}$$

$$\{a_k^l, b_m^n\} = (n(k+2) - m(l+1))\, b_{k+m}^{l+n}. \tag{3.5d}$$

It is clear from (3.5a) and (3.5b) that Ham and Eul are brigaded Lie algebras. Using the Clebsch–Gordan decomposition we are now in a position to introduce a suitable basis for Vect($\mathbf{R}^2$; 0). Define

$$A_k^l = \frac{1}{k+2}\,\iota_1(a_k^l), \qquad -1 \leqslant l \leqslant k+1, \qquad k \geqslant 0 \tag{3.6a}$$

$$B_m^n = \iota_2(b_m^n), \qquad 0 \leqslant n \leqslant m, \qquad m \geqslant 0. \tag{3.6b}$$

THEOREM 3.7. (Structure Constants of Vect($\mathbf{R}^2$; 0)).

$$[A_k^l, A_m^n] = (k+m+2)\left(\frac{n+1}{m+2} - \frac{l+1}{k+2}\right) A_{k+m}^{l+n} \tag{3.7a}$$

$$[B_k^l, B_m^n] = (m-k)\, B_{k+m}^{n+l} \tag{3.7b}$$

$$[A_k^l, B_m^n] = \frac{m(m+2)}{m+k+2}\left(\frac{n}{m} - \frac{l+1}{k+2}\right) B_{k+m}^{n+l} - kA_{k+m}^{n+l}. \tag{3.7c}$$

*Proof.* (a) and (b) follow from (3.5). For the mixed case we use the latter together with (3.4) to obtain

$$\begin{aligned}[A_k^l, B_m^n] &= \frac{1}{k+m+2}\left(-kX_{a_k^l b_m^n} \oplus \frac{m+2}{k+2}\{a_k^l, b_m^n\}\, E\right)\\ &= -kA_{k+m}^{l+n} \oplus \frac{m(m+2)}{k+m+2}\left(\frac{n}{m} - \frac{l+1}{k+2}\right) b_{k+m}^{l+n} E. \qquad \blacksquare\end{aligned}$$

For further reference we list here a few useful special cases:

$$[A_0^1, A_m^n] = (n-m-1)\, A_m^{n+1} \tag{3.8a}$$

$$[A_0^1, B_m^n] = (n-m)\, B_m^{n+1} \tag{3.8b}$$

$$[A_0^{-1}, A_m^n] = (n+1)\, A_m^{n-1} \tag{3.8c}$$

$$[A_0^{-1}, B_m^n] = nB_m^{n-1} \tag{3.8d}$$

$$[A_\mu^{-1}, A_m^n] = \frac{(\mu+m+2)(n+1)}{m+2} A_{\mu+m}^{n-1} \tag{3.8e}$$

$$[A_\mu^{-1}, B_m^n] = \frac{(m+2)\, n}{\mu+m+2} B_{m+\mu}^{n-1} - \mu A_{\mu+m}^{n-1} \tag{3.8f}$$

$$[B_\nu^0, A_m^n] = \frac{\nu(\nu+2)(n+1)}{(m+\nu+2)(m+2)} B_{m+\nu}^n + mA_{m+\nu}^n \tag{3.8g}$$

$$[B_\nu^0, B_m^n] = (m-\nu)\, B_{\nu+m}^n. \tag{3.8h}$$

## 4. First Order Normal Forms and Their Invariants

Consider a planar vector field $X$ in equilibrium at the origin with a non-zero nilpotent linearization. In the notation of Section 3 and up to linear conjugation, we may and will assume $X$ to be of the form

$$X = -x\frac{\partial}{\partial y} + \cdots = A_0^1 + \cdots, \tag{4.1}$$

where the dots represent terms in $\mathscr{F}_1$. From the work of Takens ([Ta]) we know that there is a formal near-identity transformation sending $X$ to its (first order) normal form

$$X^{(1)} = -x\frac{\partial}{\partial y} + f(y)\frac{\partial}{\partial x} + g(y)\, E,$$

which for simplicity of notation will be relabeled $X$. To place this result in our context we briefly recall the theory of first order normal forms for fields with nilpotent linearization $N$ as presented in [Cu–Sa1]. By 1. of Theorem 2.2 with $\mathscr{J} = \mathscr{F}_1$, one can always conjugate $X = N + \sum_{k \geqslant 1} X_k$ to an element $Y = N + \sum_{k \geqslant 1} Y_k$, where the $Y_k$ belong to an *arbitrary* complement to $\operatorname{im} \operatorname{ad}(N)$ in $\mathrm{Vect}_k(\mathbf{R}^n)$. Using standard properties of $\mathrm{sl}(2, \mathbf{R})$-representation theory, such a complement is automatically given by $\ker \operatorname{ad}(M)$, where $M$ is any linear nilpotent field such that $N$ and $M$

generate a Lie algebra isomorphic to sl(2, $\mathbf{R}$). In our case, and using the structure constants (3.7), one can readily check that for $N=A_0^1$ an appropriate choice is $M=A_0^{-1}$. In fact from the properties of the $x$-degree it is immediately seen that ad$(A_0^1)$ raises upper degree by 1, whereas ad$(A_0^{-1})$ lowers it by the same amount. It follows that

$$\operatorname{im}(\operatorname{ad}(N) \mid \operatorname{Vect}_k(\mathbf{R}^2)) = \operatorname{span}\{A_k^l, B_k^n\}_{l=0, n=1}^{k+1, k},$$

so that for a complement one can take the two-dimensional space

$$\ker(\operatorname{ad}(M) \mid \operatorname{Vect}_k(\mathbf{R}^2)) = \operatorname{span}\{A_k^{-1}, B_k^0\}.$$

Therefore the first order normal form has a representation

$$X = A_0^1 + \sum_{k=\mu}^{\infty} \alpha_k A_k^{-1} + \sum_{k=\nu}^{\infty} \beta_k B_k^0, \qquad 1 \leqslant \mu \leqslant \infty, 1 \leqslant \nu \leqslant \infty, \tag{4.2}$$

where it is understand that while either sum may be absent, our notation is such that $\mu < \infty \Rightarrow \alpha_\mu \neq 0$ and similarly $\nu < \infty \Rightarrow \beta_\nu \neq 0$. We leave it to the reader to check that (4.2) coincides with Takens' result.

Following [Ba–Sa] we construct a new degree $\delta^{(2)}$ on Vect($\mathbf{R}^2$) defined on bihomogeneous elements by

$$\delta^{(2)} = 2\delta^{(1)} + \min(\mu, 2\nu)\, \delta^{(x)}, \tag{4.3}$$

so that $\delta^{(2)}(A_k^l) = \delta^{(2)}(B_k^l) = 2k + l \min(\mu, 2\nu)$. The reason for this seemingly bizarre choice will become apparent in the next few sections. For now we can only say that the idea is to assign the same (new) degree to $A_0^1$ and to either $A_\mu^{-1}$ or to $B_\nu^0$ depending on whether $\mu \leqslant 2\nu$ or not. Thus

$$\delta^{(2)}(A_0^1) = \delta^{(2)}(A_\mu^{-1}) = \mu \qquad \text{if} \quad \mu \leqslant 2\nu,$$

and

$$\delta^{(2)}(A_0^1) = \delta^{(2)}(B_\nu^0) = 2\nu \qquad \text{if} \quad \mu \geqslant 2\nu.$$

In what follows we will occasionally abbreviate $\operatorname{Vect}_j^{(1)}$ and $\mathscr{F}_j^{(1)}$ as introduced in item 11 of Section 2 to $\operatorname{Vect}_j$ and $\mathscr{F}_j$, respectively, reserving superscripted notation for degrees other than the standard $\delta^{(1)}$.

PROPOSITION 4.4. *Let $Z \in \mathscr{F}_1$ be such that $Y = \exp \operatorname{ad}(Z) X$ is also in first order normal form. Then $Z \in \mathscr{F}_1^{(2)}$.*

*Proof.* Write $Z = \sum_{j \geqslant -1} Z^j$, where $Z^j$ is $\delta^{(x)}$-homogeneous of degree $j$ and observe that $Z \in \mathscr{F}_1 \Rightarrow Z^j = \sum_{k \geqslant 1} Z_k^j$ so that $\delta^{(2)}(Z^j) \geqslant 2 + j \min(\mu, 2\nu) > 0$ unless $j = -1$. Thus it suffices to show that $\delta^{(2)}(Z^{-1}) > 0$.

By (2.5) and (3) of Corollary 2.7 $[Z, X] \overset{\mathscr{F}_{\lambda+2}}{\equiv} [Z, X]_{\lambda+1}$, where $\lambda := \min(\mu, \nu)$. For the 0th order $\delta^{(x)}$-homogeneous component we then have

$$[Z^{-1}, X^1] + [Z^0, X^0] + [Z^1, X^{-1}] = \sum_{k \geqslant \lambda+1} [Z, X]_k^0.$$

From $X^0 = \sum_{k \geqslant \nu} \beta_k B_k^0$ and $X^{-1} = \sum_{k \geqslant \mu} \alpha_k A_k^{-1}$, it follows that $[Z^0, X^0] + [Z^1, X^{-1}] \in \mathscr{F}_{\nu+1} + \mathscr{F}_{\mu+1} \subseteq \mathscr{F}_{\lambda+1}$, so that $[Z^{-1}, X^1] \in \mathscr{F}_{\lambda+1}$. But $X^1 = A_0^1$, and $\operatorname{ad}(A_0^1)$ is injective on $\mathrm{Vect}_{-1}^{(x)}$ (see (3.8)). Thus $Z^{-1} \in \mathscr{F}_{\lambda+1}$ and

$$\begin{aligned}
\delta^{(2)}(Z^{-1}) &\geqslant 2(\lambda+1) - \min(\mu, 2\nu) \\
&= 2 + 2\min(\mu, \nu) - \min(\mu, 2\nu) \\
&= 2 + \begin{cases} \mu & \text{if } \mu \leqslant \nu \\ 2\nu - \mu & \text{if } \nu \leqslant \mu \leqslant 2\nu \\ 0 & \text{if } 2\nu \leqslant \mu \end{cases} \\
&\geqslant 2 \qquad \text{in all cases.} \quad \blacksquare
\end{aligned}$$

To examine the properties of the indices $\mu$, $\nu$, we again suppose that $Y$ is conjugate to $X$ via a near identity transformation and is also in first order normal form. Thus

$$Y = \exp \operatorname{ad}(Z)\, X \qquad \text{for some} \quad Z \in \mathscr{F}_1.$$

Moreover, one has the representation

$$Y = A_0^1 + \sum_{k=\mu'}^{\infty} \alpha_k' A_k^{-1} + \sum_{j=\nu'}^{\infty} \beta_j' B_j^0,\ 1 \leqslant \mu' \leqslant \infty,\ 1 \leqslant \nu' \leqslant \infty,$$

which is generally distinct from that of $X$.

PROPOSITION 4.5. *The indices $\mu$, $\nu$ have the following invariance properties*:

(a) $\min(\mu', \nu') = \min(\mu, \nu)$. *Moreover if* $m := \min(\mu, \nu) < \infty$ *then* $\alpha_m' = \alpha_m$ *and* $\beta_m' = \beta_m$.

(b) $\mu < 2\nu$ *iff* $\mu' < 2\nu'$. *In such case* $\mu' = \mu$ *and* $\alpha_\mu' = \alpha_\mu$.

(c) $\mu > 2\nu$ *iff* $\mu' > 2\nu'$. *In such case* $\nu' = \nu$ *and* $\beta_\nu' = \beta_\nu$.

(d) $\mu = 2\nu$ *iff* $\mu' = 2\nu'$. *In such case* $\mu' = \mu$, $\nu' = \nu$, *and if finite, then* $\alpha_\mu' = \alpha_\mu$ *and* $\beta_\nu' = \beta_\nu$.

*Proof.* (a) Use (3) of Corollary 2.7 with $\mathscr{L} = \mathrm{Vect}$, $\mathscr{I} = \mathscr{F}_1$, $\gamma = 0$, and $x_\gamma = A_0^1$.

(b) By Proposition 4.4, $Z \in \mathscr{F}_1^{(2)}$ so that we can apply Corollary 2.7 this time to $\mathscr{L} = \text{Vect}^{(2)}$ with $\mathscr{I} = \mathscr{F}_1^{(2)}$, $\gamma = \mu$, and $x_\gamma = A_0^1 + \alpha_\mu A_\mu^{-1}$.

(c) We argue as in (b), with $\mathscr{L} = \text{Vect}^{(2)}$, $\mathscr{I} = \mathscr{F}_1^{(2)}$, $\gamma = 2\nu$, and $x_\gamma = A_0^1 + \beta\nu B_\nu^0$.

(d) The same argument applies here with $\mathscr{L} = \text{Vect}^{(2)}$, $\mathscr{I} = \mathscr{F}_1^{(2)}$, $\gamma = \mu = 2\nu$, and $x_\gamma = A_0^1 + \alpha_\mu A_\mu^{-1} + \beta\nu B_\nu^0$. ∎

COROLLARY 4.6. *The new degree* (4.3) *is an invariant of X in its conjugacy class* mod exp ad$(\mathscr{F}_1^{(1)})$.

COROLLARY 4.7. *If $\nu < \mu \leqslant 2\nu$ then $(\mu', \nu') = (\mu, \nu)$.*

*Proof.* By (a) we may assume $\nu, \mu$ finite. When $\mu = 2\nu$ the result follows from (d). On the oter hand when $\nu < \mu < 2\nu$, (b) $\Rightarrow \mu' = \mu$, so that from (a) we get $\min(\mu', \nu') = \min(\mu, \nu) = \nu < \mu = \mu'$. But this implies $\nu = \min(\mu', \nu') = \nu'$. ∎

*Remark* 4.8. Despite the corollary, the pair $(\mu, \nu)$ is *not* an unrestricted first order normal form invariant of $X$. Indeed, one can show that any $X$ for which $\mu < \nu$ has a conjugate in first order normal form with $\nu' \neq \nu$. Similarly any field with $\mu > 4\nu$ has another form for which $\mu' \neq \mu$. True invariants will come out of our higher order normal forms below. On the other hand, a more careful analysis shows that the pair $(\mu, \nu)$ is a first order normal form invariant on the range $\nu \leqslant \mu \leqslant 3\nu + 1$, which is larger than the one given by the last corollary. This fact, however, has limited interest. What matters to us is that the trichotomy $\mu < 2\nu$, $\mu = 2\nu$, and $\mu > 2\nu$, is a true invariant of the field obtainable form any one of its first order normal forms. Because of this we will study separately the following cases:

I. $\mu < 2\nu$, $\quad \delta^{(2)}(A_0^1) = \delta^{(2)}(A_\mu^{-1}) = \mu < 2\nu = \delta^{(2)}(B_\nu^0)$.

II. $\mu > 2\nu$, $\quad \delta^{(2)}(A_0^1) = \delta^{(2)}(B_\nu^0) = 2\nu < \mu < 2\mu - 2\nu = \delta^{(2)}(A_\mu^{-1})$.

III. $\mu = 2\nu$, $\quad \delta^{(2)}(A_0^1) = \delta^{(2)}(A_\mu^{-1}) = \delta^{(2)}(B_\nu^0) = 2\nu = \mu$.

Since $\delta^{(2)}$ is a degree, we can in Case I consider $B_\nu^0$ as a perturbation of the (Hamiltonian) problem $A_0^1 + \alpha_\mu A_\mu^{-1} + \cdots$, while in Case II we consider $A_\mu^{-1}$ as a perturbation of the problem $A_0^1 + \beta_\nu B_\nu^0$. In this respect Case II seems to be the most difficult, since it cannot be formulated as a perturbation problem of either $A_\mu^{-1}$ or $B_\nu^0$, but only of a combination of the two. As usual however the exceptional situation is made slightly easier by the explicit knowledge of the relation $\mu = 2\nu$, but complicated by the extra parameter.

## 5. Case I: $\mu < 2\nu$

### 5.1. *Second Order Normal Form*

We assume $X$ to be in first order normal form (4.2) but now write $X = A + B$, where $A = A^{\mu} + \cdots + A^{\lambda} + \cdots$, $B = B^{2\nu} + \cdots + B^{\lambda} + \cdots$, and $\delta^{(2)}(A^{\lambda}) = \delta^{(2)}(B^{\lambda}) = \lambda$. We point out that in Case I the lowest $\delta^{(2)}$-homogeneous component of $X$ is

$$A^{\mu} = A_0^1 + \alpha_{\mu} A_{\mu}^{-1},$$

whereas that of $B$ is

$$B^{2\nu} = \beta_{\nu} B_{\nu}^0.$$

Now define $\Gamma := \mathrm{ad}(A_0^{-1}) \circ \mathrm{ad}(A^{\mu})$, and note that since $\delta^{(2)}(A_0^{-1}) = -\delta^{(2)}(A^{\mu}) = -\mu$, $\Gamma$ is a $\delta^{(2)}$-homogeneous operator of degree 0, and thus an endomorphism on each $\mathrm{Vect}_{\lambda}^{(2)}(\mathbf{R}^2; 0)$.

*Remark.* The purpose of introducing $\Gamma$ is to make the study of $\mathrm{im}\,\mathrm{ad}(A^{\mu})$ in the space of first order normal forms $\ker \mathrm{ad}(A_0^{-1})$, finite dimensional.

To calculate $\Gamma(A_k^l)$ and $\Gamma(B_k^l)$, recall that from (3.8) we have

$$[A_0^{-1}, [A_0^1, A_k^l]] = (l+2)(l-k-1)\, A_k^l,$$

$$[A_0^{-1}, [A_0^1, B_k^l]] = (l+1)(l-k)\, B_k^l,$$

$$[A_0^{-1}, [A_{\mu}^{-1}, A_k^l]] = \frac{l(l+1)(\mu+k+2)}{k+2}\, A_{k+\mu}^{l-2},$$

$$[A_0^{-1}, [A_{\mu}^{-1}, B_k^l]] = \frac{l(l-1)(k+2)}{k+\mu+2}\, B_{k+\mu}^{l-2} - \mu l A_{k+\mu}^{l-2}.$$

Combining these we get

$$\Gamma(A_k^l) = (l-k-1)(l+2)\, A_k^l + \alpha_{\mu} \frac{l(l+1)(k+\mu+2)}{k+2}\, A_{k+\mu}^{l-2},$$

$$\Gamma(B_k^l) = (l-k)(l+1)\, B_k^l + \alpha_{\mu} \frac{l(l-1)(k+2)}{k+\mu+2}\, B_{k+\mu}^{l-2} - \alpha_{\mu} l \mu A_{k+\mu}^{l-2}.$$

We now order the basis $\{A_k^l, B_k^l\}$ of $\mathrm{Vect}_{\lambda}^{(2)}(\mathbf{R}^2; 0)$ as follows: $B_{k'}^{l'} < A_k^l$, and $B_{k'}^{l'} < B_k^l$, $A_{k'}^{l'} < A_k^l$ whenever $l' < l$.

The above formulas show that relative to this ordered basis the matrix of $\Gamma$ is triangular, and from this we can readily determine its spectrum. To make the problem finite dimensional, let $\Gamma_\lambda := \Gamma \mid \mathrm{Vect}_\lambda^{(2)}(\mathbf{R}^2)$.

PROPOSITION 5.1.1. *$\Gamma_\lambda$ is invertible unless $\lambda \overset{(\mu+2)}{\equiv} 0$ or $\mu$ in which cases $\ker(\Gamma_\lambda)$ is one-dimensional.*

*Proof.* First we claim that 0 is an eigenvalue of multiplicity at most 1 on each $\delta^{(2)}$-homogeneous subspace of $\mathrm{Vect}(\mathbf{R}^2; 0)$. Indeed the only basic elements associated with this eigenvalue are of the form $A_k^{k+1}$ or $B_k^k$. But no two elements of such form can have the same $\delta^{(2)}$-degree as the relations

$$\delta^{(2)}(B_k^k) < \delta^{(2)}(A_k^{k+1}) < \delta^{(2)}(B_{k+1}^{k+1})$$

show. This establishes our claim.

From the algebraic multiplicity of 0 as an element of $\mathrm{spec}(\Gamma_\lambda)$ being zero or one, it follows that the same is true of its geometric multiplicity. Thus $\dim \ker(\Gamma_\lambda) \leqslant 1$. On the other hand $0 \in \mathrm{spec}(\Gamma_\lambda)$ iff $(l-k-1)(l+2)=0$, $-1 \leqslant l \leqslant k+1$, $\lambda = 2k + l\mu$ or $(l-k)(l+1)=0, 0 \leqslant l \leqslant k, \lambda = 2k + l\mu$. This clearly is the case iff $l = k+1$ or $l = k$ with $\lambda = 2k + l\mu$, which gives $\lambda \overset{(\mu+2)}{\equiv} 0$ or $\mu$. ▮

We now study $\ker(\Gamma_\lambda)$ in the non-trivial cases. Notice that the congruences 0 and $\mu \bmod(\mu+2)$ split into four cases $\bmod(2\mu+4)$: 0, $-2$, $\mu$, and $\mu+2$. These will be treated separately in the ensuing paragraphs.

PROPOSITION 5.1.2. *Let $\lambda \overset{(2\mu+4)}{\equiv} 0$, say $\lambda = k(2\mu+4)$. Then $\ker(\Gamma_\lambda)$ is spanned by a unique element $Z \overset{\mathscr{F}_{2k+1}}{\equiv} B_{2k}^{2k}$. Moreover $[A^\mu, Z]$ is a non-vanishing multiple of $A_{k(\mu+2)+\mu}^{-1}$.*

*Remark.* $\mathscr{F}_k$ refers to the standard filtration.

*Proof.* With $\alpha_\mu^m := (\alpha_\mu)^m$, the $m$th power of $\alpha_\mu$, let

$$Z = \sum_{m=0}^{k} \alpha_\mu^m b_m B_{2k+\mu m}^{2k-2m} - \sum_{m=1}^{k} \alpha_\mu^m a_m A_{2k+\mu m}^{2k-2m},$$

where the coefficients $a_m$, $b_m$, are recurrently defined by the relations

$$(\mu+2)(m+1)\, b_{m+1} = \frac{2(k-m)(2k+2+\mu m)}{2k+\mu(m+1)+2}\, b_m,$$

and

$$((\mu+2)(m+1)+1)\, a_{m+1} = \frac{2k+1-2m}{2k+2+\mu m}\,(2k+(m+1)\,\mu+2)\, a_m + \mu b_m,$$

with initial conditions $a_0=0$ and $b_0=1$. Since the $b_m$ are clearly non-negative, it follows that the $a_m$ are positive, and in particular that $a_{k+1}\neq 0$. Using the brackets in (3.8) we have

$$\begin{aligned}
[A^\mu, Z] &= -\sum_{m=1}^{k} \alpha_\mu^m a_m [A_0^1, A_{2k+\mu m}^{2k-2m}] + \sum_{m=0}^{k} \alpha_\mu^m b_m [A_0^1, B_{2k+\mu m}^{2k-2m}] \\
&\quad - \sum_{m=1}^{k} \alpha_\mu^{m+1} a_m [A_\mu^{-1}, A_{2k+\mu m}^{2k-2m}] + \sum_{m=0}^{k} \alpha_\mu^{m+1} b_m [A_\mu^{-1}, B_{2k+\mu m}^{2k-2m}] \\
&= \sum_{m=1}^{k} \alpha_\mu^m a_m ((\mu+2)m+1) A_{2k+\mu m}^{2k-2m+1} \\
&\quad - \sum_{m=0}^{k} \alpha_\mu^m b_m (\mu+2) m B_{2k+\mu m}^{2k-2m+1} \\
&\quad - \sum_{m=1}^{k} \alpha_\mu^{m+1} a_m \frac{2k+1-2m}{2k+2+\mu m} (2k+(m+1)\mu+2) A_{2k+\mu(m+1)}^{2k-2m-1} \\
&\quad - \sum_{m=0}^{k} \alpha_\mu^{m+1} b_m \mu A_{2k+(m+1)\mu}^{2k-2m-1} \\
&\quad + \sum_{m=0}^{k-1} \alpha_\mu^{m+1} b_m \frac{2(k-m)(2k+2+\mu m)}{2k+2+\mu(m+1)} B_{2k+(1+m)\mu}^{2k-2m-1} \\
&= -\alpha_\mu^{k+1} a_{k+1}((\mu+2)(k+1)+1) A_{(\mu+2)k+\mu}^{-1} \neq 0. \quad \blacksquare
\end{aligned}$$

PROPOSITION 5.1.3. *Let $\lambda \overset{(2\mu+4)}{\equiv} -2$, say $\lambda = k(2\mu+4)-2$. Then $\ker(\Gamma_\lambda)$ is spanned by a unique element $Z \overset{\mathscr{F}_{2k}}{\equiv} A_{2k-1}^{2k}$ such that $[A^\mu, Z]$ is a non-vanishing multiple of $A_{k(\mu+2)+\mu-1}^{-1}$.*

*Proof.* Since the mapping $\imath_1 : h \to X_h$ is a bigraded Lie algebra monomorphism from $\mathrm{Ham}(\mathbf{R};0)$ to $\mathrm{Vect}(\mathbf{R}^2;0)$, the result follows from Proposition 8.8a in [Ba–Sa]. $\blacksquare$

PROPOSITION 5.1.4. *Let $\lambda \overset{(2\mu+4)}{\equiv} \mu+2$, say $\lambda=(k-1)(2\mu+4)+\mu+2$. Then $\ker(\Gamma_\lambda)$ is spanned by a unique element $Z \overset{\mathscr{F}_{2k}}{\equiv} B_{2k-1}^{2k-1}$ such that $[A^\mu, Z]$ is a non-vanishing multiple of $B_{k(\mu+2)-1}^{0}$.*

*Proof.* Note that $\delta^{(2)}(B_{2k-1}^{2k-1}) = 2(2k-1)+\mu(2k-1)=\lambda$ and that the same holds for arbitrary elements of the form $X_{2k-1+\mu m}^{2k-1-2m}$. We therefore define

$$Z = B_{2k-1}^{2k-1} + \cdots = \sum_{m=0}^{k-1} \alpha_\mu^m b_m B_{2k-1+\mu m}^{2k-1-2m} - \sum_{m=1}^{k} \alpha_\mu^m a_m A_{2k-1+\mu m}^{2k-1-2m}$$

with

$$(\mu+2)(m+1)\, b_{m+1}=\frac{(2k-1-2m)(2k+1+\mu m)}{2k+1+\mu(m+1)}\, b_m,$$

$$((\mu+2)(m+1)+1)\, a_{m+1}=\frac{2(k-m)}{2k+1+\mu m}\,(2k+1+(m+1)\,\mu)\, a_m+\mu b_m,$$

$a_0=0$, and $b_0=1$. As before, the positivity of the coefficients of the recursive relations defining the $a_m$ and $b_m$ imply that $b_k \neq 0$. From (3.8) we have

$$\begin{aligned}
[A^\mu, Z] = &- \sum_{m=1}^{k} \alpha_\mu^m a_m [A_0^1, A_{2k-1+\mu m}^{2k-1-2m}] + \sum_{m=0}^{k-1} \alpha_\mu^m b_m [A_0^1, B_{2k-1+\mu m}^{2k-1-2m}] \\
&- \sum_{m=1}^{k} \alpha_\mu^{m+1} a_m [A_\mu^{-1}, A_{2k-1+\mu m}^{2k-1-2m}] \\
&+ \sum_{m=0}^{k-1} \alpha_\mu^{m+1} b_m [A_\mu^{-1}, B_{2k-1+\mu m}^{2k-1-2m}] \\
= &\sum_{m=1}^{k} \alpha_\mu^m a_m ((\mu+2)\, m+1)\, A_{2k-1+\mu m}^{2k-2m} \\
&- \sum_{m=1}^{k-1} \alpha_\mu^m b_m (\mu+2)\, m B_{2k-1+\mu m}^{2k-2m} \\
&- \sum_{m=1}^{k-1} \alpha_\mu^{m+1} a_m \frac{2(k-m)}{2k+1+\mu m} A_{2k-1+\mu(m+1)}^{2k-2-2m} \\
&- \sum_{m=0}^{k-1} \alpha_\mu^{m+1} b_m \mu A_{2k-1+(1+m)\mu}^{2k-2-2m} \\
&+ \sum_{m=0}^{k-1} \alpha_\mu^{m+1} b_m \frac{(2k-1-2m)(2k+1+\mu m)}{2k+1+\mu(m+1)} B_{2k-1(1+m)\mu}^{2k-2-2m} \\
= &\,\alpha_\mu^k b_k (\mu+2)\, k B_{(\mu+2)k-1}^0 \neq 0. \quad \blacksquare
\end{aligned}$$

**Proposition** 5.1.5. *Let* $\lambda \overset{(2\mu+4)}{\equiv} \mu$, *say* $\lambda = k(2\mu+4)+\mu$. *Then* $\ker(\Gamma_\lambda)$ *is spanned by the Hamiltonian field* $X_{(a^\mu)^{k+1}}$, *where* $a^\mu := (x^2/2) + \alpha_\mu(y^{\mu+2}/\mu+2)$ *is the Hamiltonian for the field* $A^\mu$, *and the superscript* $k+1$ *represents ordinary* $(k+1)$*th power.*

*Proof.* Obviously $X_{(a^\mu)^{k+1}} \in \ker \operatorname{ad}(A^\mu) \subseteq \ker \Gamma$. Therefore the result follows from the straightforward verification that $\delta^{(2)}(X_{(a^\mu)^{k+1}}) = \lambda$. $\blacksquare$

For further reference we summarize the results of this section in the following:

THEOREM 5.1.6. *$\Gamma_\lambda$ is invertible unless $\lambda \overset{(2\mu+4)}{\equiv} 0, -2, \mu$, or $\mu+2$. In all such cases* $\ker(\Gamma_\lambda)$ *is one-dimensional. Furthermore*:

(1) *If $\lambda \overset{(2\mu+4)}{\equiv} 0$, say $\lambda = k(2\mu+4)$, then* $\ker(\Gamma_\lambda)$ *is spanned by a unique element of the form $Z \overset{\mathscr{F}_{2k+1}}{\equiv} B^{2k}_{2k}$ such that $[A^\mu, Z]$ is a non-vanishing multiple of $A^{-1}_{k(\mu+2)+\mu}$.*

(2) *If $\lambda \overset{(2\mu+4)}{\equiv} -2$, say $\lambda = k(2\mu+4)-2$, then* $\ker(\Gamma_\lambda)$ *is spanned by a unique element of the form $Z \overset{\mathscr{F}_{2k}}{\equiv} A^{2k}_{2k-1}$, such that $[A^\mu, Z]$ is a non-vanishing multiple of $A^{-1}_{k(\mu+2)+\mu-1}$.*

(3) *If $\lambda \overset{(2\mu+4)}{\equiv} \mu+2$, say $\lambda = (k-1)(2\mu+4)+\mu+2$, then* $\ker(\Gamma_\lambda)$ *is spanned by a unique element of the form $Z \overset{\mathscr{F}_{2k+2}}{\equiv} B^{2k-1}_{2k-1}$ such that $[A^\mu, Z]$ is a non-vanishing multiple of $B^0_{k(\mu+2)-1}$.*

(4) *If $\lambda \overset{(2\mu+4)}{\equiv} \mu$, say $\lambda = k(2\mu+4)+\mu$, then* $\ker(\Gamma_\lambda)$ *is spanned by a unique element of the form $Z = X_{(a^\mu)^{k+1}} \overset{\mathscr{F}_{2k+1}}{\equiv} A^{2k+1}_{2k}$ in the kernel of* $\mathrm{ad}(A^\mu)$.

COROLLARY 5.1.7. *$\Gamma$ is injective on* $\mathrm{Vect}^{(2)}_\lambda \cap \mathrm{Vect}^{(x)}_{(\mathrm{even})}$ *unless $\lambda \overset{(2\mu+4)}{\equiv} 0$ or $-2$.*

*Proof.* $\Gamma$ is homogeneous of $\delta^{(x)}$-degree 0 (mod 2) so that by uniqueness the elements $Z \in \ker(\Gamma)$ of (1)–(4) are $\delta^{(x)}$-homogeneous of degrees 0, 0, 1, and 1, respectively. ∎

COROLLARY 5.1.8. *Let $Z^\lambda$ be $\delta^{(2)}$-homogeneous. Then $Z^\lambda \in \ker \mathrm{ad}(A^\mu)$ if and only if $\lambda \overset{(2\mu+4)}{\equiv} \mu$, say $\lambda = k(2\mu+4)+\mu$, and $Z^\lambda = cX_{(a^\mu)^{k+1}}$ for some $c \in \mathbf{R}$. Moreover*

$$Z^\lambda = \sum_{m=0}^{k+1} \frac{a_m}{m!} \alpha_\mu^m A^{2k-2m+1}_{2k+\mu m},$$

*where $a_0 = c$ and the rest of the coefficients satisfy the recursive relations*

$$a_{m+1} = \frac{2(2k+\mu(m+1)+2)(k-m+1)}{(\mu+2)(2k+\mu m+2)} a_m, \qquad 0 \leqslant m \leqslant k. \quad (5.1.9)$$

*Proof.* The form of $Z^\lambda$ is a straightforward verification using (3.6a). ∎

Before stating the main result of this section, we direct the reader's attention to the paragraph following Remark (2.3). With $\mathscr{I} := \mathscr{F}^{(1)}_1 \cap \mathscr{F}^{(2)}_1$, the normalization that follows will take place in the $\delta^{(2)}$-graded Lie algebra

$\mathrm{Vect}^{(2)}(\mathbf{R}^2; 0)$ relative to the group $\mathscr{G} := \exp \mathrm{ad}(\mathscr{J})$. To formulate it, we introduce the second order normal form space

$$\mathscr{N}^{(2)} = \mathrm{span}\{A_j^{-1}, B_k^0\}, \qquad \text{where} \quad j > \mu, j \not\equiv \mu - 1, \mu, \text{ and } k \not\equiv \mu + 1. \tag{5.1.10}$$

PROPOSITION 5.1.11. *$\mathscr{N}^{(2)} \cap \mathrm{Vect}_\lambda^{(2)}$ is strictly transverse to* $\mathrm{ad}(A^\mu)$ $(\mathrm{Vect}_{\lambda-\mu}^{(2)})$ *for $\lambda > \mu$.*

*Proof.* By [Ba–Sa, Prop. 3.4] $\ker \mathrm{ad}(A_0^{-1}) \cap \mathrm{Vect}_\lambda^{(2)}$ is transverse to

$$\mathrm{im}\, \mathrm{ad}(A^\mu) \,|\, (\mathrm{Vect}_{\lambda-\mu}^{(2)} \cap \mathscr{F}_1^{(1)})$$

for each $\lambda > \mu$. The generators of $\ker \mathrm{ad}(A_0^{-1})$ are $\{A_k^{-1}\}$ and $\{B_j^0\}$. By (1)–(3) of Theorem 5.1.6 we can remove all terms $A_{k(\mu+2)+\mu}^{-1}$, $A_{k(\mu+2)+\mu-1}^{-1}$, and $B_{j(\mu+2)-1}^0$, with $k, j \geqslant 1$, and still have transversality. The resulting space is precisely $\mathscr{N}^{(2)}$.

To see that the sum $\mathscr{N}_{\lambda+\mu}^{(2)} + [A^\mu, \mathrm{Vect}_\lambda^{(2)}]$ is direct, assume $W \in \mathscr{N}_{\lambda+\mu}^{(2)} \cap [A^\mu, \mathrm{Vect}_\mu^{(2)}]$, say $W = [A^\mu, Z]$. Since $\mathscr{N}^{(2)} \subseteq \ker \mathrm{ad}(A_0^{-1})$, we have $Z \in \ker \Gamma_\lambda$. Thus $Z$ is as described in Theorem 5.1.6. In cases (1)–(3) of that reference one must have $Z = 0$, since the images of such elements have been removed from $\mathscr{N}^{(2)}$; and in case (4) $Z \in \ker \mathrm{ad}(A^\mu)$, so that $W = [A^\mu, Z] = 0$. ∎

THEOREM 5.1.12 (Second Order Normal Form). *Let $X = -x(\partial/\partial y) + \cdots$ be a nilpotent field admitting a first order normal form as in* (4.2) *with $\mu < 2\nu$. Then a second order normal form for $X$ has the same type of representation but with $\alpha_k = 0$ for $k \overset{(\mu+2)}{\equiv} \mu - 1$ or $\mu$, and $\beta_k = 0$ for $k \overset{(\mu+2)}{\equiv} \mu + 1$ for $k \geqslant \mu + 1$.*

*Proof.* The result follows from the above Proposition and (1) of Theorem 2.2. ∎

We will now show that the pair $(\mu, \nu)$ that enters the second order normal form is an invariant of the field (still under the assumption $\mu < 2\nu$).

LEMMA 5.1.13. *Let $A = A^\mu + \cdots$ be a Hamiltonian field, and let $Z = Z^j + \cdots + Z^r$ be a solution of the equation $[Z, A] \overset{\mathscr{F}^{(2)}_{\mu+r+1}}{\equiv} 0$. Then $Z$ extends to a solution $\tilde{Z}$ of $[\tilde{Z}, A] = 0$.*

*Proof.* By Corollary 5.1.8 the centralizer of $A^\mu$ is the set of Hamiltonian fields $X_H$ where $H$ is a power series in $a^\mu$. The result follows from [Ba–Sa, Prop. 5.2]. ∎

*Remark* 5.1.14. Strictly speaking the result of [Ba–Sa] would directly apply only if $Z$ had been assumed Hamiltonian. However, for the proof in that paper to be valid here we only need that the first homogeneous component of $Z$ be Hamiltonian and in $\ker \operatorname{ad}(A^\mu)$, which is indeed the case in view of our hypothesis and Corollary 5.1.8.

LEMMA 5.1.15. *Let $0 \neq Z \in \mathscr{J}$, be such that $\pi_0([Z, X]) \in \mathcal{N}^{(2)}$. Then*

(1) *$Z = Z^\lambda + \cdots$ for some $\lambda \geqslant 1$ and $0 \neq Z^\lambda \in \ker \operatorname{ad}(A^\mu)$.*

(2) *There exists a homogeneous element $W^{\lambda+2\nu-\mu}$ such that*

$$[Z, X] \overset{\mathscr{F}^{(2)}_{\lambda+2\nu+1}}{\equiv} [W^{\lambda+2\nu-\mu}, A^\mu] + [Z^\lambda, B^{2\nu}].$$

(3) $[A_0^{-1}, [W^{\lambda+2\nu-\mu}, A^\mu] + [Z^\lambda, B^{2\nu}]] = 0.$

*Proof.* Writing $Z = Z^\lambda + \cdots$ with $Z^\lambda \neq 0$, we shall prove by induction on $r \in [\mu, 2\nu]$ that

$$[Z, X] \overset{\mathscr{F}^{(2)}_{\lambda+r+1}}{\equiv} \begin{cases} 0 & \text{if } \mu \leqslant r < 2\nu \\ [W^{\lambda+2\nu-\mu}, A^\mu] + [Z^\lambda, B^{2\nu}] & \text{if } r = 2\nu. \end{cases} \tag{5.1.16}$$

Note that (1) then follows from $r = \mu$, (2) from $r = 2\nu$, and (3) from (2) and the fact that either

$$[W^{\lambda+2\nu-\mu}, A^\mu] + [Z^\lambda, B^{2\nu}] = \pi_0([Z, X]) \in \mathcal{N}^{(2)} \subseteq \ker \operatorname{ad}(A_0^{-1}),$$

or the left hand side of this equality vanishes.

To start the induction, observe that $[Z^\lambda, A^\mu] \neq 0$ implies

$$\mathcal{N}^{(2)} \ni \pi_0([Z, X]) = [Z^\lambda, A^\mu] \in \operatorname{im} \operatorname{ad}(A^\mu),$$

which is impossible by Proposition 5.1.11. Now assume $[Z, X] \overset{\mathscr{F}^{(2)}_{r+\lambda}}{\equiv} 0$ for some $r \in [\mu. 2\nu]$. Writing $X = A + B$ with $A = A^\mu + \cdots$ and $B = B^{2\nu} + \cdots$, we clearly have $[Z, B] \overset{\mathscr{F}^{(2)}_{r+\lambda}}{\equiv} 0$ so that $[Z, A] \overset{\mathscr{F}^{(2)}_{r+\lambda}}{\equiv} 0$ as well. By Lemma 5.1.13 we can extend $Z^\lambda + \cdots + Z^{\lambda+r-1-\mu}$ to a solution $\tilde{Z} = Z - W^{\lambda+r-\mu} + \cdots$ of $[\tilde{Z}, A] = 0$. Thus

$$\begin{aligned} [Z, X] &\overset{\mathscr{F}^{(2)}_{\lambda+r+1}}{\equiv} [Z, A] + [Z^\lambda, B^r] \\ &\overset{\mathscr{F}^{(2)}_{\lambda+r+1}}{\equiv} [\tilde{Z}, A] + [W^{\lambda+r-\mu}, A] + [Z^\lambda, B^r] \\ &\overset{\mathscr{F}^{(2)}_{\lambda+r+1}}{\equiv} [W^{\lambda+r-\mu}, A^\mu] + [Z^\lambda, B^r]. \end{aligned}$$

This concludes the induction if $r = 2\nu$. If on the other hand $r < 2\nu$ then $B^r = 0$, so that $[Z^\lambda, B^r] = 0$. But in that case $[W^{\lambda+r-\mu}, A^\mu] \neq 0 \Rightarrow \mathcal{N}^{(2)} \ni \pi_0([Z, X]) = [W^{\lambda+r-\mu}, A^\mu] \in \operatorname{im} \operatorname{ad}(A^\mu)$. This contradicts Prop. 5.1.11, and we conclude $[W^{\lambda+r-\mu}, A^\mu] = 0$, so that $[Z, X] \overset{\mathscr{F}^{(2)}_{\lambda+r+1}}{\equiv} 0$. ∎

COROLLARY 5.1.17. *For $\mu < 2\nu$ the pair $(\mu, \nu)$ of the second order normal form is an absolute invariant of the nilpotent field $X$.*

*Proof.* Let both $X$ and $Y = \exp \operatorname{ad}(Z)\, X$ be in second order normal form, with $X$ as above, and $Z \in \mathscr{F}_1^{(1)}$. By Proposition 4.4 $\in \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)} = \mathscr{J}$, whereas by (2.5) the first $\delta^{(2)}$-homogeneous component of $Y$ differing from the corresponding $X$-component is $\pi_0([Z, X])$. By hypothesis this must belong to $\mathscr{N}^{(2)}$, so that applying (2) of the proposition, we conclude that $\delta^{(2)}(\pi_0([Z, X])) \geqslant \lambda + 2\nu + 1 \geqslant 2\nu + 2 > 2\nu = \delta^{(2)}(B^{2\nu})$. ∎

5.2. *Infinite Order Normal Form* (*Case* I: $\mu < 2\nu$)

PROPOSITION 5.2.1. *Let $Z^\lambda \neq 0$ be a $\delta^{(2)}$-homogeneous element in* $\ker \operatorname{ad}(A^\mu) \cap \mathscr{F}_1^{(1)}$. *Then*

(1) *$\lambda = k(2\mu + 4) + \mu$ for some $k \geqslant 1$ and $Z^\lambda = bX_{(a^\mu)^{k+1}}$ for some $b \in \mathbf{R}$;*

(2) *There exists $W \in \mathrm{Vect}^{(2)}_{\lambda + 2\nu - \mu} \cap \mathrm{Vect}^{(x)}_{(\mathrm{even})}$ and a constant $c \neq 0$ such that*

$$[W, A^\mu] + [Z^\lambda, B^{2\nu}] = cA^{-1}_{k(\mu+2)+\mu+\nu}.$$

(3) *If $\nu \overset{(\mu+2)}{\equiv} 0, -1$, then $W$ is unique.*

*Proof.* (1) Corollary 5.1.8 implies that $\lambda$ and $Z^\lambda$ are of the indicated form, and furthermore $Z^\lambda \in \mathscr{F}_1^{(1)} \Rightarrow k \geqslant 1$. (The latter is because $X_{a^\mu} = A^\mu \notin \mathscr{F}_1^{(1)}$.)

(2) Without loss of generaity we may assume that $b = 1$. Let

$$W = \sum_{m=0}^{k} c_m \alpha_\mu^m \beta_\nu A^{2k-2m}_{2k+\nu+\mu m} + \sum_{m=0}^{k} b_m \alpha_\mu^m \beta_\nu B^{2k-2m}_{2k+\nu+\mu m},$$

with $\{b_m\}_0^k$, $\{c_m\}_0^{k+1}$, defined by the recursive relations

$$((2+\mu)\, m + \nu)\, b_m = \frac{2(k-m+1)}{2k+\mu m+\nu+2}\left((2k+\mu(m-1)+\nu+2)\, b_{m-1} + \frac{\nu(\nu+2)}{m!(2k+\mu m+2)}\, a_m\right), \tag{5.2.2a}$$

$$(\nu+1+m(\mu+2))\, c_m = \frac{(2k+\mu m+\nu)(2k-2m+3)}{2k+\mu(m-1)+\nu+2}\, c_{m-1} - \mu b_{m-1} + \frac{(2k+\mu m)}{m!}\, a_m, \tag{5.2.2b}$$

where $b_{-1} = c_{-1} = 0$, and the $a_k$ are given by (5.1.9) with $a_0 = 1$. Using (3.8) we have

$$\begin{aligned}
&[A^{\mu}, W] + [B^{2\nu}, Z^{\lambda}] \\
&\quad = [A_0^1 + \alpha_\mu A_\mu^{-1}, W] + [\beta_\nu B_\nu^0, Z^\lambda] \\
&\quad = -\sum_{m=0}^{k} c_m \alpha_\mu^m \beta_\nu (\nu + 1 + m(\mu+2))\, A_{2k+\mu m+\nu}^{2k-2m+1} \\
&\qquad + \sum_{m=1}^{k+1} c_{m-1} \alpha_\mu^m \beta_\nu \frac{(2k+\mu m+\nu+2)(2k-2m+3)}{2k+\mu(m-1)+\nu+2} A_{2k+\mu m+\nu}^{2k-2m+1} \\
&\qquad - \sum_{m=0}^{k} b_m \alpha_\mu^m \beta_\nu ((\mu+2)\, m + \nu)\, B_{2k+\mu m+\nu}^{2k-2m+1} \\
&\qquad + \sum_{m=1}^{k} b_{m-1} \alpha_\mu^m \beta_\nu 2(k-m+1) \frac{(2k+\mu(m-1)+\nu+2)}{2k+\mu m+\nu+2} B_{2k+\mu m+\nu}^{2k-2m+1} \\
&\qquad -\mu \sum_{m=1}^{k+1} b_{m-1} \alpha_\mu^m \beta_\nu A_{2k+\mu m+\nu}^{2k-2m+1} + \beta_\nu \sum_{m=0}^{k+1} \frac{a_m}{m!} \alpha_\mu^m (2k+\mu m)\, A_{2k+\mu m+\nu}^{2k-2m+1} \\
&\qquad + \sum_{m=0}^{k} \frac{a_m}{m!} \alpha_\mu^m \beta_\nu \frac{2\nu(\nu+2)(k-m+1)}{(2k+\mu m+\nu+2)(2k+\mu m+2)} B_{2k+\mu m+\nu}^{2k-2m+1},
\end{aligned}$$

which in view of the coefficients' definition reduces to

$$\begin{aligned}
&\left( \frac{(2+\mu)\,k+\mu+\nu+2}{(2+\mu)\,k+\nu+2} c_k - \mu b_k + \frac{(2+\mu)\,k+\mu}{(k+1)!} a_{k+1} \right) \alpha_\mu^{k+1} \beta_\nu A_{2k+\mu(k+1)+\nu}^{-1} \\
&\quad = (\nu + 1 + (k+1)(\mu+2))\, \alpha_\mu^{k+1} \beta_\nu c_{k+1} A_{k(\mu+2)+\mu+\nu}^{-1}.
\end{aligned}$$

We need to show that $c_{k+1} \neq 0$. From (5.2.2b) we see that it suffices to prove

$$\mu b_{m-1} < \frac{a_m}{m!} (2k + \mu m)$$

for all $1 \leqslant m \leqslant k$. In order to simplify the argument we introduce new coefficients as follows:

$$\begin{aligned}
\tilde{a}_m &= \frac{a_m}{m!(2k+\mu m+2)} \\
\tilde{b}_m &= (2k + \mu m + \nu + 2)\, b_m.
\end{aligned}$$

Then

$$\tilde{a}_{m+1} = \frac{2(k-m+1)}{(m+1)(\mu+2)} \tilde{a}_m$$

and

$$\tilde{b}_m((\mu+2)\, m + \nu) = 2(k-m+1)\, \tilde{b}_{m-1} + 2\nu(\nu+2)(k-m+1)\, \tilde{a}_m.$$

As one can easily check, solutions to these recursion relations are given by

$$\tilde{a}_m = \binom{k+1}{m} \left(\frac{2}{\mu+2}\right)^m \tilde{a}_0,$$

$$\tilde{b}_m = \binom{k}{m} \left(\frac{2}{\mu+2}\right)^m 2(\nu+2)(k+1)\, \tilde{a}_0.$$

This means that we must prove for $m \geqslant 1$

$$\mu \tilde{b}_{m-1} < \tilde{a}_m (2k+\mu m)(2k+\mu m+2)(2k+\mu(m-1)+\nu+2),$$

which on account of the above explicit solutions becomes

$$\mu m(\mu+2)(\nu+2) < (2k+\mu m)(2k+\mu m+2)(2k+\mu(m-1)+\nu+2).$$

But since $k \geqslant 1$ one has when $m \geqslant 1$:

$$\mu m < 2k + \mu m,$$

$$\mu + 2 < 2k + \mu m + 2,$$

and

$$\nu + 2 < 2k + \mu(m-1) + \nu + 2,$$

which establishes (2).

(3) Clearly the difference between any two solutions belongs to $\ker(\Gamma) \cap \mathrm{Vect}^{(2)}_{\lambda+2\nu-\mu} \cap \mathrm{Vect}^{(x)}_{(\mathrm{even})}$. By (1), $\lambda + 2\nu - \mu \overset{(2\mu+4)}{\equiv} 2\nu$. The result follows from our hypothesis $\nu \overset{(\mu+2)}{\not\equiv} 0, -1$, together with Corollary 5.1.7. ▮

To properly state our uniqueness result for Case I: $\mu < 2\nu$, the pair $(\mu, \nu)$ will henceforth refer to the second normal form invariant given by Corollary 5.1.17.

*Remark* 5.2.3. Note that $v \overset{(\mu+2)}{\not\equiv} -1$. (This follows from Theorem 5.1.12 and $\mu+1 \overset{(\mu\pm 2)}{\equiv} -1$).

Let

$$\mathcal{N}^{(\infty)} = \operatorname{span}\{A_j^{-1}, B_k^0\}, \tag{5.2.4}$$

where $j > \mu$, $j \not\equiv \mu - 1$, $j \not\equiv \mu + v$ when $j > \mu + v$, and $k \not\equiv \mu + 1$. Let $\mathscr{R} := \prod_\lambda \mathscr{R}_\lambda$, where the $\mathscr{R}_\lambda$ are the removable spaces $\pi_0([X, \mathscr{J})] \cap \mathrm{Vect}_\lambda^{(2)}$ introduced in (2.1). We have

PROPOSITION 5.2.5. (1) *$\mathcal{N}^{(\infty)} \cap \mathrm{Vect}_\lambda^{(2)}$ is transverse to $\mathscr{R}_\lambda$ for $\lambda > \mu$.*

(2) *If $v \overset{(\mu+2)}{\not\equiv} 0$ and $Z \in \mathscr{J}$, then $\pi_0([Z, X]) \in \mathcal{N}^{(\infty)} \Rightarrow Z = 0$.*

(3) *If $v \overset{(\mu+2)}{\not\equiv} 0$ then the transversality in* (1) *is strict.*

*Proof.* (1) Proposition 5.1.11 together with the definition (5.1.10) of $\mathcal{N}^{(2)}$, imply that it suffices to prove $A^{-1}_{k(\mu+2)+\mu+v} \in \mathscr{R}$. Let $Z^\lambda$, $W$, be as in Proposition 5.2.1. Using the Clebsch–Gordan decomposition write $X = A + B$ where $A = X_a$ for some Hamiltonian $a \in \mathbf{R}[[x, y]]$. We have $A = A^\mu + \cdots$ and $A^\mu = X_{a^\mu}$. Now the power $k+1$ of the Hamiltonian is of the form $a^{k+1} = (a^\mu)^{k+1} + \cdots$, where the omitted terms represent a Hamiltonian whose associated field has $\delta^{(2)}$ higher than $\lambda = \delta^{(2)}(Z^\lambda) = \delta^{(2)} X_{(a^\mu)^{k+1}}$. Let $Z = bX_{a^{k+1}} = Z^\lambda + \cdots$ (cf. (1) of Proposition 5.2.1). It is clear that $[Z, A] = [Z^\lambda, A^\mu] = 0$, so that $[Z + W, X] = [Z^\lambda, W] + [B^{2v}, A^\mu] + \cdots = cA^{-1}_{k(\mu+2)+\mu+v} + \cdots$, by (2) of Proposition 5.2.1. But this establishes $A^{-1}_{k(\mu+2)+\mu+v} \in \mathscr{R}$ as we wanted.

(2) Assume $\pi_0([Z, X]) \in \mathcal{N}^{(\infty)}$ for some $Z \neq 0$. Then by Lemma 5.1.15 one has $Z = Z^\lambda + \cdots$ for some $0 \neq Z^\lambda \in \ker \operatorname{ad}(A^\mu)$, and

$$[Z, X] \overset{\mathscr{F}^{(2)}_{\lambda+2v+1}}{\equiv} [W^{\lambda+2v-\mu}, A^\mu] + [Z^\lambda, B^{2v}]$$

for an appropriate homogeneous $W^{\lambda+2v-\mu}$. Since $v \overset{(\mu+2)}{\not\equiv} 0, -1$, we can apply (3) and (2) of Proposition 5.2.1, to conclude that

$$[W^{\lambda+2v-\mu}, A^\mu] + [Z^\lambda, B^{2v}] = cA^{-1}_{k(\mu+2)+\mu+v}$$

for some $c \neq 0$. But this implies

$$A^{-1}_{k(\mu+2)+\mu+v} = \pi_0((1/c)[Z, X]) \in \mathcal{N}^{(\infty)},$$

in contradiction with the definition of $\mathcal{N}^{(\infty)}$.

(3) This is immediate from (2). ∎

THEOREM 5.2.6 (Infinite Order Normal Form). *Every nilpotent field with $\mu < 2\nu$ is formally conjugate via a near identity transformation to a field of the form*

$$X^{(\infty)} = A_0^1 + \alpha_\mu A_\mu^{-1} + \alpha_{\mu+\nu} A_{\mu+\nu}^{-1} + \sum_{\substack{j>\mu \\ j \overset{(\mu+2)}{\not\equiv} \mu-1,\mu,\mu+\nu}} \alpha_j A_j^{-1} + \sum_{\substack{j \geqslant \nu \\ j \overset{(\mu+2)}{\not\equiv} \mu+1}} \beta_j B_j^0$$

*If $\nu \overset{(\mu+2)}{\not\equiv} 0$ then this form is unique, and its symmetry group among near identity transformations is trivial. Specifically, if $Y = \exp \operatorname{ad}(Z)\, X^{(\infty)}$ is also in infinite order normal form, then $Z = 0$ and $Y = X^{(\infty)}$.*

*Proof.* For existence first transform to first order normal form in $\mathrm{Vect}(\mathbf{R}^2; 0)$ with its standard filtration $\mathscr{F}^{(1)}$, and then apply (1) of Theorem 2.2 to $\mathcal{N}^{(\infty)}$ relative to the Lie subalgebra $\mathscr{J}$ of $\mathrm{Vect}^{(2)}(\mathbf{R}^2; 0)$. Uniqueness *relative to the group* $\exp \operatorname{ad}(\mathscr{F}_1^{(1)})$ (when $\nu \overset{(\mu+2)}{\not\equiv} 0$), follows from Proposition 4.4 together with (2) of Theorem 2.2. Finally, the last statement is a consequence of (2) of Proposition 5.2.5, together with the fact that $Y = X^{(\infty)}$ and (2.5) imply $\pi_0([Z, X]) = 0$. ∎

## 6. CASE II: $2\nu < \mu$

### 6.1. *Second Order Normal Form*

Since $\delta^{(2)}$ as defined by (4.3) has values in the even integers when $2\nu < \mu$, division by 2 will simplify notation in this chapter. Redefine (4.3) as follows:

$$\delta^{(2)}(A_k^l) = \delta^{(2)}(B_k^l) = k + \nu l.$$

Assume $X$ to be in first order normal form (4.2)

$$X = A_0^1 + \sum_{k=\mu}^{\infty} \alpha_k A_k^{-1} + \sum_{k=\nu}^{\infty} \beta_k B_k^0, \qquad 1 \leqslant \nu < \infty,\ 2\nu < \mu \leqslant \infty,$$

and write $X = X^\nu + X^\lambda + \cdots + X^j + \cdots$, where $\delta^{(2)}(X^j) = j$,

$$X^\nu = A_0^1 + \beta_\nu B_\nu^0,$$

and $X^\lambda$ represents *the first non-vanishing $\delta^{(2)}$-homogeneous component of $X$ beyond $X^\nu$* (which is supposed to exist whenever this notation is used. Otherwise we write $X = X^\nu$).

Let $\Gamma := \mathrm{ad}(A_0^{-1}) \circ \mathrm{ad}(X^\nu)$. Since $\delta^{(2)}(A_0^{-1}) = -\nu = -\delta^{(2)}(X^\nu)$, $\Gamma$ is a homogeneous operator of degree 0, and thus an endomorphism of the finite-dimensional vector space

$$\mathrm{Vect}_j^{(2)}(\mathbf{R}^2; 0) := \{ Y \in \mathrm{Vect}(\mathbf{R}^2; 0)) \mid \delta^{(2)}(Y) = j \}.$$

The action of $\Gamma$ on a basis of $\mathrm{Vect}_j^{(2)}(\mathbf{R}^2; 0)$ is described in the following formulas:

$$\begin{aligned}\Gamma(A_k^l) &= (l-k-1)(l+2)\, A_k^l + \beta_\nu k(l+1)\, A_{k+\nu}^{l-1} \\ &\quad + \frac{\beta_\nu \nu(\nu+2)(l+1)}{(k+2)(k+\nu+2)}\, B_{k+\nu}^{l-1}, \\ \Gamma(B_k^l) &= (l+1)(l-k)\, B_k^l + \beta_\nu l(k-\nu)\, B_{k+\nu}^{l-1},\end{aligned}$$

with $\delta^{(2)}(A_k^l) = \delta^{(2)}(B_k^l) = j$. Ordering the basis $\{A_k^l, B_k^l\}$ of $\mathrm{Vect}_j^{(2)}(\mathbf{R}^2; 0)$ as before (i.e., $B_{k'}^{l'} < A_k^l$, and $B_{k'}^{l'} < B_k^l, A_{k'}^{l'} < A_k^l$ whenever $l' < l$), we immediately see that $\Gamma$ is upper triangular with eigenvalues

$$(l+2)(l-k-1), \qquad -1 \leqslant l \leqslant k+1, \qquad k + \nu l = j,$$

and

$$(l+1)(l-k), \qquad 0 \leqslant l \leqslant k, \qquad k + \nu l = j.$$

Setting $k = j - \nu l$ we can thus write down a list of the roots $z_l$, $w_l$ of the characteristic polynomial of $\Gamma_j := \Gamma \mid \mathrm{Vect}_j^{(2)}(\mathbf{R}^2; 0)$ as follows:

$$z_l = (l+2)(l(\nu+1) - (j+1)), \ -1 \leqslant l \leqslant \left[\frac{j+1}{\nu+1}\right],$$

and

$$w_l = (l+1)((\nu+1)\, l - j), \ 0 \leqslant l \leqslant \left[\frac{j}{\nu+1}\right].$$

Thus

$$z_l = 0 \Leftrightarrow j \overset{(\nu+1)}{\equiv} -1, \ l = \frac{j+1}{\nu+1},$$

and

$$w_l = 0 \Leftrightarrow l = \left[\frac{j}{\nu+1}\right] \Leftrightarrow j \overset{(\nu+1)}{\equiv} 0, \ l = \frac{j}{\nu+1}.$$

PROPOSITION 6.1.1. *$\Gamma_j$ is invertible unless $j \overset{(v+1)}{\equiv} 0$ or $-1$. In these two cases* $\ker(\Gamma_j)$ *is one-dimensional.*

*Proof.* This follows from the above together with the observation that whenever $j \overset{(v+1)}{\equiv} 0$ or $-1$, the algebraic multiplicity of zero as a root of the characteristic polynomial is one. ∎

We now turn to the study of $\mathrm{ad}(X^v)$. For the rest of this section the relation $\equiv$ with no over-writings will indicate congruence $\mathrm{mod}(\mathrm{im}\,\mathrm{ad}(X^v \,|\, \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)}))$, and we will use the letters $a, b$ to indicate generic *positive* constants whose value may change from equation to equation.

PROPOSITION 6.1.2. *Let $\delta^{(2)}(B_r^k) > v$. Then $r \geqslant 1$ and for $1 \leqslant j \leqslant k$ there exist constants $b = b(r, k, j) > 0$ such that*

$$B_r^k \equiv b(\beta_v)^j \left( \prod_{n=-1}^{j-2} (r + nv) \right) B_{r+jv}^{k-j}. \tag{6.1.3}$$

*Moreover,*

$$B_r^k = \begin{cases} b(\beta_v)^k B_{r+kv}^0 & \text{if } r > v \\ 0 & \text{if } r = v \\ -b(\beta_v)^k B_{r+kv}^0 & \text{if } 1 \leqslant r < v. \end{cases} \tag{6.1.4}$$

*Proof.* The only basic element $B_r^k$ with $r = 0$ is $B_0^0$ (cf. (3.6b)), so that $\delta^{(2)}(B_r^k) > v$ implies $r \geqslant 1$. Using (3.8) we obtain

$$[X^v, B_r^{k-1}] = -(r - k + 1) B_r^k + \beta_v(r - v) B_{r+v}^{k-1}.$$

Since $r \geqslant 1$ and $\delta^{(2)}(B_r^{k-1}) = \delta^{(2)}(B_r^k) - v \geqslant 1$ we have $B_r^{k-1} \in \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)}$ so that (6.1.3) is established when $j = 1$. The cases $j > 1$ follows by induction on $j$, and (6.1.4) is a consequence of (6.1.3) with $j = k$. ∎

PROPOSITION 6.1.5. *Let $\delta^{(2)}(A_r^k) > v$. Then $r \geqslant 1$ and*

$$A_r^k \equiv a(\beta_v)^{k+1} A_{r+(k+1)v}^{-1} + b'(\beta_v)^k B_{r+kv}^0 \tag{6.1.6}$$

*for some constants $a > 0$ and $b' > 0$ unless $k = 0$, in which case $b' = 0$.*

*Proof.* In view of (3.6a) if $r = 0$ then $k \leqslant 1$ so that $\delta^{(2)}(A_r^k) > v$ implies $r \geqslant 1$. Using (3.8) we obtain

$$[X^v, A_r^{k-1}] = -(r - k + 2) A_r^k + \beta_v r A_{r+v}^{k-1} + \frac{\beta_v v(v+2) k}{(r+2)(v+r+2)} B_{v+r}^{k-1}.$$

Since $r \geqslant 1$ and $\delta^{(2)}(A_r^{k-1}) = \delta^{(2)}(A_r^k) - v \geqslant 1$ we have $A_r^{k-1} \in \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)}$. This establishes (6.1.6) when $k=0$ as well as the case $j=1$ of the following relations,

$$A_r^k \equiv a(B_v)^j A_{r+jv}^{k-j} + b(\beta_v)^j B_{r+jv}^{k-j}, \tag{6.1.7}$$

where $a = a(r, k, j)$ and $b = b(r, k, j)$ are positive constants and $1 \leqslant j \leqslant k$. In view of (6.1.3) we see by induction on $j$ that (6.1.7) holds in all cases. Finally (6.1.6) is a consequence of (6.1.7) with $j = k$ and (6.1.6) with $k = 0$. ∎

PROPOSITION 6.1.8. *Let $B$ be $\delta^{(2)}$-homogeneous with $\delta^{(2)}(B) = \delta^{(2)}(A_r^k) > v$. Then for all $a' \in \mathbf{R}$ there exist constants $c$, $d$, and an element $Z \in \mathscr{F}_1^{(1)} \cap \mathscr{F}_2^{(2)}$ such that*

$$a'A_r^k + B = cA_{r+(k+1)v}^{-1} + dB_{r+kv}^0 + [X^v, Z].$$

*Moreover if $a' \neq 0$ then $c \neq 0$ for at least one such representation. Finally if $a' = 0$, $B = b'B_r^k$ with $b' \neq 0$, and $r \neq v$, then $c = 0$ and $d \neq 0$.*

*Proof.* Decompose $B$ into bihomogeneous components and apply (6.1.6) to $a'A_r^k$ and (6.1.4) to the components of $B$. If $a' = 0$ and $B$ is bihomogeneous, the result follows from (6.1.4). ∎

PROPOSITION 6.1.9. *For each $j \overset{(v+1)}{\equiv} 0$, $j \geqslant 0$ and $j \neq v(v+1)$, there exists a unique $Z^j \in \ker(\Gamma_j)$ such that $[X^v, Z^j] = B_{j+v}^0$. Every element of the form $B_k^0$ with $k \overset{(v+1)}{\equiv} v$, $k > v$, $k \neq v(v+2)$ is in the image of* $\operatorname{ad}(X^v) | \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)}$.

*Proof.* For $j \overset{(v+1)}{\equiv} 0$, $j \neq v(v+1)$, $j \geqslant 0$, set $k = j/(v+1) \in \mathbf{N}$. We have $[X^v, B_k^k] = \beta_v(k-v) B_{k+v}^k$. Since $\delta^{(2)}(B_k^k) = j$, this proves existence when $k = 0$. If $k > 0$, then $k + v > v$, so we can apply (6.1.4) to $B_{k+v}^k$ to conclude that

$$[X^v, B_k^k] \equiv cB_{k(v+1)+v}^0 \qquad \text{with} \quad c \neq 0.$$

Thus for some $Z$, $[X^v, Z] = cB_{k(v+1)+v}^0$. Taking the $j$th $\delta^{(2)}$-component of $Z$ gives the desired existence modulo a non-vanishing constant. Since $0 \neq Z^j \in \ker(\Gamma_j)$, uniqueness follows from the one-dimensionality of this space (Proposition 6.1.1). Finally the second statement of the proposition is a consequence of the first. ∎

*Remark.* The exclusion of $k = v$ is due to the fact that although $B_v^0 = [X^v, -(v\beta_v)^{-1} B_0^0]$, $B_0^0$ is not in $\mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)}$.

PROPOSITION 6.1.10. *For each* $j \overset{(\nu+1)}{\equiv} -1$ *with* $j > \nu$, *there exists a unique* $Z^j \in \ker(\Gamma_j)$ *such that* $[X^\nu, Z^j] = A^{-1}_{j+2\nu} + cB^0_{j+\nu}$. *For this* $Z^j$ *the coefficient* $c$ *of* $B^0_{j+\nu}$ *does not vanish. For every* $k \overset{(\nu+1)}{\equiv} \nu - 1$ *with* $k > 2\nu$ *we have* $A^{-1}_{k+\nu} \equiv cB^0_k$ *for some* $c \in \mathbf{R}$.

*Proof.* Let $k := (j-\nu)/(\nu+1) \in \mathbf{N}$, and consider $A^{k+1}_k$ (whose $\delta^{(2)}$-degree is $j$). Using (3.8) we have

$$[X^\nu, A^{k+1}_k] = \beta_\nu \left( kA^{k+1}_{k+\nu} + \frac{\nu(\nu+2)}{\nu+k+2} B^{k+1}_{\nu+k} \right).$$

Applying (6.1.6) and (6.1.4) we obtain

$$[X^\nu, A^{k+1}_k] = \beta_\nu k \left( a\beta^{k+2}_\nu A^{-1}_{j+2\nu} + b\beta^{k+1}_\nu B^0_{j+\nu} \right.$$
$$\left. + \frac{\nu(\nu+2)}{\nu+k+2} b'\beta^{k+1}_\nu B^0_{j+\nu} \right) + [X^\nu, Z'],$$

with positive constants $a$, $b$, and $b'$. From this the existence of a homogeneous $Z^j$ with the desired properties readily follows. Since $0 \neq Z^j \in \ker(\Gamma_j)$, uniqueness follows from its one-dimensionality (Proposition 6.1.1). Finally, the last statement of the proposition is a consequence of the first. ∎

PROPOSITION 6.1.11.

$$\ker \operatorname{ad}(X^\nu) = \langle X^\nu \rangle \oplus \langle B^\nu_\nu \rangle.$$

*Proof.* Recall that $\Gamma = \operatorname{ad}(A^{-1}_0) \circ \operatorname{ad}(X^\nu)$, so that $\ker \operatorname{ad}(X^\nu) \subseteq \ker(\Gamma)$. By Proposition 6.1.1, $\ker(\Gamma_j) \neq 0$ only if $j \overset{(\nu+1)}{\equiv} 0$ or $\nu$. If $j \overset{(\nu+1)}{\equiv} 0$ but $j \neq \nu(\nu+1)$, we see from Proposition 6.1.9 that $[X^\nu, Z^j] \neq 0$ for some $Z^j \in \ker(\Gamma_j)$. Since the latter is one-dimensional, this implies that for such $j$, $\operatorname{ad}(X^\nu)$ is injective on $\ker(\Gamma_j)$. Likewise, Proposition 6.1.10 implies the injectivity of $\operatorname{ad}(X^\nu)$ on $\mathrm{Vect}^{(2)}_j$ for $j \overset{(\nu+1)}{\equiv} \nu$ and $j > \nu$. The only possibilities left are $j = \nu$ and $\nu(\nu+1)$. Since $\ker \operatorname{ad}(X^\nu) \cap \mathrm{Vect}^{(2)}_j$ is at most one-dimensional, the result follows from $\operatorname{ad}(X^\nu)(X^\nu) = 0$, $\delta^{(2)}(X^\nu) = \nu$, and $\operatorname{ad}(X^\nu)(B^\nu_\nu) = 0$, $\delta^{(2)}(B^\nu_\nu) = \nu(\nu+1)$. ∎

In order to further normalize $X$ we will specify for each $k > \nu$ the following subspace $\mathscr{N}^{(2)}_k$ of $\mathrm{Vect}^{(2)}_k$:

$$\mathscr{N}^{(2)}_k := \begin{cases} \langle A^{-1}_{k+\nu} \rangle & \text{if } k \overset{(\nu+1)}{\equiv} \nu, k \neq \nu(\nu+2) \\ \langle B^0_k \rangle & \text{if } k \overset{(\nu+1)}{\equiv} \nu - 1, k \neq 2\nu \\ \langle A^{-1}_{k+\nu} \rangle \oplus \langle B^0_k \rangle & \text{in all other cases.} \end{cases} \tag{6.1.12}$$

PROPOSITION 6.1.13. *For all $k > \nu$, $\mathcal{N}_k^{(2)}$ is strictly transversal to* $\mathrm{ad}(X^\nu)(\mathrm{Vect}_{k-\nu}^{(2)} \cap \mathscr{F}_1^{(1)})$.

*Proof.* Let $\mathscr{I}_k = \mathrm{im}(\mathrm{ad}(X^\nu) | \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)})$. By (6.1.4) and (6.1.6) we have

$$\mathrm{Vect}_k^{(2)} = \mathscr{I}_k + (\langle A_{k+\nu}^{-1} \rangle \oplus \langle B_k^0 \rangle)$$

and by Propositions 6.1.9 and 6.1.10 we can drop from the above $\langle B_k^0 \rangle$ when $k \overset{(\nu+1)}{\equiv} \nu$ but $k \neq \nu(\nu+2)$, and either $\langle A_{k+\nu}^{-1} \rangle$ or $\langle B_k^0 \rangle$, say $\langle A_{k+\nu}^{-1} \rangle$, when $k \overset{(\nu+1)}{\equiv} \nu - 1$ but $k \neq 2\nu$. Thus $\mathrm{Vect}_k^{(2)} = \mathscr{I}_k + \mathcal{N}_k^{(2)}$. To show that this sum is direct suppose that $[X^\nu, Z^{k-\nu}] \in \mathcal{N}_k^{(2)}$ for some $k > \nu$. Since $\mathcal{N}_k^{(2)} \subseteq \ker(\mathrm{ad}(A_0^{-1}) \circ \mathrm{ad}(X^\nu)) = \ker(\Gamma)$, we have by Proposition 6.1.2 that $k - \nu \overset{(\nu+1)}{\equiv} 0$ or $-1$. Moreover, by Proposition 6.1.11 if $k - \nu = \nu$ or $\nu(\nu+1)$, $[X^\nu, Z^{k-\nu}] = 0$, so the transversality is strict. If $k - \nu \overset{(\nu+1)}{\equiv} 0$ but $k \neq \nu(\nu+2)$ then $\mathcal{N}_k^{(2)} = \langle A_{k+\nu}^{-1} \rangle$ and by Proposition 6.1.9 $[X^\nu, Z^{k-\nu}] \in \langle B_k^0 \rangle \cap \langle A_{k+\nu}^{-1} \rangle = 0$. Finally if $k - \nu \overset{(\nu+1)}{\equiv} -1$ but $k \neq 2\nu$ then $[X^\nu, Z^{k-\nu}] \in \mathcal{N}_k^{(2)} = \langle B_k^0 \rangle$ implies (Prop. 6.1.10) that $Z^{k-\nu} = 0$. ∎

THEOREM 6.1.14 (Second Order Normal Form). *Every nilpotent field $X$ with $2\nu < \mu$ can be conjugated to a field of the form*

$$X^{(2)} = X^\nu + \sum_{k \geqslant \lambda} X^k, \qquad X^k \in \mathcal{N}_k^{(2)}, \ \nu < \lambda \leqslant \infty$$

(*where as usual $X^\lambda \neq 0$ if $\lambda < \infty$.*) *Both $\lambda$ and $X^\lambda$ are invariants of the original field $X$ relative to the group of near identity transformations of* $(\mathbf{R}^2; 0)$.

*Proof.* By passing to first order normal form we may assume that $X = X^\nu = \cdots$. In this context existence follows from Propositions 6.1.13 and Theorem 2.2 with $\mathscr{L} = \mathrm{Vect}^{(2)}$ and $\mathscr{J} = \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)}$. Invariance of $X^\lambda$ *relative to* $\exp \mathrm{ad}(\mathscr{F}_1)$ follows from Proposition 4.4 and (3) of Corollary 2.7 again applied with $\mathscr{J} = \mathscr{F}_1^{(1)} \cap \mathscr{F}_1^{(2)}$ ∎

## 6.2. *Infinite Order Normal Form* (*Case* II: $2\nu < \mu$)

Unlike the case $\mu < 2\nu$ where the transition from second order normal form to infinite order normal form involves the removal of infinitely many terms, we will show below that in the present case we obtain uniqueness by removing a single additional term whose nature depends on the invariant $X^\lambda$ of Theorem 6.1.14.

As before we write $X$ using the $\delta^{(2)}$ grading in the form $X = X^\nu + X^\lambda + \cdots$, (or $X = X^\nu$) but assume $X$ to be in second order normal form. Note that when $\lambda < \infty$ then

$$X^\lambda = A^\lambda + B^\lambda = \alpha_{\lambda+\nu} A_{\lambda+\nu}^{-1} + \beta_\lambda B_\lambda^0, \qquad \alpha_{\lambda+\nu}^2 + \beta_\lambda^2 \neq 0.$$

Let $A\colon \langle B^{\nu}_{\nu}\rangle \oplus \mathrm{Vect}^{(2)}_{\lambda+\nu^2} \to \mathrm{Vect}^{(2)}_{\lambda+\nu(\nu+1)}$ be the operator defined by

$$A(aB^{\nu}_{\nu}+Z^{\lambda+\nu^2}) = [Z^{\lambda+\nu^2}, X^{\nu}] + [aB^{\nu}_{\nu}, X^{\lambda}].$$

With $\mathscr{R}_k = \mathscr{R}_k(X^{\nu}, \mathscr{F}^{(1)}_1 \cap \mathscr{F}^{(2)}_1)$ as defined in (2.1) we have

PROPOSITION 6.2.1. $\mathrm{im}(A) \subseteq \mathscr{R}_{\lambda+\nu(\nu+1)}$.

*Proof.* If $Z \in \langle B^{\nu}_{\nu}\rangle \oplus \mathrm{Vect}^{(2)}_{\lambda+\nu^2}$ we have

$$[Z, X] = [Z, X^{\nu} + X^{\lambda} + \cdots] = A(Z) + h.o.t. \quad \blacksquare$$

PROPOSITION 6.2.2. *There exists* $Z_0 \in \langle B^{\nu}_{\nu}\rangle \oplus \mathrm{Vect}^{(2)}_{\lambda+\nu^2}$ *with non-zero* $B^{\nu}_{\nu}$*-component such that for some* $b \in \mathbf{R}$ *one has*

$$A(Z_0) = \begin{cases} A^{-1}_{\nu(\nu+2)+\lambda} + bB^0_{\lambda+\nu(\nu+1)} & \text{if } \alpha_{\lambda+\nu} \neq 0 \\ B^0_{\lambda+\nu(\nu+1)} & \text{if } \alpha_{\lambda+\nu} = 0 \text{ but } \beta_\lambda \neq 0 \end{cases} \tag{6.2.3}$$

*Proof.* Use (3.8) to obtain

$$[B^{\nu}_{\nu}, X^{\lambda}] = \alpha_{\lambda+\nu}\left((\lambda+\nu)\, A^{\nu-1}_{\lambda+2\nu} - \frac{\nu(\nu+2)}{\lambda+2\nu+2} B^{\nu-1}_{\lambda+2\nu}\right) + \beta_\lambda(\lambda-\nu)\, B^{\nu}_{\lambda+\nu}$$

and apply Proposition 6.18 to the right hand side. $\blacksquare$

Next consider the operator

$$\Delta := \mathrm{ad}(A^{-1}_0) \circ A\colon \langle B^{\nu}_{\nu}\rangle \oplus \mathrm{Vect}^{(2)}_{\lambda+\nu^2} \to \mathrm{Vect}^{(2)}_{\lambda+\nu^2},$$

and observe that $A(\ker(\Delta)) \subseteq \ker \mathrm{ad}(A^{-1}_0) \cap \mathrm{Vect}^{(2)}_{\lambda+\nu(\nu+1)} = \langle A^{-1}_{\lambda+\nu(\nu+2)}\rangle \oplus \langle B^0_{\lambda+\nu(\nu+1)}\rangle$.

PROPOSITION 6.2.4. *With* $Z_0$ *as in* (6.2.3) *we have*

(1) $\ker(\Delta) = \langle Z_0\rangle \oplus \ker(\Gamma_{\lambda+\nu^2})$ *and in particular* $\dim \ker(\Delta) = 1 + \dim \ker(\Gamma_{\lambda+\nu^2})$.

(2) $A\colon \ker(\Delta) \to \langle A^{-1}_{\lambda+\nu(\nu+1)}\rangle \oplus \langle B^0_{\lambda+\nu^2}\rangle$ *is injective unless* $\lambda = 2\nu$ *and* $\alpha_{\lambda+\nu} \neq 0$, *or* $\lambda = \nu(\nu+2)$ *and* $\alpha_{\lambda+\nu} = 0$.

(3) $A\colon \ker(\Delta) \to \langle A^{-1}_{\lambda+\nu(\nu+2)}\rangle \oplus B^0_{\lambda+\nu(\nu+1)}\rangle$ *is an isomorphism if* $\lambda \overset{(\nu+1)}{\equiv} \nu$ *or* $\nu - 1$ (*except when* $\lambda = 2\nu$ *and* $\alpha_{\lambda+\nu} \neq 0$ *or when* $\lambda = \nu(\nu+2)$ *and* $\alpha_{\lambda+\nu} = 0$); *a space transversal to* $A(\ker(\Delta))$ *in* $\ker \mathrm{ad}(A^{-1}_0) \cap \mathrm{Vect}^{(2)}_{\lambda+\nu^2}$ *is* $\langle B^0_{\lambda+\nu^2}\rangle$ *if* $\alpha_{\lambda+\nu} \neq 0$, *and* $\langle A^{-1}_{\lambda+\nu(\nu+1)}\rangle$ *if* $\alpha_{\lambda+\nu} = 0$.

*Proof.* (1) Since $\Delta \mid \mathrm{Vect}^{(2)}_{\lambda-\nu^2} = \Gamma_{\lambda+\nu^2}$, the representation $\ker(\Delta) = \langle Z_0\rangle \oplus \ker(\Gamma_{\lambda+\nu^2})$ follows from the fact that the $B^{\nu}_{\nu}$-component of $Z_0$ does not vanish.

(2) By (6.2.3) $\Lambda(Z_0) \neq 0$. This together with (1) implies injectivity whenever $\ker(\Gamma_{\lambda+\nu^2}) = 0$. By Proposition 6.1.2 the only other cases to consider are $\lambda + \nu^2 \overset{(\nu+1)}{\equiv} 0$ or $-1$, i.e., $\lambda \overset{(\nu+1)}{\equiv} \nu$ or $\nu - 1$. Assume $\lambda \overset{(\nu+1)}{\equiv} \nu$. By (6.1.12) and $X^\lambda \in \mathcal{N}_\lambda^{(2)}$ we see that $\alpha_{\lambda+\nu} \neq 0$ (unless $\lambda = \nu(\nu+2)$ and $\alpha_{\lambda+\nu} = 0$, a case that we have ruled out). It follows from (6.2.3) that $\mathrm{im}(\Lambda \mid \ker(\Delta))$ contains an element with non-vanishing $A$-component as well as $B_{\lambda+\nu(\nu+1)}$. (Use Proposition 6.1.9 with $j = \lambda + \nu^2$ and note that $\Lambda(Z^{\lambda+\nu^2}) = [Z^{\lambda+\nu^2}, X^\nu]$). It follows that $\Lambda\colon \ker(\Delta) \to \langle A^{-1}_{\lambda+\nu(\nu+2)} \rangle \oplus \langle B^0_{\lambda+\nu(\nu+1)} \rangle$ is an isomorphism (both spaces are two-dimensional) which proves what we want when $\lambda \overset{(\nu+1)}{\equiv} \nu$. Finally, assume $\lambda \overset{(\nu+1)}{\equiv} \nu - 1$. A second check with (6.1.12) will establish that now $\alpha_{\lambda+\nu} = 0$. (Remember that we have ruled out the case $\lambda = 2\nu$ with $\alpha_{\lambda+\nu} \neq 0$.) By (6.2.3) $B^0_{\lambda+\nu(N+1)} \in \mathrm{im}(\Lambda \mid \ker(\Delta))$ and by Proposition 6.1.10 (with $j = \lambda + \nu^2 \overset{(\nu+1)}{\equiv} -1$) this image also contains an element with non-vanishing $A$-component. Thus $\Lambda\colon \ker \Delta \to \langle A^{-1}_{\lambda+\nu(\nu+2)} \rangle \oplus \langle B^0_{\lambda+\nu(\nu+1)} \rangle$ is again an isomorphism, and this establishes 2.

(3) The surjectivity follows from (1) and (2) (as well as from the proof of (2)), and the last statement is an immediate consequence of (6.2.3). ∎

We summarize the information of item (3) of the above Proposition in Table I, where the slots are occupied with $\mathscr{C}_{\lambda+\nu(\nu+1)}$, a space transversal to $\Lambda(\ker(\Delta))$ in $\ker \mathrm{ad}(A_0^{-1}) \cap \mathrm{Vect}^{(2)}_{\lambda+\nu(\nu+1)} = \langle A^{-1}_{\lambda+\nu(\nu+2)} \rangle \oplus \langle B^0_{\lambda+\nu(\nu+1)} \rangle$, and where $A = A^{-1}_{\lambda+\nu(\nu+2)}$ and $B = B^0_{\lambda+\nu(\nu+1)}$. Observe that since $X_\lambda \in \mathcal{N}_\lambda^{(2)}$, the omitted items in Table I cannot occur (cf. (6.1.12)).

We now introduce the following subspaces of $\mathrm{Vect}^{(2)}$:

$$\mathcal{N}_k^{(\infty)} = \begin{cases} \mathscr{C}_{\lambda+\nu(\nu+1)} \text{ (cf. Table I)} & \text{if } k = \lambda + \nu(\nu+1) \\ \langle A^{-1}_{k+\nu} \rangle & \text{if } k \overset{(\nu+1)}{\equiv} \nu,\ k \neq \nu(\nu+2),\ \lambda + \nu(\nu+1) \\ \langle B^0_k \rangle & \text{if } k \overset{(\nu+1)}{\equiv} \nu - 1,\ k \neq 2\nu,\ \lambda + \nu(\nu+1) \\ \langle A^{-1}_{k+\nu} \rangle \oplus \langle B^0_k \rangle & \text{in all other cases.} \end{cases}$$

TABLE I

| $\alpha_{\lambda+\nu}$ \ $\lambda$ | $\equiv \nu - 1$ | | $\equiv \nu$ | | $\not\equiv \nu - 1,\ \nu$ |
|---|---|---|---|---|---|
| | $\neq 2\nu$ | $2\nu$ | $\neq \nu(\nu+2)$ | $\nu(\nu+2)$ | |
| $\neq 0$ | — | $\langle B \rangle$ | 0 | 0 | $\langle B \rangle$ |
| 0 | 0 | 0 | — | $\langle A \rangle$ | $\langle A \rangle$ |

PROPOSITION 6.2.5. *For $k > \nu$ the $\mathcal{N}_k^{(\infty)}$ are strictly transverse to the $\mathcal{R}_k$.*

*Proof.* When $k \neq \lambda + \nu(\nu+1)$, $\mathcal{N}_k^{(\infty)}$ coincides with $\mathcal{N}_k^{(2)}$ so that transversality follows from Proposition 6.1.13. On the other hand when $k = \lambda + \nu(\nu+1)$ we have (by the same proposition)

$$\begin{aligned}
\mathrm{Vect}^{(2)}_{\lambda+\nu(\nu+1)} &= \mathcal{N}^{(2)}_{\lambda+\nu(\nu+1)} \oplus \mathrm{ad}(X^\nu)(\mathrm{Vect}^{(2)}_{\lambda+\nu^2}) \\
&\subseteq (\langle A^{-1}_{\lambda+\nu(\nu+2)}\rangle \oplus \langle B^0_{\lambda+\nu(\nu+1)}\rangle) + \mathcal{R}_{\lambda+\nu(\nu+1)} \\
&\overset{\text{(by Proposition 6.2.4)}}{\subseteq} (\mathcal{C}_{\lambda+\nu(\nu+1)} + \Lambda(\ker(\Delta))) + \mathcal{R}_{\lambda+\nu(\nu+1)} \\
&\subseteq \mathcal{C}_{\lambda+\nu(\nu+1)} + \mathrm{im}(\Lambda) + \mathcal{R}_{\lambda+\nu(\nu+1)} \\
&\overset{\text{(by Proposition 6.2.1)}}{\subseteq} \mathcal{C}_{\lambda+\nu(\nu+1)} + \mathcal{R}_{\lambda+\nu(\nu+1)},
\end{aligned}$$

which proves transversality when $k = \lambda + \nu(\nu+1)$.

We now show that the sum $\mathcal{N}_j^{(\infty)} + \mathcal{R}_j$ is direct for all $j > \nu$. For suppose $0 \neq [Z, X] = [Z, X]^j + \cdots$, with $[Z, X]^j \in \mathcal{N}_j^{(\infty)}$ for some $Z \in \mathcal{F}_1^{(1)} \cap \mathcal{F}_1^{(2)}$ (cf. (2.1)). We can write $Z = Z^{j_0} + \text{h.o.t.}$ with $Z^{j_0} \neq 0$ for some $j_0 \geqslant 1$. By (2) of Proposition 2.6, $Z^k \in \ker \mathrm{ad}(X^\nu)$ for $j_0 \leqslant k < j_0 + \lambda - \nu$. (Note that since $\mathcal{N}_j^{(\infty)} \subseteq \mathcal{N}_j^{(2)}$, the hypothesis $[X^\nu, \mathrm{Vect}^{(2)}_{j-\nu} \cap \mathcal{F}_1^{(1)}] = 0$ is met by Proposition 6.1.13.) By Proposition 6.1.11 and the fact that $X^\nu \notin \mathcal{F}_1^{(1)}$, we conclude that $j = \nu(\nu+1)$, that $Z^{j_0} = cB^\nu_\nu$ for some $c \neq 0$, and that $Z^k = 0$ for $\nu(\nu+1) < k < \nu(\nu+1) + \lambda - \nu$. Moreover by Proposition 2.6, $j \geqslant \lambda + j_0 = \lambda + \nu(\nu+1)$, and

$$\begin{aligned}
&[Z^{\lambda+\nu^2}, X^\nu] + c[B^\nu_\nu, X^\lambda] \\
&\quad = [Z, X]^{\lambda+\nu(\nu+1)} \begin{cases} = 0 & \text{if } j > \lambda + \nu(\nu+1) \\ \in \mathcal{N}^{(\infty)}_{\lambda+\nu(\nu+1)} & \text{if } j = \lambda + \nu(\nu+1) \end{cases}
\end{aligned}$$

From Table I we see that $\mathcal{N}^{(\infty)}_{\lambda+\nu(\nu+1)} = \mathcal{C}_{\lambda+\nu(\nu+1)} = 0$ under the current assumptions, so that $[Z^{\lambda+\nu^2}, X^\nu] + c[B^\nu_\nu, X^\lambda] = \Lambda(Z^{\lambda+\nu^2} + cB^\nu_\nu) = 0$. By (2) of Proposition 6.2.4 we must have $Z^{\lambda+\nu^2} + cB^\nu_\nu = 0$, but this contradicts $c \neq 0$. ▌

COROLLARY 6.2.6 (of the proof). *Let* $[Z, X] = 0$ *for some* $Z \in \mathcal{F}_1^{(1)}$. *Then* $Z = 0$ *unless* $X^{(1)} = X^\nu$ *in which case* $Z \in \langle B^\nu_\nu \rangle$. ▌

*Remark.* The transversality of Proposition 6.2.5 in the special cases $\lambda = 2\nu$ but $\alpha_{\lambda+\nu} \neq 0$ and $\lambda = \nu(\nu+2)$ but $\alpha_{\lambda+\nu} = 0$ need not hold. In our next (main) result we refer to these cases as *exceptional.*

THEOREM 6.2.7 (Infinite Order Normal Form). *Let $X$ be a vector field admitting a first order normal form $X^{(1)} = X^\nu + \cdots$, with dots representing* *higher order $\delta^{(2)}$-terms. Then $X$ is conjugate to $X^{(\infty)} = X^\nu$ or to a field of the form*

$$X^{(\infty)} = X^\nu + X^\lambda + \sum_{k>\lambda} X^k,$$

*with $0 \neq X^\lambda \in \ker \mathrm{ad}(A_0^{-1}) \cap \mathrm{Vect}^{(2)}_\lambda$ and $X^k \in \mathcal{N}^{(\infty)}_k$. Moreover in the non-exceptional cases the above representation is unique. More precisely, the following strong uniqueness result holds: if $Y^{(\infty)} = \exp \mathrm{ad}(Z)\, X$ is another such form then $Z = 0$ unless $X = X^\nu$ in which case $Z \in \langle B^\nu_\nu \rangle$. In either case $Y^{(\infty)} = X^{(\infty)}$.*

*Proof.* Existence and uniqueness relative to the group $\mathcal{G} := \exp \mathrm{ad}(\mathcal{F}^{(1)}_1 \cap \mathcal{F}^{(2)}_1)$ follows from Theorem 2.2 and the strict transversality of the $\mathcal{N}^{(\infty)}_k$. Uniqueness relative to the group of near identity transformations follows from Proposition 4.4 together with uniqueness relative to $\mathcal{G}$. Finally, the last two statements are consequences of Corollary 6.2.6. ∎

## 7. Case III: $\mu = 2\nu$

Our results in this case are incomplete. Since the methods parallel previous sections we shall be succinct. As before define

$$\Gamma = \mathrm{ad}(A_0^{-1}) \circ \mathrm{ad}(A_0^1 + \alpha_\mu A_\mu^{-1} + \beta_\nu B_\nu^0).$$

We have

$$\begin{aligned}
\Gamma A^k_k = {} & (l-k-1)(l+2)\, A^l_k + \beta_\nu k(l+1)\, A^{l-1}_{k+\nu} \\
& + \frac{\alpha_\mu (k+2\nu+2)\, l(l+1)}{(k+2)} A^{l-2}_{k+2\nu} \\
& + \beta_\nu \nu(\nu+2)\, l(l+1)/((k+2)(k+\nu+2))\, B^{l-1}_{k+n}, \\
\Gamma B^l_k = {} & (l+1)(l-k)\, B^l_k + \frac{\alpha_\mu l(l-1)(k+2)}{(k+2\nu+2)} B^{l-2}_{k+2\nu} \\
& - 2\alpha_\mu l\nu A^{l-2}_{k+2\nu} + \beta_\nu l(k-\nu)\, B^{l-1}_{k+\nu}.
\end{aligned}$$

With $\delta^{(2)}$ as in (4.3), observe that $\Gamma$ is homogeneous of degree 0 and therefore maps each homogeneous subspace into itself. We set $\Gamma_\lambda := \Gamma | \mathrm{Vect}^{(2)}_\lambda$.

Proposition 7.1. *$\Gamma_\lambda$ is invertible unless $\lambda \overset{(\mu+2)}{\equiv} 0$ or $\mu$. In such cases* $\dim \ker(\Gamma_\lambda) = 1$.

*Proof.* Use the above formulas with an appropriate ordering of the basic elements. ▌

PROPOSITION 7.2. (1) *If* $\lambda \overset{(\mu+2)}{\equiv} 0$, *say* $\lambda = k(\mu+2)$, *then* $\ker(\Gamma_\lambda)$ *is spanned by a unique element* $Z$ *of the form* $B_k^k + \cdots$.

(2) *If* $\lambda \overset{(\mu+2)}{\equiv} \mu$, *say* $\lambda = k(\mu+2)+\mu$, *then* $\ker(\Gamma_\lambda)$ *is spanned by a unique element* $Z$ *of the form* $A_k^{k+1} + \cdots$.

*Proof.* (1) For convenience we write the lowest order operator as $X^\mu = A_0^1 + \alpha_{2\nu}\beta_\nu^2 A_{2\nu}^{-1} + \beta_\nu B_\nu^0$, and let

$$Z = B_k^k + \cdots = \sum_{m=2}^{k+1} c_m \beta_\nu^m A_{k+\nu m}^{k-m} + \sum_{m=0}^{k} b_m \beta_\nu^m B_{k+\nu m}^{k-m}.$$

Then

$$\begin{aligned}
[X^\mu, Z] = &-\sum_{m=2}^{k+1} c_m \beta_\nu^m ((\nu+1)m+1)\, A_{k+\nu m}^{k-m+1} \\
&-\sum_{m=1}^{k} b_m \beta_\nu^m (\nu+1)\, m B_{k+\nu m}^{k-m+1} \\
&+\alpha_{2\nu}\beta_\nu^2 \sum_{m=2}^{k} c_m \beta_\nu^m \frac{(k+(2+m)\,\nu+2)(k-m+1)}{(k-\nu m+2)} A_{k+(m+2)\nu}^{k-m-1} \\
&+\alpha_{2\nu}\beta_\nu^2 \sum_{m=2}^{k-1} b_m \beta_\nu^m \frac{(k+\nu m+2)(k-m)}{(k+(m+2)\,\nu+2)} B_{k+(m+2)\nu}^{k-m-1} \\
&-\alpha_{2\nu} \sum_{m=0}^{k} b_m \beta_\nu^{m+2} 2\nu A_{k+(m+2)\nu}^{k-m-1} \\
&+\beta_\nu \sum_{m=2}^{k} c_m \beta_\nu^m \frac{\nu(\nu+2)(k-m+1)}{(k+(m+2)\,\nu+2)(k+m\nu+2)} B_{k+(m+1)\nu}^{k-m} \\
&+\sum_{m=2}^{k+1} c_m \beta_\nu^{m+1} (k+\nu m)\, A_{k+(m+1)\nu}^{k-m} \\
&+\beta_\nu \sum_{m=0}^{k} b_m \beta_\nu^m (k+(m-1)\,\nu)\, B_{k+(m+1)\nu}^{k-m} \\
= &-\sum_{m=2}^{k+1} c_m \beta_\nu^m ((\nu+1)\, m+1)\, A_{k+\nu m}^{k-m+1} \\
&-\sum_{m=1}^{k} b_m \beta_\nu^m (\nu+1)\, m B_{k+\nu m}^{k-m+1}
\end{aligned}$$

$$+\alpha_{2\nu}\sum_{m=4}^{k+2} c_{m-2}\beta_\nu^m \frac{(k+\nu m+2)(k-m+3)}{(k+\nu(m-2)+2)} A_{k+\nu m}^{k-m+1}$$
$$+\alpha_{2\nu}\sum_{m=2}^{k+1} b_{m-2}\beta_\nu^m \frac{(k+\nu(m-2)+2)(k-m+2)}{(k+m\nu+2)} B_{k+\nu m}^{k-m+1}$$
$$-\alpha_{2\nu}\sum_{m=2}^{k+2} b_{m-2}\beta_\nu^m 2\nu A_{k+\nu m}^{k-m+1}$$
$$+\sum_{m=3}^{k+1} c_{m-1}\beta_\nu^m \frac{\nu(\nu+2)(k-m+2)}{(k+(m-1)\,\nu+2)(k+m\nu+2)} B_{k+\nu m}^{k-m+1}$$
$$+\sum_{m=3}^{k+2} c_{m-1}\beta_\nu^m(k+\nu(m-1))\, A_{k+\nu m}^{k-m+1}$$
$$+\sum_{m=1}^{k+1} b_{m-1}\beta_\nu^m(k+(m-2)\,\nu)\, B_{k+\nu m}^{k-m+1}.$$

For this expression to be in first order normal form, one must have

$$((\nu+1)\,m+1)\,c_m=\alpha_{2\nu}c_{m-2}\frac{(k+\nu m+2)(k-m+3)}{(k+\nu(m-2)+2)}-2\nu\alpha_{2\nu}b_{m-2}$$
$$+(k+\nu(m-1))\,c_{m-1}$$
$$(\nu+1)\,mb_m=\alpha_{2\nu}b_{m-2}\frac{(k+\nu(m-2)+2)(k-m+2)}{(k+m\nu+2)}$$
$$+c_{m-1}\frac{\nu(\nu+2)(k-m+2)}{(k+(m-1)\,\nu+2)(k+m\nu+2)}$$
$$+b_{m-1}(k+(m-2)\,\nu)$$

and then the commutator equals

$$[X^\mu, Z]=c_{k+2}((\nu+1)(k+2)+1)\,\beta_\nu^{k+2}A^{-1}_{(\nu+1)k+2\nu}$$
$$+\beta_\nu^{k+1}b_{k+1}(\nu+1)(k+1)\,B^0_{(\nu+1)k+\nu}.$$

(2) Let

$$Z=A_k^{k+1}+\cdots=\sum_{m=0}^{k+2} c_m\beta_\nu^m A_{k+\nu m}^{k-m+1}+\sum_{m=1}^{k+1} b_m\beta_\nu^m B_{k+\nu m}^{k-m+1}.$$

Then

$$
\begin{aligned}
[X^{\mu}, Z] = &-\sum_{m=1}^{k+2} c_m \beta_\nu^m (\nu+1)\, m A_{k+\nu m}^{k-m+2} \\
&-\sum_{m=1}^{k+1} b_m \beta_\nu^m ((\nu+1)\, m-1)\, B_{k+\nu m}^{k-m+2} \\
&+\alpha_{2\nu} \sum_{m=2}^{k} c_m \beta_\nu^{m+2} \frac{(k+(2+m)\,\nu+2)(k-m+2)}{(k+\nu m+2)} A_{k+(m+2)\nu}^{k-m} \\
&+\alpha_{2\nu} \sum_{m=1}^{k} b_m \beta_\nu^{m+2} \frac{(k+\nu m+2)(k-m+1)}{(k+(m+2)\,\nu+2)} B_{k+(m+2)\nu}^{k-m} \\
&-\alpha_{2\nu} \sum_{m=1}^{k+1} b_m \beta_\nu^{m+2} 2\nu A_{k+(m+2)\nu}^{k-m} \\
&+\sum_{m=0}^{k+1} c_m \beta_\nu^{m+1} \frac{\nu(\nu+2)(k-m+2)}{(k+(m+1)\,\nu+2)(k+m\nu+2)} B_{k+(m+1)\nu}^{k-m+1} \\
&+\sum_{m=0}^{k+2} c_m \beta_\nu^{m+1} (k+\nu m)\, A_{k+(m+1)\nu}^{k-m+1} \\
&+\sum_{m=1}^{k+1} b_m \beta_\nu^{m+1} (k+(m-1)\,\nu)\, B_{k+(m+1)\nu}^{k-m+1} \\
= &-\sum_{m=1}^{k+2} c_m \beta_\nu^m (\nu+1)\, m A_{k+\nu m}^{k-m+2} \\
&-\sum_{m=1}^{k+1} b_m \beta_\nu^m ((\nu+1)\, m-1)\, B_{k+\nu m}^{k-m+2} \\
&+\alpha_{2\nu} \sum_{m=2}^{k+3} c_{m-2} \beta_\nu^m \frac{(k+\nu m+2)(k-m+4)}{(k+\nu(m-2)+2)} A_{k+\nu m}^{k-m+2} \\
&+\alpha_{2\nu} \sum_{m=3}^{k+2} b_{m-2} \beta_\nu^m \frac{(k+\nu(m-2)+2)(k-m+3)}{(k+m\nu+2)} B_{k+\nu m}^{k-m+2} \\
&-\alpha_\nu \sum_{m=3}^{k+3} b_{m-2} \beta_\nu^m 2\nu A_{k+\nu m}^{k-m+2} \\
&+\sum_{m=1}^{k+2} c_{m-1} \beta_\nu^m \frac{\nu(\nu+2)(k-m+3)}{(k+(m-1)\,\nu+2)(k+m\nu+2)} B_{k+\nu m}^{k-m+2} \\
&+\sum_{m=1}^{k+3} c_{m-1} \beta_\nu^m (k+\nu(m-1))\, A_{k+\nu m}^{k-m+2} \\
&+\sum_{m=2}^{k+2} b_{m-1} \beta_\nu^m (k+(m-2)\,\nu)\, B_{k+\nu m}^{k-m+2}.
\end{aligned}
$$

For this expression to be in first order normal form, one must have

$$(\nu+1)\,mc_m=\alpha_{2\nu}c_{m-2}\frac{(k+\nu m+2)(k-m+4)}{(k+\nu(m-2)+2)}-2\nu\alpha_{2\nu}b_{m-2}$$
$$+(k+\nu(m-1))\,c_{m-1}$$
$$((\nu+1)\,m-1)\,b_m=\alpha_{2\nu}b_{m-2}\frac{(k+\nu(m-2)+2)(k-m+3)}{(k+m\nu+2)}$$
$$+c_{m-1}\frac{\nu(\nu+2)(k-m+3)}{(k+(m-1)\,\nu+2)(k+m\nu+2)}$$
$$+b_{m-1}(k+(m-2)\,\nu)$$

and then the commutator equals

$$[X^\mu, Z]=c_{k+3}(\nu+1)(k+3)\,\beta_\nu^{k+3}A^{-1}_{(\nu+1)k+3\nu}$$
$$+\beta_\nu^{k+1}b_{k+2}((\nu+1)(k+2)-1)\,B^0_{(\nu+1)k+2\nu}. \quad \blacksquare$$

*Remark.* In both cases one now has the following options:

(1) Scaling $x$ and $y$ (this is equivalent to expanding the generating algebra of the group of near-identity transformations to include $B_0^0$) in such a way that $|\alpha_{2\nu}|=1$. Then one can try to prove that the commutators are not zero by the techniques used in the previous sections.

(2) Prove that the resultant of the two coefficients appearing in the commutator (as functions of $\alpha_{2\nu}$) is not zero. This has been checked in the second case for $k \leqslant 10$.

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