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Content evaluation and class symmetric functions

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Abstract

In this article we study the evaluation of symmetric functions on the alphabet of contents of a partition. Applying this notion of *content evaluation* to the computation of central characters of the symmetric group, we are led to the definition of a new basis of the algebra Λ of symmetric functions over $\mathbb{Q}(n)$ that we call the basis of *class symmetric functions*.

By definition this basis provides an algebra isomorphism between Λ and the Farahat–Higman algebra FH governing for all n the products of conjugacy classes in the center $\mathscr{Z}_n$ of the group algebra of the symmetric group $\mathfrak{S}_n$. We thus obtain a calculus of all connexion coefficients of $\mathscr{Z}_n$ inside Λ. As expected, taking the homogeneous components of maximal degree in class symmetric functions, we recover the symmetric functions introduced by Macdonald to describe *top* connexion coefficients.

We also discuss the relation of class symmetric functions to the asymptotic of central characters and of the enumeration of standard skew young tableaux. Finally we sketch the extension of these results to Hecke algebras.

Keywords: Symmetric groups; Symmetric functions; Conjugacy classes; Hecke algebras; Content; Tableaux; Shift-symmetric functions

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1. Introduction

A starting point for the present work can be found in early investigations, by Frobenius [3] and Ingram [6], of the so-called *central characters* of the symmetric group. Given a partition μ, the central character ω_μ evaluated at a partition λ of n is a normalization of the irreducible character χ^λ of the symmetric group $\mathfrak{S}_n$ evaluated at a permutation σ of reduced cycle type μ,

$$\omega_\mu^\lambda = \frac{|\mathscr{C}_\mu(n)|}{f(\lambda)} \chi^\lambda(\sigma), \tag{1}$$

where $\mathscr{C}_\mu(n)$ stands for the conjugacy class of permutation with reduced cycle type μ, and $f(\lambda)$ is the dimension of the irreducible representation indexed by the partition λ. For μ the empty partition, ω_μ is identically equal to 1 since $f(\lambda) = \chi^\lambda_{1^n}$, and the first interesting case is $\mu = 1$, i.e. for the class $\mathscr{C}_1(n)$ of transpositions: it is then well known (see [3,6] or [15, Chapter 1]) that $\frac{|\mathscr{C}_1(n)|}{f(\lambda)} \chi^\lambda_{21^{n-2}}$ is obtained by summing the numbers $j - i$ over all positions (i,j) in the Ferrers diagram of λ:

$$\omega_1^\lambda = \frac{\binom{n}{2}}{f(\lambda)} \chi^\lambda_{21^{n-2}} = \sum_{(i,j)\in\lambda} (j - i). \tag{2}$$

The numbers $j - i$ are called the *contents* of λ. See Table 1 for the contents of the partitions λ of 4 and the corresponding values ω_1^λ, computed out of the values of the characters in Table 2.

In view of Formula (2), it is natural to ask for a general expression of central characters. Ingram [6] computed a few other similar expressions, but the idea to use systematically *content evaluation* was brought to our attention by the work of

Table 1
Contents of λ

λ	Content	ω_1^λ	ω_2^λ	$C(\lambda)$	$p_1(C(\lambda))$	$p_2(C(\lambda))$
(4)	0 1 2 3	6	8	$\{0,1,2,3\}$	6	14
$(3,1)$	−1 / 0 1 2	2	0	$\{-1,0,1,2\}$	2	6
$(2,2)$	−1 0 / 0 1	0	−4	$\{-1,0,0,1\}$	0	2
$(2,1^2)$	−2 / −1 / 0 1	−2	0	$\{-2,-1,0,1\}$	−2	6
(1^4)	−3 / −2 / −1 / 0	−6	8	$\{-3,-2,-1,0\}$	−6	14

Table 2
Character table of S_4

$\lambda\backslash\mu$	(4)	(3,1)	(2,2)	$(2,1^2)$	(1^4)
(4)	1	1	1	1	1
(3,1)	−1	0	−1	1	3
(2,2)	0	−1	2	0	2
$(2,1^2)$	1	0	−1	−1	3
(1^4)	−1	1	1	−1	1

Table 3
Class polynomials $\omega_\mu(\mathbf{x})$ of $\mathfrak{S}_n$ indexed with reduced partitions $\mu, |\mu|\leqslant 6$ and $\ell(\mu)\leqslant 3$

$\omega_1 = p_1$	$\omega_2 = p_2 - \binom{n}{2}$
$\omega_3 = p_3 - (2n-3)p_1$	$\omega_{1^2} = \frac{1}{2}p_{1^2} - \frac{3}{2}p_2 + \binom{n}{2}$
$\omega_4 = p_4 - 2p_{1^2} - (3n-10)p_2 + 5\binom{n}{3} - 3\binom{n}{2}$	$\omega_{21} = p_{21} - 4p_3 - (\binom{n}{2} - 6n + 8)p_1$
$\omega_5 = p_5 - 6p_{21} - (4n-25)p_3 + 2(3n-4)(n-5)p_1$	
$\omega_{31} = p_{31} - 5p_4 + (12n-35)p_2 - (2n-11)p_{1^2} - 16\binom{n}{3} + 8\binom{n}{2}$	
$\omega_{2^2} = \frac{1}{2}p_{2^2} - \frac{5}{2}p_4 - (\binom{n}{2} - (6n-15))p_2 + 3p_{1^2} + 3\binom{n}{4} - 6\binom{n}{3} + 3\binom{n}{2}$	
$\omega_{1^3} = \frac{1}{6}p_{1^3} + \frac{10}{3}p_3 - \frac{3}{2}p_{21} + (\binom{n}{2} - 4n + 5)p_1$	
$\omega_6 = p_6 - \frac{9}{2}p_{(2^2)} - \frac{5}{2}(2n-21)p_4 - 8p_{31} + \frac{1}{2}(21n^2 - 241n + 504)p_2 + 2(7n-36)p_{1^2}$ $-49\binom{n}{4} + 91\binom{n}{3} - 40\binom{n}{2}$	
$\omega_{41} = p_{41} - 6p_5 + 10(2n-11)p_3 - (3n-40)p_{21} - 2p_{1^3} + (5\binom{n}{3} - 53\binom{n}{2} + 24(5n-6))p_1$	
$\omega_{32} = p_{32} - 6p_5 - \frac{1}{2}(n-5)(n-36)p_3 - (2n-27)p_{21} + (6\binom{n}{3} - 47\binom{n}{2} + 20(5n-6))p_1$	
$\omega_{21^2} = \frac{1}{2}p_{21^2} + 15p_4 - 4p_{31} - \frac{3}{2}p_{2^2} + \frac{5}{2}(\binom{n}{2} - 6(2n-5))p_2 - \frac{1}{2}(\binom{n}{2} - 6n + 26)p_{1^2}$ $-6\binom{n}{4} + 30\binom{n}{3} - 13\binom{n}{2}$	

Katriel. Let $C(\lambda) = \{j - i : (i,j) \in \lambda\}$ be the multiset of contents of the partition λ and let $p_k(C(\lambda)) = \sum_{(i,j)\in\lambda}(j-i)^k$ be the kth power sum symmetric function evaluated at the contents of λ (see Table 1). Katriel [8–11] conjectured that the central characters ω_r^λ can be written as

$$\omega_r^\lambda = p_r(C(\lambda)) + \sum_{|\nu|<r} \Omega_r^\nu(n) p_\nu(C(\lambda)),$$

where $\Omega_r^\nu(n)$ are polynomials in n of degree at most $(r - |\nu|)/2$ and that $\Omega_r^\nu(n)$ is equal to zero if r and $|\nu|$ do not have the same parity.

We show in this article that the conjecture of Katriel is true and more precisely that the degree of the polynomial $\Omega_r^\nu(n)$ is at most $(r - |\nu|)/2 + 1 - \ell(\nu)$. Moreover, we extend that result and show that for each partition μ, there corresponds a symmetric function, called *class symmetric function*, and denoted $\omega_\mu(\mathbf{x})$ such that when ω_μ is evaluated on the contents of a partition λ, we obtain the central character ω_μ^λ. The first few $\omega_\mu(\mathbf{x})$ are presented in Table 3 for $|\mu|\leqslant 6$ and $\ell(\mu)\leqslant 3$. A preliminary version of these results was presented at the 12th FPSAC conference [5].

These class symmetric functions in fact form a new basis of the algebra Λ of symmetric functions over $\mathbb{Q}(n)$. This is the key to obtain an isomorphism between Λ

and the Farahat–Higman algebra *FH* governing products of conjugacy classes in the center $\mathscr{Z}_n$ of the group algebra of the symmetric group $\mathfrak{S}_n$. This main result allows to study *content evaluation* of symmetric functions and to show, for example, that given a family $(a_\lambda)_{\lambda \vdash n}$ of rational values indexed by partitions of n there exists a unique symmetric polynomial $f(x_1, \dots, x_n)$ such that $f(C(\lambda)) = a_\lambda$. Then we study the relation of these results with known results about the asymptotic of skew young tableaux. We recover for instance the fact that if A is a fixed number and λ is a partition of n with $\lambda_1 \leqslant A\sqrt{n}$ and $\ell(\lambda) \leqslant A\sqrt{n}$ and if μ is a partition of k then

$$\frac{f(\lambda/\mu)}{f(\lambda)} = \frac{f(\mu)}{k!} + O(n^{-1/2}),$$

as $n \to \infty$. This result first appeared in [22]. More generally our results appear connected to a number of recent attempts to develop original approaches to the study of representation theory of the symmetric group (in particular see [1,12,19]).

Our exposition is organized as follows. We start with some definitions. Then we prove the main theorem and give some of its consequences. Next, we use class symmetric functions to give asymptotic enumeration of certain classes of tableaux. We end by showing how these results extend to Hecke algebras.

2. Definitions

2.1. Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a finite non-increasing sequence of positive integers λ_i, called the parts of λ, such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$. When the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, we say that λ is a partition of n into k parts and we write $\lambda \vdash n$ or $|\lambda| = n$ and $\ell(\lambda) = k$. The conjugate of a partition λ is denoted by λ' and $\lambda'_i = |\{j\colon \lambda_j \geqslant i\}|$.

The refinement partial order $<_R$ on integer partitions of size n is the partial order induced by the inclusion order on set partitions: $\alpha <_R \beta$ if α is obtained by breaking some parts from β. The inclusion partial order $\subset$ on partitions is the partial order such that $\alpha \subset \beta$ if and only if $\alpha_i \leqslant \beta_i$ for $1 \leqslant i \leqslant \ell(\alpha)$. The (partial) *natural ordering* on partitions is defined for $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_\ell)$ by: $\alpha \leqslant \beta$ if $|\alpha| < |\beta|$ or if $\alpha_1 + \cdots + \alpha_i \leqslant \beta_1 + \cdots + \beta_i$ for all $i = 1, \dots, \ell$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n, we define its *reduced partition* $\tilde{\lambda}$ as the partition $(\lambda_1 - 1, \lambda_2 - 1, \dots)$, i.e. each part is reduced by one and zero parts are removed. Observe that for all $n \geqslant 1$, this is a bijection between partitions of n and partitions μ satisfying $|\mu| + \ell(\mu) \leqslant n$ (including the empty partition ε). The reduced cycle type of a permutation σ is defined accordingly. Given a partition μ, we denote by $\mathscr{C}_\mu(n)$ the conjugacy class of permutations of $\mathfrak{S}_n$ with reduced cycle type μ. For instance $\mathscr{C}_1(n)$ is the class of transpositions of $\mathfrak{S}_n$, $\mathscr{C}_2(n)$ the class of 3-cycles, and so on. In view of the previous observation, $\mathscr{C}_\mu(n) \neq \emptyset$ if and only if $n \geqslant |\mu| + \ell(\mu)$, and $(\mathscr{C}_\mu(n))_{|\mu| + \ell(\mu) \leqslant n}$ is the family of all conjugacy classes of $\mathfrak{S}_n$. We also identify each conjugacy class $\mathscr{C}_\mu(n)$ with the formal sum $\sum_{\sigma\in\mathscr{C}_\mu(n)}\sigma$, which is an element the center $\mathscr{Z}_n$ of the group algebra of the symmetric group $\mathfrak{S}_n$. The family of all conjugacy classes $(\mathscr{C}_\mu(n))_{|\mu|+\ell(\mu)\leqslant n}$ then forms a basis of $\mathscr{Z}_n$.

2.2. Characters

A linear representation of a group G is a group homomorphism $M: G\to Aut(V)$ from G to the group $Aut(V)$ of automorphisms of a vector space V. It can be seen as a map that associates to each group element $x\in G$ an invertible matrix $M(x)$ that acts on V. The character of a representation M is the homomorphism $\chi: G\to\mathbb{Q}$ given by $\chi(x)=\mathrm{tr}(M(x))$, i.e. that associates to each $x\in G$ the trace of the matrix $M(x)$. Unlike the characters, the entries of the matrices $M(x)$ are dependent on the choice of the basis of V. When the basis of V is chosen so that all the matrices $M(x)$ are block-diagonal with indecomposable blocks, these irreducible blocks are the irreducible representations of G. The irreducible representations of the symmetric group $\mathfrak{S}_n$ and their corresponding characters are indexed with partitions λ of n, and we write $\chi^\lambda(\sigma)=\mathrm{tr}(M^\lambda(\sigma))$ where λ is the indexing partition of the irreducible representation.

Characters are obviously class functions and we write χ^λ_μ for the common value $\chi^\lambda(\sigma)$ for $\sigma\in\mathscr{C}_{\tilde{\mu}}(n)$. We also view the character χ^λ as an element $\sum_\sigma\chi^\lambda(\sigma)\,\sigma$ in $\mathscr{Z}_n$. It is well known that the family $(\chi^\lambda)_{\lambda\vdash n}$ forms a basis of $\mathscr{Z}_n$, and that the matrix $[\chi^\lambda_\mu]_{\lambda,\mu\vdash n}$, called the character table, is invertible. Table 2 shows the character table of the group $\mathfrak{S}_4$.

2.3. Farahat–Higman algebra

The fact that the family $(\mathscr{C}_\mu(n))_{|\mu|+\ell(\mu)\leqslant n}$ forms a basis of the center $\mathscr{Z}_n$ of the group algebra of the symmetric group $\mathfrak{S}_n$ allows to define the linearization coefficients $c^\lambda_{\mu,\nu}(n)$ of the products $\mathscr{C}_\mu(n)*\mathscr{C}_\nu(n)$ in $\mathscr{Z}_n$ for all n:

$$\mathscr{C}_\mu(n)*\mathscr{C}_\nu(n)=\sum_\lambda c^\lambda_{\mu,\nu}(n)\mathscr{C}_\lambda(n).$$

Farahat and Higman [2] proved that the coefficients $c^\lambda_{\mu,\nu}(n)$ are polynomials in n. Let the Farahat–Higman algebra FH be the algebra over $\mathbb{Q}(n)$ with basis $(\mathscr{C}_\mu)_\mu$ (considered as formal elements) and product

$$\mathscr{C}_\mu*\mathscr{C}_\nu=\sum_\lambda c^\lambda_{\mu,\nu}(n)\mathscr{C}_\lambda.$$

This algebra was introduced in [15, Chapter I.7, Example 24]. The following conditions are necessary for the connexion coefficient $c^\lambda_{\mu,\nu}(n)$ to be non-zero polynomials:

1. $|\mu|+|\nu|$ and $|\lambda|$ have the same parity,
2. $|\lambda|\leqslant|\mu|+|\nu|$,
3. if $|\lambda|=|\mu|+|\nu|$ then $(\mu\cup\nu)<_R\lambda$ in the refinement order on partitions.

Proposition 2.1. *The coefficient $c^{\lambda}_{\mu,\nu}(n)$ is a polynomial in n of degree bounded by*

$$\frac{1}{2}(|\mu| + |\nu| - |\lambda|) + \ell(\mu) + \ell(\nu) - \ell(\lambda). \tag{3}$$

Proof (sketch of). First we show that this bound is correct for any coefficient $c^{\lambda}_{1,\nu}(n)$ in a product $\mathscr{C}_1 * \mathscr{C}_\nu$. This is done by a verification of the six possible cycle types obtained when we multiply $\sigma \in \mathscr{C}_\nu$ by a transposition. After this verification, we observe that when $|\mu| = k$ then (3) is an upper bound for the product $\mathscr{C}_1 * \cdots * \mathscr{C}_1 * \mathscr{C}_\nu$ of $\mathscr{C}_1$ with itself k times and with $\mathscr{C}_\nu$. Since $\mathscr{C}_\mu * \mathscr{C}_\nu$ is a component of the latter product, the coefficient $c^{\lambda}_{\mu,\nu}(n)$ is also bounded by (3). $\square$

Observe that this bound is not always tight: for instance $c^{(1,1)}_{(1),(3)}(n) = 4$ is independent of n, in contradiction with the (false) statement of [15, Chapter I.7, Example 24] that $c^{\lambda}_{\mu,\nu}(n)$ should be independent of n if and only if $|\mu| + |\nu| = |\lambda|$.

Similarly we define the coefficients $b^{\nu}_{\mu}(n)$, for $\mu = (\mu_1, \dots, \mu_k) = 1^{m_1} \cdots n^{m_k}$ a partition (see [15, Chapter I.7, Example 24, 25]):

$$\begin{aligned} \mathscr{C}_{\mu_1} * \cdots * \mathscr{C}_{\mu_k} &= \left(\prod_i m_i!\right) \mathscr{C}_\mu + \sum_{\mu <_R \nu} b^{\nu}_{\mu}(n) \mathscr{C}_\nu + \sum_{|\nu| < |\mu|} b^{\nu}_{\mu}(n) \mathscr{C}_\nu, \\ &= \left(\prod_i m_i!\right) \mathscr{C}_\mu + \sum_{\nu' < \mu'} b^{\nu}_{\mu}(n) \mathscr{C}_\nu, \end{aligned} \tag{4}$$

(recall that ν' is the conjugate of ν and that $\nu < \mu$ refers to the natural ordering as previously defined). The connexion coefficient $b^{\nu}_{\mu}(n)$ is zero if $|\mu|$ and $|\nu|$ have different parity and is a polynomial in n of degree bounded by $\frac{1}{2}(|\mu| - |\nu|) + \ell(\mu) - \ell(\nu)$.

2.4. Classical symmetric functions

The elementary and homogeneous symmetric functions can be defined by their generating functions (see [15, Chapter I])

$$\prod_{i \geqslant 1} (1 + tx_i) = \sum_{m \geqslant 0} e_m(\mathbf{x}) t^m, \quad \prod_{i \geqslant 1} \frac{1}{1 - tx_i} = \sum_{m \geqslant 0} h_m(\mathbf{x}) t^m$$

with $\mathbf{x} = x_1, x_2, \dots$. For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, these definitions are extended multiplicatively: $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$. The power sum symmetric functions are classically defined by

$$p_m(\mathbf{x}) = \sum_{i \geqslant 1} x_i^m \quad \text{and} \quad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$$

and this definition is extended to differences of alphabets in the plethystic notation by

$$p_m(A - B) = \sum_{\alpha \in A} \alpha^m - \sum_{\beta \in B} \beta^m. \tag{5}$$

A general relation between elementary, homogeneous and power sum symmetric functions then reads, for two alphabets A and B,

$$\frac{\prod_{\alpha \in A} (1 - t\alpha)}{\prod_{\beta \in B} (1 - t\beta)} = \sum_{m \geqslant 0} h_m(A - B)\, t^m, \quad \text{where} \quad h_m(A - B) = \sum_{\mu \vdash m} \frac{1}{z_\mu} p_\mu(A - B), \tag{6}$$

where for a partition $\mu = 1^{m_1} \cdots p^{m_p}$, we write as usual $z_\mu = m_1! 1^{m_1} \cdots m_p! p^{m_p}$.

2.5. Content evaluation

Recall that $C(\lambda) = \{j - i \mid (i,j) \in \lambda\}$ is the multiset of *contents* of the partition λ. Given a symmetric function $f(x_1, x_2, \ldots)$ and a partition $\lambda \vdash n$ we define its *content evaluation* $f(C(\lambda)) = f(c_1, \ldots, c_n, 0, 0, \ldots)$ where the $(c_i)_{1 \cdots n}$ form an arbitrary enumeration of the elements of $C(\lambda)$. For instance, the content evaluation of power sums is particularly simple to express

$$p_k(C(\lambda)) = \sum_{(i,j) \in \lambda} (j - i)^k.$$

The classical content polynomial of a partition λ of n,

$$c_\lambda(t) = \prod_{(i,j) \in \lambda} (t + (j - i)) = \sum_{k=0}^{n} e_k(C(\lambda))\, t^{n-k},$$

is just the generating function of elementary symmetric functions evaluated on the contents of λ (modulo some power of t).

Observe now that (see also [15, p. 15])

$$\frac{c_\lambda(t+k)}{c_\lambda(t+k-1)} = \prod_{(i,j) \in \lambda} \frac{t + k + (j - i)}{t + k - 1 + (j - i)} = \prod_{i=1}^{\ell(\lambda)} \frac{t + \lambda_i + k - i}{t + k - i}.$$

We shall be interested in expansions in power series of $y = 1/t$ of the latter expression, which is conveniently rewritten with the notation $c(\alpha) = j - i$ for the content of the box $\alpha=(i,j)$:

$$\frac{c_\lambda(1/y+k)}{c_\lambda(1/y+k-1)}=\prod_{\alpha\in\lambda}\frac{1+yk+yc(\alpha)}{1+y(k-1)+yc(\alpha)}$$
$$=\left(\frac{1+yk}{1+y(k-1)}\right)^n\prod_{\alpha\in\lambda}\frac{1+\frac{yc(\alpha)}{1+yk}}{1+\frac{yc(\alpha)}{1+y(k-1)}}. \tag{7}$$

2.6. Power content symmetric functions and shift-symmetric functions

The evaluations $p_k(C(\lambda))$ can also be expressed as functions in the parts λ_i of λ. For example we deduce from Fig. 1:

$$p_1[C(\lambda)]=\sum_i\binom{\lambda_i}{2}-(i-1)\lambda_i,$$

$$p_2[C(\lambda)]=\sum_i\binom{\lambda_i+1}{3}+\binom{\lambda_i}{3}-(i-1)\lambda_i(\lambda_i-1).$$

We call these polynomials in the λ_i the *power-content* polynomials. More precisely we define the kth power-content function $pc_k(\mathbf{x})$ as the unique function such that for all partitions λ

$$pc_k(\lambda_1,\lambda_2,\ldots)=p_k(C(\lambda)).$$

The functions $pc_k(x_1,x_2,\ldots)$ are not symmetric but they are *shift symmetric*, in the sense of Okounkov and Olshanski [19]: a function $p(x_1,x_2,\ldots)$ is shift symmetric if it satisfies one of the following two equivalent conditions:

1. $p(x_1,\ldots,x_i,x_{i+1},\ldots)=p(x_1,\ldots,x_{i+1}-1,x_i+1,\ldots)\quad\forall i\geqslant 1.$
2. $p(x_1,x_2,\ldots)$ becomes a symmetric polynomial under the change of variables $x'_i:=x_i-i+k$ with k an arbitrary constant.

Theorem 2.1. *For any positive integer* $k, pc_k(x_1,\ldots,x_i,x_{i+1},\ldots)$ *is a shift-symmetric function.*

λ_k	$-k+1$	$\cdots$	λ_k-k				$\binom{\lambda_k-k+1}{2}-\binom{k}{2}$
$\vdots$	$\vdots$		$\vdots$				$\vdots$
λ_3	-2	-1	0	$\cdots$	λ_3-3		$\binom{\lambda_3-2}{2}-\binom{3}{2}$
λ_2	-1	0	1	$\cdots$	λ_2-2		$\binom{\lambda_2-1}{2}-1$
λ_1	0	1	2	3	$\cdots$	λ_1-1	$\binom{\lambda_1}{2}$

Fig. 1. The sum of contents of λ along lines.

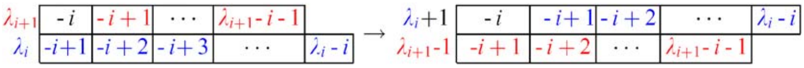

Fig. 2. Shifted transposition of $(\lambda_i, \lambda_{i+1})$.

Proof. This is obvious when we observe the effect on the contents of the exchange of two adjacent rows λ_i, λ_{i+1} of a diagram λ.

The set of contents in the two rows on the left of Fig. 2 remains unchanged after we perform the shifted transposition. This means that the sum of the kth power of these entries is invariant under a shifted transposition and proves the theorem. $\square$

The algebra Λ^* of shift-symmetric functions was studied by Okounkov and Olshanski [19]. The set $\{pc_\lambda = pc_{\lambda_1} pc_{\lambda_2} \cdots\}$ of power content functions forms a basis of Λ^*. As any symmetric function $g(C(\lambda))$ in the contents of a partition λ becomes a shift-symmetric polynomial in the parts of λ we are allowed to define a map $\pi : \Lambda \to \Lambda^*$ by $\pi(p_k) = pc_k$.

3. The basis of class symmetric functions

We are now in a position to state our main result.

Theorem 3.1. *There exists a basis $(\omega_\mu(\mathbf{x}))_\mu$ of the algebra Λ over $\mathbb{Q}(n)$ of symmetric functions in the indeterminates $\mathbf{x} = \{x_1, x_2, \ldots\}$ such that the content evaluations of the symmetric functions $\omega_\mu(\mathbf{x})$ yield central characters: for any partition μ, for all $n \geqslant 1$, $\lambda \vdash n$,*

$$\omega_\mu(C(\lambda)) = \frac{|\mathscr{C}_\mu(n)|}{f(\lambda)} \chi^\lambda(\sigma) \quad \textit{for any } \sigma \in \mathscr{C}_\mu(n), \tag{8}$$

where χ^λ and $f(\lambda)$ are the character and dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by λ. In other terms each ω_μ represents simultaneously for all n the eigenvalues of the multiplicative action of the $\mathscr{C}_\mu(n)$ on the center $\mathscr{Z}_n$ of the group algebra of $\mathfrak{S}_n$: for all $n \geqslant 1$ and $\lambda \vdash n$,

$$\mathscr{C}_\mu(n) * \chi^\lambda = \omega_\mu(C(\lambda))\, \chi^\lambda. \tag{9}$$

Moreover, the map $c : \mathscr{C}_\mu \mapsto \omega_\mu(\mathbf{x})$ is an algebra isomorphism between the Farahat–Higman algebra FH and the algebra Λ: for all partitions μ, ν,

$$c(\mathscr{C}_\mu * \mathscr{C}_\nu) = c(\mathscr{C}_\mu) c(\mathscr{C}_\nu) = \omega_\mu(\mathbf{x}) \omega_\nu(\mathbf{x}).$$

The symmetric functions $\omega_\mu(\mathbf{x})$ thus provide a calculus in Λ for the connexion coefficients of the algebra FH, or, equivalently, for any fixed n, for connexion *coefficients of the symmetric group $\mathfrak{S}_n$: for all partitions μ, ν,*

$$\omega_\mu(\mathbf{x})\omega_\nu(\mathbf{x}) = \sum_{|\lambda| \leqslant |\mu|+|\nu|} c_{\mu,\nu}^{\lambda}(n)\, \omega_\lambda(\mathbf{x}).$$

Remark. For any fixed n, we thus obtain a polynomial calculus that provides all connexion coefficients of $\mathscr{Z}_n$ as linearizations. As opposed to this, the polynomial representation introduced by Macdonald [15, Chapter I.7, Example 25] was restricted to the so-called *top connexion coefficients* of $\mathscr{Z}_n$ (see also Goulden and Jackson's account of Macdonald's result in [4]). In fact, the homogeneous components of maximal degree in our class polynomials ω_μ are the polynomials g_λ studied there.

3.1. Class symmetric functions for cycles

We need some preliminary results before proving Theorem 3.1.

Lemma 3.1. *The symmetric function $\omega_\mu(\mathbf{x})$ defined by Relation* (8) *with $\mu = (r)$ for all $\lambda \vdash n$ have an expansion in the power sum basis*

$$\omega_r(\mathbf{x}) = p_r(\mathbf{x}) + \sum_{|\nu|<r} \Omega_r^{\nu}(n)\, p_\nu(\mathbf{x}),$$

where the $\Omega_r^{\nu}(n)$ are polynomials in n of degree at most $(r - |\nu|)/2 + 1 - \ell(\nu)$.

Proof. Let $(t)_r$ and $(t)^{(r)}$ denote, respectively, the rth descending and ascending factorials of t. The following expression is due to Ingram [6] (see also [15, p. 118]): for $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, the central character

$$\frac{|\mathscr{C}_r(n)|}{f(\lambda)} \chi^\lambda(\sigma) \quad \text{with } \sigma \in \mathscr{C}_r(n)$$

is equal to the coefficient of t^{-1} in the expansion in powers of $1/t$ of

$$\frac{(-1)^{r+1}(t)^{(r+1)}}{(r+1)^2} \prod_{i=1}^{n} \frac{t + r + 1 + \lambda_i + n - i}{t + \lambda_i + n - i}.$$

Using the remarks of Section 2.4, this is also the coefficient of t^{-1} in the expansion in powers of $1/t$ of

$$\frac{(-1)^{r+1}(t+n)^{(r+1)}}{(r+1)^2} \frac{c_\lambda(t+n+r+1)}{c_\lambda(t+n+r)} \frac{c_\lambda(t+n-1)}{c_\lambda(t+n)}.$$

The coefficient of t^{-1} in the expansion in powers of $1/t$ of a rational function is stable under translation of the form $t \to t - x$ so that we can as well look for the coefficient of t^{-1} in

$$\frac{(-1)^{r+1}(t)^{(r+1)}}{(r+1)^2}\frac{c_\lambda(t+r+1)}{c_\lambda(t+r)}\frac{c_\lambda(t-1)}{c_\lambda(t)}.$$

Upon setting $t=-1/y$ and using Eq. (7) the central character thus reads

$$\frac{-1}{(r+1)^2}[y^{r+2}]\prod_{s=0}^{r}(1-sy)\left(\frac{(1-y(r+1))(1+y)}{1-yr}\right)^n\prod_{\alpha\in\lambda}\frac{\left(1-\frac{yc(\alpha)}{1-y(r+1)}\right)\left(1-\frac{yc(\alpha)}{1+y}\right)}{\left(1-\frac{yc(\alpha)}{1-yr}\right)(1-yc(\alpha))},$$

where $[y^r]f(y)$ denotes the coefficient of y^r in the power series expansion of $f(y)$. Now consider the alphabet $A=C(\lambda)\times[A_r+A_0-A_{-1}-A_{r+1}]$ where $A_k=\{\frac{1}{1-ky}\}$ is a one-letter alphabet. Using the plethystic formulas (5) and (6), the last product can be expanded to give

$$\frac{-1}{(r+1)^2}[y^{r+2}]\prod_{s=0}^{r}(1-sy)\left(\frac{(1-y(r+1))(1+y)}{1-yr}\right)^n \left(1+\sum_{\nu,|\nu|>0}\frac{p_\nu[A_r+A_0-A_{-1}-A_{r+1}]\,p_\nu(C(\lambda))\,y^{|\nu|}}{z_\nu}\right).$$

This expression is an expansion in power sums evaluated on contents. The coefficients of Lemma 3.1 should therefore be defined as

$$\Omega_r^\varepsilon(n)=\frac{-1}{(r+1)^2}[y^{r+2}]\prod_{s=0}^{r}(1-sy)\left(1-\frac{y^2(r+1)}{1-yr}\right)^n,$$

where ε is the empty partition; otherwise ($|\nu|>0$):

$$\Omega_r^\nu(n)=\frac{-1}{(r+1)^2}[y^{r+2-|\nu|}]\prod_{s=0}^{r}(1-sy) \times\left(1-\frac{y^2(r+1)}{1-yr}\right)^n\frac{p_\nu[A_r+A_0-A_{-1}-A_{r+1}]}{z_\nu}.$$

This is indeed a polynomial in n, as shown by expansion in the binomial basis of the nth power:

$$\Omega_r^\nu(n)=\frac{1}{z_\nu}\sum_{i=0}^{\left\lfloor\frac{r+2-|\nu|}{2}\right\rfloor}(-1)^{i+1}\binom{n}{i}(r+1)^{i-2}[y^{r+2-|\nu|-2i}] \times\frac{\prod_{s=0}^{r}(1-sy)}{(1-ry)^i}p_\nu[A_r+A_0-A_{-1}-A_{r+1}].$$

Observe moreover that

$$\begin{aligned}
p_k[A_r + A_0 - A_{-1} - A_{r+1}] &= (1-ry)^{-k} + 1 - (1+y)^{-k} - (1-(r+1)y)^{-k} \\
&= \sum_{j\geqslant 1} \binom{k+j-1}{j} \left(\frac{y}{1-ry}\right)^{j+k} + 1 - (1+y)^{-k} \\
&= \sum_{j\geqslant 0} \binom{k+j-1}{j} (r^j + 0^j - (-1)^j - (r+1)^j) y^j \\
&= \sum_{j>i>0} \binom{k+j-1}{j} \binom{j}{i} (r+1)^i (-1)^{j-i} y^j,
\end{aligned}$$

so that $p_k[A_r + A_0 - A_{-1} - A_{r+1}] = -(k+1)k(r+1)y^2 + O(y^3)$. This implies that there exists a constant c such that $p_v[A_r + A_0 - A_{-1} - A_{r+1}] = cy^{2\ell(v)} + O(y^{2\ell(v)+1})$. In other terms, the polynomial $\Omega_r^v(n)$ has degree at most $\frac{r-|v|}{2} + 1 - \ell(v)$. This allows to compute the leading term and verify that $\Omega_r^r(n) = 1$. $\square$

3.2. The general case

Now let us consider the general central characters. Inverting the triangular relation (4) yields

$$\mathscr{C}_\mu = \frac{1}{\prod_i m_i!} \mathscr{C}_{\mu_1} * \cdots * \mathscr{C}_{\mu_k} + \sum_{v' < \mu'} d_\mu^v(n) \mathscr{C}_{v_1} * \cdots * \mathscr{C}_{v_\ell}, \tag{10}$$

where $v = (v_1, \dots, v_\ell)$ and the coefficients $d_\mu^v(n)$ satisfy the same properties as the $b_\mu^v(n)$, that is they have degree at most $(|\mu| - |v|)/2 + \ell(\mu) - \ell(v)$ and they are zero if $|\mu|$ and $|v|$ have different parity.

In view of relation (9), formula (10) yields

$$\omega_\mu(\mathbf{x}) = \frac{1}{\prod_i m_i!} \omega_{\mu_1}(\mathbf{x}) \cdots \omega_{\mu_k}(\mathbf{x}) + \sum_{v' < \mu'} d_\mu^v(n) \omega_{v_1}(\mathbf{x}) \cdots \omega_{v_\ell}(\mathbf{x}). \tag{11}$$

Using now Lemma 3.1, relation (11) allows us to obtain a general expansion for our class symmetric functions, which proves that they form a basis of Λ, and thus concludes the proof of Theorem 3.1.

Lemma 3.2. *The symmetric functions $\omega_\mu(\mathbf{x})$ defined by relation* (8) *for all $\lambda \vdash n$ have an expansion in the power sum basis*

$$\omega_\mu(\mathbf{x}) = \frac{1}{\prod_i m_i!} p_\mu(\mathbf{x}) + \sum_{v' < \mu'} \Omega_\mu^v(n) p_v(\mathbf{x}),$$

where the $\Omega_\mu^v(n)$ are polynomials in n.

Now we study the degree of these polynomials.

Lemma 3.3. *The $\Omega_\mu^\nu(n)$ are polynomials in n of degree at most $\frac{|\mu|-|\nu|}{2}+\ell(\mu)-\ell(\nu)$.*

Proof. We know that

$$\omega_\mu(\mathbf{x}) = \sum_{\alpha' \leqslant \mu'} d_\mu^\alpha(n)\omega_{\alpha_1}(\mathbf{x})\cdots\omega_{\alpha_\ell}(\mathbf{x}),$$

where the $d_\mu^\alpha(n)$ are polynomials in n of degree at most $\frac{1}{2}(|\mu|-|\alpha|)+\ell(\mu)-\ell(\alpha)$ and that

$$\omega_r(\mathbf{x}) = p_r + \sum_{|\nu|<r} \Omega_r^\nu(n)p_\nu(\mathbf{x}),$$

where the $\Omega_r^\nu(n)$ are polynomials in n of degree at most $(r-|\nu|)/2+1-\ell(\nu)$.

Therefore,

$$\begin{aligned}\omega_\mu(\mathbf{x}) &= \sum_{\alpha' \leqslant \mu'} d_\mu^\alpha(n) \sum_{|\nu^{(1)}|\leqslant\alpha_1} \cdots \sum_{|\nu^{(\ell)}|\leqslant\alpha_\ell} \left(\prod_{i=1}^{\ell} \Omega_{\alpha_i}^{\nu^{(i)}}(n)\right) p_{\cup_{i=1}^{\ell}\nu^{(i)}}(\mathbf{x}) \\ &= \sum_{|\nu|\leqslant|\mu|} \sum_{\substack{\nu^{(1)},\dots,\nu^{(\ell)} \\ \cup_{i=1}^{\ell}\nu^{(i)}=\nu}} \sum_{\substack{\alpha,\alpha_i \geqslant |\nu^{(i)}| \\ \alpha' \leqslant \mu'}} d_\mu^\alpha(n) \left(\prod_{i=1}^{\ell} \Omega_{\alpha_i}^{\nu^{(i)}}(n)\right) p_\nu(\mathbf{x}).\end{aligned}$$

As each $\Omega_{\alpha_i}^{\nu^{(i)}}(n)$ has degree at most $(\alpha_i - |\nu^{(i)}|)/2+1-\ell(\nu^{(i)})$ then $\prod_{i=1}^{\ell}\Omega_{\alpha_i}^{\nu^{(i)}}(n)$ has degree at most $\frac{1}{2}(|\mu|-|\alpha|)+\ell(\mu)-\ell(\alpha)$.

Hence

$$\Omega_\nu^\mu(n) = \sum_{\substack{\nu^{(1)},\dots,\nu^{(\ell)} \\ \cup_{i=1}^{\ell}\nu^{(i)}=\nu}} \sum_{\substack{\alpha,\alpha_i' \geqslant \nu^{(i)} \\ \alpha' \leqslant \mu'}} d_\mu^\alpha(n) \prod_{i=1}^{\ell} \Omega_{\alpha_i}^{\nu^{(i)}}(n)$$

has degree at most $(|\mu|-|\nu|)/2+\ell(\mu)-\ell(\nu)$. $\square$

3.3. Asymptotic of central characters

We show here that the asymptotic of central characters is a simple consequence of the two previous subsections. Let λ be a partition of n and let A be a constant such that $\lambda_1 \leqslant A\sqrt{n}$ and $\ell(\lambda)\leqslant A\sqrt{n}$. This implies that $p_r[C(\lambda)] = O(n^{r/2+1})$ and $p_\nu[C(\lambda)] = O(n^{|\nu|/2+\ell(\nu)})$.

Moreover we know from Lemmas 3.2 and 3.3 that

$$\omega_\mu(C(\lambda)) = \frac{1}{\prod_i m_i!} p_\mu(\mathscr{C}(\lambda)) + \sum_{\mu <_R \nu} \Omega_\mu^\nu p_\nu(C(\lambda)) + \sum_{|\nu|<|\mu|} \Omega_\mu^\nu p_\nu(C(\lambda)),$$

where Ω_μ^ν is a polynomial in n of degree at most $(|\mu| - |\nu|)/2 + \ell(\mu) - \ell(\nu)$. Therefore,

Proposition 3.1. *If $\lambda \vdash n$ is such that $\lambda_1 \leqslant A\sqrt{n}$ and $\ell(\lambda) \leqslant A\sqrt{n}$ for some constant A then*

$$\omega_\mu(C(\lambda)) = O(n^{|\mu|/2+\ell(\mu)}). \tag{12}$$

as $n \to \infty$.

Remark. This asymptotic was studied by Biane [1], building on work of Vershik and Kerov [23], when $\lambda^1, \lambda^2, \ldots$ is a sequence such that $\lambda^n \vdash n$, and such that the λ^ns when rescaled by a factor $\sqrt{n}$ have area 1 and converge uniformly to some limit.

4. Consequences of the isomorphism

We now state some interesting consequences of the isomorphism c:

Corollary 4.1. *Given a family of rational values $(a_\lambda)_{\lambda \vdash n}$, there is a unique symmetric polynomial $f(x_1, \ldots, x_n)$ with rational coefficients that satisfies $f(C(\lambda)) = a_\lambda$ for all partitions $\lambda \vdash n$.*

Proof. Take $f(\mathbf{x}) = c\left(\sum_\nu a_\nu \chi^\nu / h_\nu\right)$. Then for all partitions $\lambda \vdash n$, $a_\lambda \chi^\lambda = \sum_\nu a_\nu \chi^\nu / h_\nu \cdot \chi^\lambda = f(C(\lambda))\chi^\lambda$. The first equality is true by idempotency of the irreducible characters, the second is a consequence of Theorem 3.1. □

Theorem 4.1. *Let $(e_k(\mathbf{x}))_{k \geqslant 1}$ denote the elementary symmetric functions. Then*

$$e_k(\mathbf{x}) = \sum_{\nu \vdash k} \omega_\nu(\mathbf{x}).$$

Proof. The Jucys–Murphy elements J_i of the group algebra of the symmetric group $\mathfrak{S}_n$ are defined by

$$J_i = (1, i) + (2, i) + \cdots + (i-1, i), \quad \text{for all } i = 2, \ldots, n.$$

These elements were introduced independently by Jucys [7] and Murphy [18] and they satisfy two fundamental properties first recognized by Jucys and pointed out to us by Lascoux [13] (see also [17,20]).

First, the elementary symmetric functions in the J_i algebraically span the center $\mathscr{Z}_n$ of the group algebra of $\mathfrak{S}_n$ and are sums of conjugacy classes of permutations with a given number of cycles

$$e_k(J_1, \dots, J_n) = \sum_{\mu \vdash k} \mathscr{C}_\mu(n). \tag{13}$$

Next consider a symmetric function $f(\mathbf{x})$. The evaluation of f on the Jucys–Murphy elements belongs to the center of the group algebra of $\mathfrak{S}_n$ and satisfies

$$f(J_1, \dots, J_n) * \chi^\lambda = f(C(\lambda))\chi^\lambda. \tag{14}$$

Our interest in the latter properties is clear: translating Eq. (13) in Λ through Eq. (14) gives for all integers n and partitions $\lambda \vdash n$,

$$e_k(C(\lambda)) = \sum_{\mu \vdash k} \omega_\mu(C(\lambda)). \tag{15}$$

The theorem follows from Eq. (15) and the application of Corollary 4.1. $\square$

Finally, observe that the Jucys–Murphy elements yield another way to state the relation between FH and Λ:

$$\mathscr{C}_\mu(n) = \omega_\mu(J_1, \dots, J_n).$$

5. Tableaux

We now use the previous results to obtain the asymptotic and ordinary enumeration of some tableaux: skew young tableaux and semi-standard tableaux. In this section we use non-reduced partitions. When reduced partition are needed we use the $\tilde{\mu}$ notation.

5.1. Skew young tableaux

If $\alpha \subset \lambda$ then the diagram of α is contained in the diagram of λ. Let λ/α denote the skew diagram obtained by the difference of the Ferrers diagrams λ and α. A standard tableau of skew shape λ/α is a filling of the positions of the Ferrers diagram λ/α with the numbers from 1 to $|\lambda| - |\alpha|$ in increasing order from left to right and from bottom to top. Fig. 3 shows a standard tableau of skew shape $\lambda/\alpha = (5,4,2,1)/(2,1)$.

A combinatorial interpretation of the numbers $f(\lambda/\alpha)$ was given by Okounkov et al. [19] and linked to shift symmetric functions. We now want to exhibit these numbers $f(\lambda/\alpha)$ in terms of our power content functions. The key observation is that the irreducible characters χ^λ_μ are linear combinations of values $f(\lambda/\alpha)$. More

7	8			
×	3	5	6	
×	×	1	2	4

Fig. 3. A standard tableau of skew shape $(5, 4, 2, 1)/(2, 1)$.

precisely, we have

$$\chi^{\lambda}_{(\mu,1^{n-|\mu|})} = \sum_{\alpha \vdash |\mu|} \chi^{\alpha}_{\mu} f(\lambda/\alpha). \tag{16}$$

This equality is a straightforward consequence of the application of the Murnaghan–Nakyama rule for the evaluation of the irreducible characters $\chi^{\lambda}_{(\mu,1^{n-|\mu|})}$. For each partition μ of weight $|\mu| = k$, there is an identity similar to (16) and this establishes a system of linear equations. The coefficients matrix $[\chi^{\alpha}_{\mu}]$ of this system is invertible so that we can express the $f(\lambda/\alpha)$ as linear combinations of the characters $\chi^{\lambda}_{\mu,1^{n-\mu}}$. It follows that the normalized numbers $\frac{(n)_k}{f(\lambda)} f(\lambda/\alpha)$ are linear combinations of central characters:

$$f(\lambda/\alpha) = \sum_{\mu \vdash k} \frac{\chi^{\alpha}_{\mu}}{z_{\mu}} \chi^{\lambda}_{\mu 1^{n-|\mu|}}.$$

See also [22]. We then get

$$\frac{(n)_k}{f(\lambda)} f(\lambda/\alpha) = \sum_{\mu \vdash k} \chi^{\alpha}_{\mu} \binom{n-k+m_1(\mu)}{m_1(\mu)} \underbrace{\frac{|\mathscr{C}_{\tilde{\mu}}(n)|}{f(\lambda)} \chi^{\lambda}_{\mu 1^{n-k}}}_{\omega_{\tilde{\mu}}(C(\lambda))},$$

where $m_1(\mu)$ is the number of 1 in μ. The following fact follows. For every partition $\alpha \vdash k$, there exists a unique symmetric polynomial $g_{\alpha}(\mathbf{x})$ and a corresponding shift symmetric polynomial $\pi(g_{\alpha}(\mathbf{x}))$ such that for all partitions λ

$$g_{\alpha}(C(\lambda)) = \pi(g_{\alpha})(\lambda_1, \lambda_2, \ldots) = \frac{(n)_k}{f(\lambda)} f(\lambda/\alpha).$$

We present in Table 4 the normalized numbers $\frac{(n)_k}{f(\lambda)} f(\lambda/\alpha)$ of standard skew tableaux as shift symmetric polynomials $\pi(g_{\alpha})(\lambda_1, \lambda_2, \ldots)$.

We can also use our results to get the asymptotics of $f(\lambda/\alpha)$. This asymptotic was studied by Stanley [22]. Let λ be a partition of n and A a constant such that $\lambda_1 \leqslant A\sqrt{n}$ and $\ell(\lambda) \leqslant A\sqrt{n}$.

Table 4
$f(\lambda/\alpha)$ as a polynomial in the parts of λ

$\frac{(n)_2}{f(\lambda)}f(\lambda/2) = pc_1 + \binom{n}{2}$	$\frac{(n)_2}{f(\lambda)}f(\lambda/1^2) = -pc_1 + \binom{n}{2}$
$\frac{(n)_3}{f(\lambda)}f(\lambda/3) = pc_2 + (n-2)pc_1 + \binom{n}{3} - \binom{n}{2}$	$\frac{(n)_3}{f(\lambda)}f(\lambda/21) = -pc_2 + 2\binom{n}{3} + \binom{n}{2}$
$\frac{(n)_3}{f(\lambda)}f(\lambda/1^3) = pc_2 - (n-2)pc_1 + \binom{n}{3} - \binom{n}{2}$	
$\frac{(n)_4}{f(\lambda)}f(\lambda/4) = pc_3 + \frac{1}{2}(2n+9)pc_2 + \frac{1}{2}pc_1^2 + (\binom{n}{2} - 4n + 6)pc_1 + \binom{n}{4} - 3\binom{n}{3} + 2\binom{n}{2}$	
$\frac{(n)_4}{f(\lambda)}f(\lambda/31) = -pc_3 + \frac{3}{2}pc_2 - \frac{1}{2}pc_1^2 + \binom{n}{2}pc_1 + 3\binom{n}{4} - \binom{n}{2}$	
$\frac{(n)_4}{f(\lambda)}f(\lambda/2^2) = -npc_2 + pc_1^2 + 2\binom{n}{4} + 3\binom{n}{3} + \binom{n}{2}$	
$\frac{(n)_4}{f(\lambda)}f(\lambda/21^2) = pc_3 + \frac{3}{2}pc_2 - \frac{1}{2}pc_1^2 - \binom{n}{2}pc_1 + 3\binom{n}{4} - \binom{n}{2}$	
$\frac{(n)_4}{f(\lambda)}f(\lambda/1^4) = -pc_3 + \frac{1}{2}(2n+9)pc_2 + \frac{1}{2}pc_1^2 - (\binom{n}{2} - 4n + 6)pc_1 + \binom{n}{4} - 3\binom{n}{3} + 2\binom{n}{2}$	

Proposition 5.1. *For every partition α of k, we have*

$$\frac{f(\lambda/\alpha)}{f(\lambda)} = \frac{f(\alpha)}{k!} + \frac{\chi^{\alpha}_{21^{k-2}}p_1(C(\lambda))}{n^2(k-2)!} + O(1/n).$$

as $n \to \infty$, with $p_1(C(\lambda)) = O(n^{3/2})$.

Proof. We use the fact that

$$\frac{f(\lambda/\alpha)}{f(\lambda)} = \frac{1}{(n)_k}\sum_{\mu \vdash k}\chi^{\alpha}_{\mu}\binom{n-k+m_1(\mu)}{m_1(\mu)}\omega_{\tilde{\mu}}(C(\lambda)).$$

We know from Proposition 3.1 that

$$\omega_{\tilde{\mu}}(C(\lambda)) = O(n^{|\mu|/2+\ell(\mu)/2-m_1(\mu)}).$$

Then

$$\frac{1}{(n)_k}\binom{n-k+m_1(\mu)}{m_1(\mu)}\omega_{\tilde{\mu}}(C(\lambda)) = \begin{cases} \dfrac{1}{k!} & \text{if } \mu = 1^k, \\ \dfrac{p_1(C(\lambda))}{n^2(k-2)!} + O(1/n) & \text{if } \mu = 21^{k-2}, \\ O(1/n) & \text{otherwise.} \end{cases}$$

As noted before $\chi^{\mu}_{1^k} = f(\mu)$. The result follows. $\square$

Remark. As noted by Stanley [22], together with results of Logan and Shepp [14] or Vershik and Kerov [23], this implies a result of McKay et al. [16]. Let $N(n)$ be the number of Young tableaux with n cells or, equivalently, the number of involutions in the symmetric group $\mathfrak{S}_n$. Let $N(n,\alpha)$ be the number of Young tableaux with

n cells that contain a fixed standard Young tableau of shape $\alpha \vdash k$; that is $N(n,\alpha) = \sum_{\lambda \vdash n} f(\lambda/\alpha)$. The result states that

$$N(n,\alpha) \sim \frac{N(n)f(\alpha)}{k!}.$$

See [22] for details.

5.2. Semi-standard tableaux

A semi-standard tableau of shape λ and distribution μ is a filling of a Ferrers diagram of shape λ with positive integers such that the number of occurrences of an integer i in λ is given by the part μ_i of μ and the numbers in the diagram are strictly increasing from bottom to top and weakly increasing from left to right. We show in Fig. 4 a semi-standard tableau of shape $\lambda = (5,4,2,1)$ and distribution $\mu = (3,3,2,2,2)$.

The numbers $K(\lambda,\mu)$ of semi-standard tableaux of shape λ and distribution μ are known as the Kostka numbers. In particular $K(\lambda, 1^n)$ is the number $f(\lambda)$ of standard tableaux of shape λ and the following relation between Kostka numbers and the numbers of skew tableaux is a straightforward consequence of their definition: for any partition α,

$$K(\lambda, (\alpha, 1^{n-|\alpha|})) = \sum_{\gamma \vdash |\alpha|} K(\gamma,\alpha) f(\lambda/\gamma).$$

Therefore, the normalized Kostka numbers $\frac{(n)_{|\alpha|}}{f(\lambda)} K(\lambda, \alpha 1^{n-|\alpha|})$ are obtained by the evaluation of linear combinations of power-content functions. For example, we have

$$\frac{(n)_4}{f(\lambda)} K(\lambda, 2^2 1^{n-4}) = -3pc_2 + pc_{1^2} + \binom{n-2}{2} pc_1 + \frac{n(n-1)(n^2-5n+10)}{4}.$$

Here is a direct consequence of Proposition 5.1. Let k be a fixed integer. Let λ be a partition of n with $\lambda_1 \leqslant A\sqrt{n}$ and $l(\lambda) \leqslant A\sqrt{n}$ for some constant A.

5				
4	5			
2	2	3	4	
1	1	1	2	3

Fig. 4. A semi-standard tableau.

Table 5
Polynomial expansion of $\frac{(n)_k}{f^\lambda} K_{\lambda,\mu}$ in $\lambda_1, \lambda_2 \ldots, \lambda_m$

$\frac{(n)_4}{f^\lambda} K(\lambda, 2^2 1^{n-4})$	$=$	$-3pc_2 + pc_{1^2} + 2\binom{n-2}{2}pc_1 + 6\binom{n}{4} + 2\binom{n}{2}$
$\frac{(n)_5}{f^\lambda} K(\lambda, 321^{n-5})$	$=$	$-4pc_3 + pc_1pc_2 + (\binom{n}{2} - 6n + 18)pc_2 + (n-4)pc_1^2$
		$+(4\binom{n}{3} - 9\binom{n}{2} + 18n - 24)pc_1 + 10\binom{n}{5} - 6\binom{n}{4} + 9\binom{n}{3} - 5\binom{n}{2}$
$\frac{(n)_6}{f^\lambda} K(\lambda, 421^{n-6})$	$=$	$-5pc_4 + pc_1pc_3 + (\binom{n}{2} - 8n + 40)pc_3 + \frac{1}{2}pc_1^3$
		$+(3\binom{n}{3} - \frac{27}{2}\binom{n}{2} + 48n - 110)pc_2 + (\frac{3}{2}\binom{n}{2} - 8n + 26)pc_1^2$
		$+(7\binom{n}{4} - 23\binom{n}{3} + 54\binom{n}{2} - 96n + 120)pc_1$
		$+15\binom{n}{6} - 30\binom{n}{5} + 36\binom{n}{4} - 43\binom{n}{3} + 20\binom{n}{2}$
$\frac{(n)_6}{f^\lambda} K(\lambda, 3^2 1^{n-6})$	$=$	$-5pc_4 + pc_{2^2} - 8(n-5)pc_3 + (2n-10)pc_{21}$
		$+(2\binom{n}{2} - 8n + 26)pc_{1^2} - (2\binom{n}{3} - 14\binom{n}{2} + 48n - 110)pc_2$
		$+(8\binom{n}{4} - 22\binom{n}{3} + 54\binom{n}{2} - 96n + 120)pc_1 + 20\binom{n}{6}$
		$-20\binom{n}{5} + 42\binom{n}{4} - 42\binom{n}{3} + 20\binom{n}{2}$
$\frac{(n)_6}{f^\lambda} K(\lambda, 2^3 1^{n-6})$	$=$	$20pc_3 - 9\sigma_1\sigma_2 + pc_{1^3} - 9\binom{n-4}{2}pc_2 + 3\binom{n-4}{2}pc_{1^2}$
		$+(18\binom{n}{4} - 36\binom{n}{3} + 60\binom{n}{2} - 96n + 120)pc_1$
		$+90\binom{n}{6} + 36\binom{n}{4} - 36\binom{n}{3} + 18\binom{n}{2}$

Table 6
Generators and relations of $\mathfrak{S}_n$ and $H_n(q_1, q_2)$

$\mathfrak{S}_n$	$\mathbf{H}_n(q_1, q_2)$
set of generators : $\{t_1, t_2, \ldots, t_{n-1}\}$	set of generators : $\{g_1, g_2, \ldots, g_{n-1}\}$
1. $t_i^2 = 1$ *or* $(t_i - 1)(t_i + 1) = 0$	1. $(g_i - q_1)(g_i - q_2) = 0$
2. $t_it_j = t_jt_i, \quad \lvert j-i \rvert > 1$	2. $g_ig_j = g_jg_i, \quad \lvert j-i \rvert > 1$
3. $t_it_{i+1}t_i = t_{i+1}t_it_{i+1}, \ \lvert j-i \rvert > 1$	3. $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, \ \lvert j-i \rvert > 1$

Proposition 5.2.

$$\frac{K(\lambda, \alpha 1^{n-k})}{f(\lambda)} = \frac{1}{k!} \sum_{\gamma \vdash k} K(\gamma, \alpha) f(\gamma) + o(1),$$

as $n \to \infty$.

In Table 5, we present a list of these polynomials.

6. Extension to Hecke algebras

The Hecke algebra of a reflection group W is a q-deformation of the group algebra $\mathbb{Q}[W]$ of W. The Hecke algebra $H_n(q_1, q_2)$ of $\mathfrak{S}_n$ is obtained by modifying one of the Coxeter relations defining $\mathfrak{S}_n$ (see Table 6).

The algebra $H_n(q_1, q_2)$ is semi-simple if and only if $q_1 \neq 0$ and $q_2 \neq 0$ and its irreducible representations are indexed with partitions of n. The dimension of an irreducible representation of $H_n(q_1, q_2)$ indexed by λ is equal to $f(\lambda)$ as in $\mathfrak{S}_n$. The characters $\chi_\mu^\lambda(q_1, q_2)$ of $H_n(q_1, q_2)$ form a square matrix of polynomials in q_1, q_2 and $H_n(1, -1) = \mathbb{Q}[\mathfrak{S}_n]$. There is a recursive rule for the evaluation of the irreducible characters $\chi^\lambda_\mu(q_1, q_2)$ (see [21]) similar to the Murnaghan–Nakayama rule with the exception that the border strips of length μ_i removed from λ are possibly disconnected.

More precisely, let $\beta \subset \lambda$ be a sub-diagram of λ such that λ/β is a *border strip* of λ i.e. a subset lying on the northeast most part of λ that does not contain any 2×2 block of boxes. Let $CC(\lambda/\lambda')$ be the set of connected components of a border strip λ/λ' and $cc(\lambda/\lambda') = card(CC(\lambda/\lambda'))$. Then the q-Murnaghan–Nakayama recursive rule states

$$\chi^\lambda_\mu(q_1, q_2) = \sum_{\beta \vdash n-\mu_i} \left((q_1 + q_2)^{cc(\lambda-\beta)-1} \prod_{x \in CC(\lambda-\beta)} q_1^{r(x)} q_2^{c(x)} \right) \chi^\beta_{\mu-\mu_i}(q_1, q_2),$$

where the sum is taken over all sub-diagrams β such that λ/β is a border strip of length μ_i possibly disconnected and $r(x), c(x)$ are the numbers of rows and columns in a connected border strip x.

We want to evaluate irreducible characters of the form $\chi^\lambda_{\mu 1^{n-|\mu|}}(q_1, q_2)$ with $\mu \vdash k \leqslant n$. Applying the recursive rule, we see that the contribution of the removal of the $1's$ from λ is one. This implies an identity similar to (16):

$$\chi^\lambda_{(\mu, 1^{n-|\mu|})}(q_1, q_2) = \sum_{\alpha \vdash |\mu|} \chi^\alpha_\mu(q_1, q_2) f(\lambda/\alpha) \tag{17}$$

and normalizing again the irreducible characters we have

$$\frac{(n)_{|\mu|}}{f(\lambda)} \chi^\lambda_{(\mu, 1^{n-|\mu|})}(q_1, q_2) = \sum_{\alpha \vdash |\mu|} \chi^\alpha_\mu(q_1, q_2) \frac{(n)_{|\mu|}}{f(\lambda)} f(\lambda/\alpha). \tag{18}$$

This tells us that there also exists symmetric functions with coefficients in $\mathbb{Q}[q_1, q_2, n]$ which give the value of irreducible characters of Hecke algebras. We also call these symmetric functions *class symmetric functions* of $H_n(q_1, q_2)$.

Theorem 6.1. *For each partition μ and its corresponding reduced partition $\tilde{\mu}$, there exists a symmetric function $\omega_{\tilde{\mu},q}(\mathbf{x})$ with coefficients in $\mathbb{Q}[q_1, q_2, n]$ such that*

$$\frac{(n)_{|\mu|}}{f(\lambda)} \chi^\lambda_{(\mu, 1^{n-|\mu|})}(q_1, q_2) = \omega_{\tilde{\mu},q}(C(\lambda)). \tag{19}$$

We also get

Lemma 6.1. *The class symmetric functions $\omega_{\mu,q}(\mathbf{x})$ for all $\lambda \vdash n$ have an expansion in the power sum basis*:

$$\omega_{\mu,q}(\mathbf{x}) = \sum_{\nu' \leqslant \mu'} \Omega^\nu_\mu(n, q_1, q_2)\, p_\nu(\mathbf{x}),$$

where the $\Omega^\nu_\mu(n, q_1, q_2)$ are polynomials in n, q_1 and q_2.

Table 7
Class polynomials $\omega_{\mu,q}(\mathbf{x})$ in $H_n(q_1, q_2)$

$$
\begin{aligned}
\omega_{1,q} &= (q_1-q_2)p_1+\tbinom{n}{2}(q_1+q_2)\\
\omega_{2,q} &= (q_1^2-q_1q_2+q_2^2)p_2+(n-2)(q_1^2-q_2^2)p_1+\tbinom{n}{3}(q_1+q_2)^2-\tbinom{n}{2}(q_1^2-q_1q_2+q_2^2)\\
\omega_{1^2,q} &= (q_1-q_2)^2(p_1^2-3p_2)+2\tbinom{n-2}{2}(q_1^2-q_2^2)p_1+6\tbinom{n}{4}(q_1+q_2)^2+2\tbinom{n}{2}(q_1-q_2)^2\\
\omega_{3,q} &= (q_1^3-q_1^2q_2+q_1q_2^2-q_2^3)p_3+((n-3)(q_1^3+q_2^3)-\tfrac{3}{2}(q_1-q_2)^2(q_1+q_2))p_2\\
&\quad+\tfrac{1}{2}(q_1-q_2)^2(q_1+q_2)p_1^2+(\tbinom{n-2}{2}(q_1+q_2)(q_1^2-q_2^2)-(2n-3)(q_1-q_2)(q_1^2+q_2^2))p_1\\
&\quad+\tbinom{n}{4}(q_1+q_2)^3-\tbinom{n}{2}[(n-4)(q_1^3+q_2^3)+q_1^2q_2+q_1q_2^2]\\
\omega_{21,q} &= (q_1-q_2)(q_1^2-q_1q_2+q_2^2)(p_{21}-4p_3)\\
&\quad+(\tbinom{n-3}{2}(q_1^3+q_2^3)-3(n-4)(q_1+q_2)(q_1-q_2)^2)p_2+(n-4)(q_1+q_2)(q_1-q_2)^2p_{1^2}\\
&\quad+(q_1-q_2)(4\tbinom{n-2}{3}(q_1+q_2)^2-(\tbinom{n}{2}-6n+8)(q_1^2-q_1q_2+q_2^2))p_1\\
&\quad+10\tbinom{n}{5}(q_1+q_2)^3+2\tbinom{n}{2}(n-4)(q_1+q_2)(q_1-q_2)^2-\tbinom{n}{2}\tbinom{n-3}{2}(q_1^3+q_2^3)\\
\omega_{4,q} &= (q_1^4-q_1^3q_2+q_1^2q_2^2-q_1q_2^3+q_2^4)(p_4-2p_{1^2}-2p_2)+(q_1-q_2)(q_1^3+q_2^3)p_{12}\\
&\quad+[(n-4)(q_1^4-q_2^4)-4(q_1-q_2)(q_1^3+q_2^3)]p_3+[\tfrac{1}{2}(n-4)(q_1^2-q_2^2)^2]p_{1^2}\\
&\quad+[\tbinom{n-3}{2}(q_1+q_2)(q_1^3+q_2^3)-(n-4)(\tfrac{3}{2}(q_1^4+q_2^4)+3(q_1-q_2)(q_1^3-q_2^3)]p_2\\
&\quad+[(q_1-q_2)(\tbinom{n-2}{3}(q_1+q_2)^3-(\tbinom{n}{2}-6n+8)(q_1^3+q_2^3))-(n-4)(2n-3)(q_1^4-q_2^4)]p_1\\
&\quad+\tbinom{n}{5}(q_1+q_2)^4+\tbinom{n}{2}\left(\frac{(5n-19)(q_1^5+q_2^5)}{3(q_1+q_2)}+(n-4)(q_1^2-q_2^2)^2-\tbinom{n-3}{2}(q_1+q_2)(q_1^3+q_2^3)\right)
\end{aligned}
$$

Note that the normalization used here is slightly different than the one used in Section 3. We present in Table 7, some class polynomials $\omega_{\mu,q}$ of $H_n(q_1, q_2)$.

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