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Effect of cross-diffusion on the stationary problem of a prey–predator model with a protection zone ☆

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ABSTRACT

This paper is concerned with the stationary problem of a prey–predator cross-diffusion system with a protection zone for the prey. We discuss the existence and non-existence of coexistence states of the two species by using the bifurcation theory. As a result, it is shown that the cross-diffusion for the prey has beneficial effects on the survival of the prey when the intrinsic growth rate of the predator is positive. We also study the asymptotic behavior of positive stationary solutions as the cross-diffusion coefficient of the prey tends to infinity.

1. Introduction

In ecosystems, whether different species can coexist or not is determined by the combination of various factors, such as natural environments, interactions between the species, and behavioral patterns. Therefore, it is important to investigate what effect the above factors will have on coexistence problems. From this viewpoint, we study the following Lotka–Volterra prey–predator model:

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$$\text{(P)}\quad \begin{cases} u_t = \Delta\big[\big(1+k\rho(x)v\big)u\big] + u\big(\lambda - u - b(x)v\big), & (x,t)\in\Omega\times(0,\infty),\\ v_t = \Delta v + v(\mu + cu - v), & (x,t)\in\Omega\setminus\overline{\Omega}_0\times(0,\infty),\\ \partial_n u = 0, & (x,t)\in\partial\Omega\times(0,\infty),\\ \partial_n v = 0, & (x,t)\in\partial(\Omega\setminus\overline{\Omega}_0)\times(0,\infty),\\ u(x,0) = u_0(x)\geqslant 0, & x\in\Omega,\\ v(x,0) = v_0(x)\geqslant 0, & x\in\Omega\setminus\overline{\Omega}_0. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ ($N \leqslant 3$) with smooth boundary $\partial\Omega$ and Ω_0 is a subdomain of Ω with smooth boundary $\partial\Omega_0$; n is the outward unit normal vector on the boundary and $\partial_n = \partial/\partial n$; $k \geqslant 0$, $\lambda > 0$, $c > 0$ and $\mu \in \mathbb{R}$ are all constants; $\rho > 0$ and $b > 0$ in $\Omega \setminus \Omega_0$, whereas $\rho = b = 0$ in Ω_0 because v is not defined in Ω_0. Furthermore, we make the following assumption for technical reasons: if $N = 2$ or 3, then $\overline{\Omega}_0 \subset \Omega$; if $N = 1$ and $\Omega = (a_1, a_2)$ for $a_1 < a_2$, then $\Omega_0 = (a_1, a)$ or $\Omega_0 = (a, a_2)$ for some $a \in (a_1, a_2)$. In (P), unknown functions $u(x,t)$ and $v(x,t)$ denote the population densities of prey and predator respectively; λ and μ denote the intrinsic growth rates of the respective species; $b(x)$ and c denote the coefficients of prey–predator interaction; the zero-flux boundary condition means that no individuals cross the boundary.

In the first equation of (P), $k\Delta[\rho(x)vu]$ is usually called a cross-diffusion term which was originally proposed by Shigesada et al. [20] to model the habitat segregation phenomena between two competing species. The cross-diffusion $k\Delta[\rho(x)vu]$ means that the movement of the prey species is affected by population pressure from the predator species. Then the cross-diffusion coefficient k represents the sensitivity of the prey species to population pressure from the predator species. See [1,2, 12,17,21] and references therein for studies on the time-global solvability of cross-diffusion systems.

In (P), the predator species cannot enter the subregion Ω_0 of the habitat Ω, whereas the prey species can enter and leave Ω_0 freely. Namely, Ω_0 is a predation-free zone for the prey species and such a subregion Ω_0 is called a protection zone. One can think that there is a barrier along $\partial\Omega_0$ that blocks the predator but not the prey (see [4–6] for further details). In the case where cross-diffusion is absent, Du et al. [4–6] have studied the effects of a protection zone on Lotka–Volterra competition model [4], Leslie prey–predator model [5], and Holling type II prey–predator model [6] respectively. They have proved that if the size of the protection zone is larger than a certain critical patch size, which is common to three models, then a fundamental change occurs in the dynamical behavior of each of three models.

In this paper, we study the effects of cross-diffusion on the set of positive stationary solutions of (P). Let $\Omega_1 := \Omega \setminus \overline{\Omega}_0$. The stationary problem associated with (P) is given by

$$\text{(SP)}\quad \begin{cases} \Delta\big[\big(1+k\rho(x)v\big)u\big] + u\big(\lambda - u - b(x)v\big) = 0 & \text{in } \Omega,\\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega_1,\\ \partial_n u = 0 & \text{on } \partial\Omega,\\ \partial_n v = 0 & \text{on } \partial\Omega_1. \end{cases}$$

When $\Omega_0 = \emptyset$, there are some studies on prey–predator models with cross-diffusion analogous to (SP) (see e.g. [8–11,18,22]). From now on, we always assume that

$$\rho(x) = \chi_{\Omega\setminus\Omega_0}(x) := \begin{cases} 1 & \text{if } x\in\Omega\setminus\Omega_0,\\ 0 & \text{if } x\in\Omega_0, \end{cases} \quad \text{and} \quad b(x) = \begin{cases} \beta & \text{if } x\in\Omega\setminus\Omega_0,\\ 0 & \text{if } x\in\Omega_0, \end{cases} \tag{1.1}$$

where β is a positive constant.

Our first goal is to understand the effects of cross-diffusion on the existence and non-existence of positive solutions of (SP). From an ecological viewpoint, a positive solution of (SP) means a coexistence state of prey and predator. When $\Omega_0 = \emptyset$, it is known that for any $k \geqslant 0$, (SP) has no positive solution if $\lambda \leqslant \beta\mu$ (the proof is essentially the same as that of Lemma 3.3 appearing in Section 3). This, together with the fact that the semitrivial solution $(\lambda, 0)$ is linearly unstable for any $\mu > 0$, implies that when no protection zone is present, the prey species cannot survive if the intrinsic growth rate of the predator is relatively high compared with the intrinsic growth rate of the prey. On the other hand, in the case where a protection zone is present, we will show that there exists a certain threshold prey growth rate for survival, denoted by $\lambda_\infty^*(k, \Omega_0)$. To be more specific, we will show that if $\lambda < \lambda_\infty^*(k, \Omega_0)$, then (SP) has no positive solution for large μ, whereas if $\lambda \geqslant \lambda_\infty^*(k, \Omega_0)$, then (SP) has at least one positive solution for any $\mu > 0$. Here, it is noted that in the absence of cross-diffusion, $\lambda_\infty^*(0, \Omega_0)$ is given by $\lambda_1^D(\Omega_0)$, where $\lambda_1^D(\Omega_0)$ is the first eigenvalue of $-\Delta$ over Ω_0 with the homogeneous Dirichlet boundary condition (the boundary condition should be replaced by $\phi(a) = \phi'(a_i) = 0$ for $i = 1$ or 2 if $N = 1$, but we use the same symbol $\lambda_1^D(\Omega_0)$; and the fact $\lambda_\infty^*(0, \Omega_0) = \lambda_1^D(\Omega_0)$ follows from Theorem 2.1 and related results in [6] since their proofs are still valid when $m = 0$ there). Then it is interesting to examine the dependence of the threshold prey growth rate $\lambda_\infty^*(k, \Omega_0)$ on k. We will show that $\lambda_\infty^*(k, \Omega_0)$ decreases as k increases and satisfies $\lim_{k\to\infty} \lambda_\infty^*(k, \Omega_0) = 0$. Namely, in the limiting case where $k \to \infty$, the prey species can coexist with the predator species regardless of the values of $\lambda > 0$ and $\mu > 0$. Moreover, we will also show that for any fixed $k > 0$, $\lambda_\infty^*(k, \Omega_0)$ converges to zero as Ω_0 is enlarged to the entire Ω. This is a strong contrast to the no cross-diffusion case, where the threshold prey growth rate $\lambda_1^D(\Omega_0)$ satisfies $\lambda_1^D(\Omega_0) \geqslant \lambda_1^D(\Omega) > 0$ for any $\Omega_0 \subset \Omega$. Therefore, we can say that the cross-diffusion for the prey has beneficial effects on the survival of the prey species when a protection zone is present.

Our second goal is to understand the asymptotic behavior of positive solutions of (SP) as $k \to \infty$. When $\mu \geqslant 0$, for any positive solution (u_k, v_k) of (SP), we will show that as $k \to \infty$, u_k converges to λ in Ω_0 and converges to 0 in $\overline{\Omega}_1$, while v_k converges to μ in $\overline{\Omega}_1$. This convergence result means that when $\mu \geqslant 0$, the prey species concentrates in the protection zone as $k \to \infty$ and when $\mu > 0$ in particular, the two species become spatially segregated as $k \to \infty$. On the other hand, when $\mu < 0$, we will show that along a sequence $k_i \to \infty$, $(u_{k_i}, k_i v_{k_i})$ converges to a pair of positive functions which satisfies a certain limiting system. We will also analyze the bifurcation structure of positive solutions of the limiting system.

It is expected that for large k, positive steady-states with the habitat segregation property mentioned above are stable. But the stability problem is difficult due to the presence of cross-diffusion and it remains open.

Let O be any bounded domain in $\mathbb{R}^N$ with smooth boundary. We denote the usual norm of $L^p(O)$ for $p \in [1, \infty)$ by

$$\|\psi\|_{p,O} = \left(\int_O |\psi(x)|^p \, dx \right)^{1/p}.$$

Furthermore, for $q \in L^\infty(O)$, we denote by $\lambda_1^N(q, O)$ the first eigenvalue of $-\Delta + q$ over O with the homogeneous Neumann boundary condition. We will omit O in the notation if there is no ambiguity. As is well known, the following properties hold:

(i) the mapping $q \mapsto \lambda_1^N(q, O) : L^\infty(O) \to \mathbb{R}$ is continuous,
(ii) $\lambda_1^N(0, O) = 0$,
(iii) if $q_1 \geqslant q_2$ and $q_1 \not\equiv q_2$, then $\lambda_1^N(q_1, O) > \lambda_1^N(q_2, O)$.

This paper is organized as follows. In Section 2, we will state the main results of this paper. In Section 3, we will prove Lemma 2.1 stated in Section 2. Moreover, we will show some non-existence result and a priori estimates of positive solutions. In Section 4, we will obtain positive solutions from the viewpoint of the local bifurcation theory. In Section 5, we will accomplish the proof of our main results.

2. Main results

We introduce a new unknown function U by

$$U = \big(1 + k\rho(x)v\big)u. \tag{2.1}$$

Then (SP) is rewritten in the following form:

$$\text{(EP)}\quad \begin{cases} \Delta U + f_1(\lambda, U, v) = 0 & \text{in } \Omega, \\ \Delta v + f_2(U, v) = 0 & \text{in } \Omega_1, \\ \partial_n U = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega_1, \end{cases}$$

where

$$\begin{cases} f_1(\lambda, U, v) = \dfrac{U}{1 + k\rho(x)v}\left(\lambda - \dfrac{U}{1 + k\rho(x)v} - b(x)v\right), \\ f_2(U, v) = v\left(\mu + \dfrac{cU}{1 + kv} - v\right). \end{cases} \tag{2.2}$$

Define

$$E = C_n^1(\overline{\Omega}) \times C_n^1(\overline{\Omega}_1), \tag{2.3}$$

where $C_n^1(\overline{O}) = \{w \in C^1(\overline{O})\colon\ \partial_n w = 0 \text{ on } \partial O\}$. We say that (u, v) is a positive solution of (SP) if $(U, v) \in E$ is a positive solution of (EP) and u is defined by (2.1). It is shown by elliptic regularity theory (see e.g. [7]) that $(U, v) \in C^{1,\theta}(\overline{\Omega}) \times C^2(\overline{\Omega}_1)$ for any $\theta \in (0, 1)$ if $(U, v) \in E$ is a positive solution of (EP). Before stating our main results, we state the following lemma.

Lemma 2.1. *For any fixed k and Ω_0, there exists a continuous and strictly increasing function $\lambda^*(\mu)$ with respect to $\mu \geqslant 0$ such that $\lambda^*(0) = 0$, $\lambda^*(\mu) < \beta\mu$ for any $\mu > 0$, $\lim_{\mu\to\infty} \lambda^*(\mu) \leqslant \lambda_1^D(\Omega_0)$ and*

$$\left\{(\lambda, \mu) \in [0, \infty)^2\colon\ \lambda_1^N\left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu}, \Omega\right) = 0\right\} = \big\{\big(\lambda^*(\mu), \mu\big)\colon\ \mu \geqslant 0\big\}.$$

Our first result is the following theorem concerning the existence of coexistence states of (SP) with fixed k and Ω_0.

Theorem 2.2. *The following results hold true*:

(i) *Suppose that $\mu \geqslant 0$. Then* (SP) *has at least one positive solution if and only if $\lambda > \lambda^*(\mu)$.*
(ii) *Suppose that $\mu < 0$. Then* (SP) *has at least one positive solution if $\lambda > -\mu/c$.*

From Theorem 2.2, we can draw the coexistence region of (SP) in the $\lambda\mu$-plane (see Fig. 1).

Our next concern is to examine the dependence of the coexistence region on k and Ω_0. We write $\lambda^*(\mu, k, \Omega_0)$ instead of $\lambda^*(\mu)$ to express the dependence on k and Ω_0 explicitly in Theorem 2.3 below. Moreover, we define $\lambda^*_\infty(k, \Omega_0) := \lim_{\mu\to\infty} \lambda^*(\mu, k, \Omega_0) \leqslant \lambda_1^D(\Omega_0)$. Then $\lambda^*_\infty(k, \Omega_0)$ is the threshold prey growth rate in the sense stated in Section 1. We can obtain the following theorem.

Theorem 2.3. *The following results hold true*:

(i) *Suppose that $\mu > 0$. Then $\lambda^*(\mu, k, \Omega_0)$ is strictly decreasing with respect to k.*
(ii) *Let $S = \{\phi \in H^1(\Omega)\colon\ \int_{\Omega_0} \phi^2\,dx > 0\}$. For any $k > 0$,*

$$\lambda^*_\infty(k, \Omega_0) = \inf_{\phi\in S} \frac{\int_\Omega |\nabla\phi|^2\,dx + \frac{\beta}{k}\int_{\Omega\setminus\Omega_0} \phi^2\,dx}{\int_{\Omega_0} \phi^2\,dx} \leqslant \frac{\beta|\Omega \setminus \Omega_0|}{k|\Omega_0|}.$$

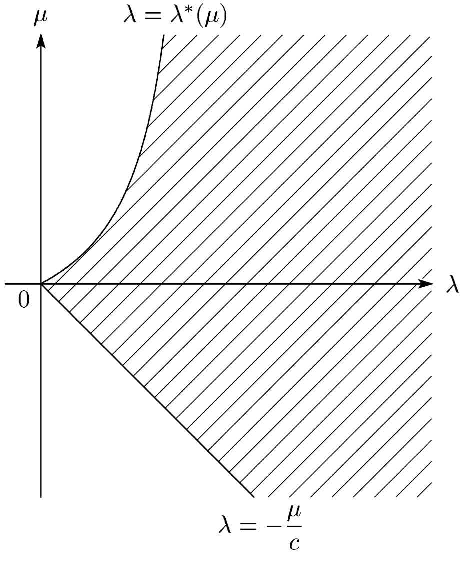

Fig. 1. Coexistence region of (SP) with fixed k and Ω_0.

Part (i) of Theorem 2.3 means that the coexistence region becomes larger as k increases, and part (ii) of Theorem 2.3 means that the threshold prey growth rate $\lambda^*_\infty(k, \Omega_0)$ decreases to 0 as $k \to \infty$ or Ω_0 is enlarged to the entire Ω.

Concerning the asymptotic behavior of positive solutions of (SP) as $k \to \infty$, the following theorem holds.

Theorem 2.4. *Let* (u_k, v_k) *be any positive solution of* (SP) *for each k.*

(i) *Suppose that* $\mu \geqslant 0$. *Then*

$$\lim_{k\to\infty} (u_k, u_k, v_k) = (\lambda, 0, \mu) \quad \text{in } C^1(\Omega_0) \times C^1(\overline{\Omega}_1) \times C^1(\overline{\Omega}_1).$$

Moreover, $\lim_{k\to\infty} kv_k = \infty$ *uniformly in* $\overline{\Omega}_1$ *even when* $\mu = 0$.

(ii) *Suppose that* $\lambda > -\mu/c > 0$ *and let* $\{k_i\}_{i=1}^\infty$ *be any sequence with* $\lim_{i\to\infty} k_i = \infty$. *Then, by passing to a subsequence if necessary,*

$$\lim_{i\to\infty} u_{k_i} = \bar{u} \quad \text{uniformly in } \overline{\Omega}, \qquad \lim_{i\to\infty} (v_{k_i}, k_i v_{k_i}) = (0, \bar{w}) \quad \text{in } C^1(\overline{\Omega}_1)^2,$$

where $(\bar{u}, \bar{w})$ *is a positive solution of*

$$\begin{cases} \Delta\big[\big(1+\rho(x)\bar{w}\big)\bar{u}\big] + \bar{u}(\lambda - \bar{u}) = 0 & \text{in } \Omega, \\ \Delta\bar{w} + \bar{w}(\mu + c\bar{u}) = 0 & \text{in } \Omega_1, \\ \partial_n \bar{u} = 0 & \text{on } \partial\Omega, \\ \partial_n \bar{w} = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{2.4}$$

We can analyze the bifurcation structure of positive solutions of the limiting system (2.4).

Theorem 2.5. *The set of positive solutions of* (2.4) *with bifurcation parameter* μ *contains an unbounded connected set* Γ *in* $\mathbb{R} \times L^\infty(\Omega) \times C^1(\overline{\Omega}_1)$ *satisfying the following properties*:

(i) *Γ bifurcates from $\{(\mu, \bar{u}, \bar{w}) = (\mu, \lambda, 0)$: $\mu \in \mathbb{R}\}$ at $\mu = -c\lambda$,*
(ii) *$(-c\lambda, 0) \subset \{\mu$: $(\mu, \bar{u}, \bar{w}) \in \Gamma\} \subset (\tilde{\mu}, 0)$ for some $\tilde{\mu} \in (-\infty, -c\lambda]$,*
(iii) *$\lim_{\mu\to 0} \bar{u}_\mu = \lambda$ in $C^1(\Omega_0)$ and $\lim_{\mu\to 0}(\bar{u}_\mu, \bar{w}_\mu) = (0, \infty)$ uniformly in $\overline{\Omega}_1$, where $(\mu, \bar{u}_\mu, \bar{w}_\mu) \in \Gamma$.*

Although $\bar{u} \notin C^1(\overline{\Omega})$ for $(\mu, \bar{u}, \bar{w}) \in \Gamma$ in Theorem 2.5, we can verify from Proposition 5.5 in Section 5 that $\bar{u}|_{\Omega_0} \in C^1(\Omega_0)$ and $\bar{u}|_{\overline{\Omega}_1} \in C^1(\overline{\Omega}_1)$. Finally, we remark that (iii) of Theorem 2.5 is compatible with (i) of Theorem 2.4.

3. Preliminaries and a priori estimates

In this section, we will prove Lemma 2.1. Moreover, we will derive some non-existence result and a priori estimates of positive solutions.

We first recall the following maximum principle (see Lou and Ni [15]) and Harnack inequality (see Lin et al. [13] and Lou and Ni [16]).

Lemma 3.1. *Suppose that $g \in C(\overline{O} \times \mathbb{R})$, where O is a bounded domain in $\mathbb{R}^N$ with smooth boundary. If $w \in C^2(O) \cap C^1(\overline{O})$ satisfies*

$$\Delta w(x) + g\big(x, w(x)\big) \leqslant 0 \quad \text{in } O, \qquad \partial_n w \geqslant 0 \quad \text{on } \partial O,$$

and $w(x_0) = \min_{\overline{O}} w$, then $g(x_0, w(x_0)) \leqslant 0$.

Lemma 3.2. *Let $f \in L^p(O)$ with $p > \max\{N/2, 1\}$, where O is a bounded domain in $\mathbb{R}^N$ with smooth boundary, and let w be a non-negative solution of $\Delta w + f(x)w = 0$ in O subject to the homogeneous Neumann boundary condition. Then there exists a positive constant $C_\# = C_\#(p, N, O, \|f\|_{p,O})$ such that*

$$\max_{\overline{O}} w \leqslant C_\# \min_{\overline{O}} w.$$

We will prove Lemma 2.1.

Proof of Lemma 2.1. Suppose that $\mu > 0$. By the continuity and monotone increasing property of $\lambda_1^N(q)$ with respect to $q \in L^\infty(\Omega)$ and the fact $\lambda_1^N(0) = 0$, we see that

$$\lambda \mapsto \lambda_1^N\left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu}\right) : [0, \infty) \to \mathbb{R}$$

is a continuous and strictly decreasing function satisfying

$$\lambda_1^N\left(\frac{b(x)\mu}{1 + k\rho(x)\mu}\right) > 0 \quad \text{and} \quad \lambda_1^N\left(\frac{b(x)\mu - \beta\mu}{1 + k\rho(x)\mu}\right) < 0,$$

where we have used the assumption (1.1) about $b(x)$. It thus follows from the intermediate value theorem that there exists a unique $\lambda^*(\mu) \in (0, \beta\mu)$ such that

$$\lambda_1^N\left(\frac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu}\right) = 0.$$

Since

$$\mu \mapsto \lambda_1^N\left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu}\right) : [0, \infty) \to \mathbb{R}$$

is continuous and strictly increasing, we can verify that $\lambda^*(\mu)$ is continuous and strictly increasing with respect to $\mu \geqslant 0$ and satisfies $\lambda^*(0) = 0$.

We finally show $\lim_{\mu\to\infty} \lambda^*(\mu) \leqslant \lambda_1^D(\Omega_0)$ by the argument used in the proof of Theorem 2.1 in [6]. From the variational characterization of the first eigenvalue, we have

$$0 = \lambda_1^N\left(\frac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu}\right) = \inf_{\phi\in\Theta} \int_\Omega \left(|\nabla\phi|^2 + \frac{b(x)\mu - \lambda^*(\mu)}{1 + k\rho(x)\mu}\phi^2\right) dx, \tag{3.1}$$

where $\Theta = \{\phi \in H^1(\Omega)\colon\ \|\phi\|_{2,\Omega} = 1\}$. Let ϕ_1 satisfy

$$-\Delta\phi_1 = \lambda_1^D(\Omega_0)\phi_1 \quad \text{in } \Omega_0, \qquad \phi_1 = 0 \quad \text{on } \partial\Omega_0, \qquad \int_{\Omega_0} \phi_1^2\, dx = 1,$$

where $\phi_1 = 0$ on $\partial\Omega_0$ should be replaced by $\phi_1(a) = \phi_1'(a_i) = 0$ for $i = 1$ or 2 if $N = 1$. Define $\tilde{\phi}_1 \in H^1(\Omega)$ as follows:

$$\tilde{\phi}_1 \equiv \phi_1 \quad \text{in } \Omega_0, \qquad \tilde{\phi}_1 \equiv 0 \quad \text{in } \Omega \setminus \Omega_0.$$

Setting $\phi = \tilde{\phi}_1$ in (3.1), we obtain

$$0 \leqslant \int_{\Omega_0} \left(|\nabla\phi_1|^2 - \lambda^*(\mu)\phi_1^2\right) dx = \lambda_1^D(\Omega_0) - \lambda^*(\mu)$$

for any $\mu \geqslant 0$. Therefore, $\lim_{\mu\to\infty} \lambda^*(\mu) \leqslant \lambda_1^D(\Omega_0)$. $\square$

We will derive the following non-existence result of positive solutions.

Lemma 3.3. *If $\mu \geqslant 0$ and $\lambda \leqslant \lambda^*(\mu)$, then* (EP) *has no positive solution.*

Proof. Let (U, v) be any positive solution of (EP) with $\mu \geqslant 0$ and define u by (2.1). Then U is a positive solution of

$$-\Delta U + \frac{-\lambda + u + b(x)v}{1 + k\rho(x)v} U = 0 \quad \text{in } \Omega, \qquad \partial_n U = 0 \quad \text{on } \partial\Omega.$$

In addition, by applying Lemma 3.1 to the second equation of (EP), we have

$$\mu + \frac{cU(x_0)}{1 + kv(x_0)} - \min_{\overline{\Omega}_1} v \leqslant 0,$$

where $v(x_0) = \min_{\overline{\Omega}_1} v$. Namely, $v > \mu$ in $\overline{\Omega}_1$. Hence we find that

$$0 = \lambda_1^N\left(\frac{-\lambda + u + b(x)v}{1 + k\rho(x)v}\right) > \lambda_1^N\left(\frac{b(x)v - \lambda}{1 + k\rho(x)v}\right) > \lambda_1^N\left(\frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu}\right).$$

On the other hand, it follows from Lemma 2.1 that

$$\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\rho(x)\mu}\right)\geqslant 0$$

for any $\lambda\leqslant\lambda^*(\mu)$. Therefore, (EP) has no positive solution if $\mu\geqslant 0$ and $\lambda\leqslant\lambda^*(\mu)$. $\square$

We will derive the following a priori estimates of positive solutions.

Lemma 3.4. *Let $\theta\in(0,1)$. Then there exists a positive constant C independent of k such that any positive solution (U,v) of* (EP) *satisfies*

$$\|U\|_{C^{1,\theta}(\overline{\Omega})}\leqslant C \quad \textit{and} \quad \|v\|_{C^{1,\theta}(\overline{\Omega}_1)}\leqslant C.$$

Proof. Let (U,v) be any positive solution of (EP) and define u by (2.1). Integrating the first equation of (EP) over Ω, we have

$$\int_\Omega u\big(\lambda-u-b(x)v\big)\,dx=0.$$

Then by the Schwarz inequality, we see that

$$\int_\Omega u^2\,dx\leqslant\lambda\int_\Omega u\,dx\leqslant\lambda|\Omega|^{1/2}\|u\|_{2,\Omega},$$

where $|\Omega|$ denotes the measure of Ω. Hence

$$\|u\|_{2,\Omega}\leqslant\lambda|\Omega|^{1/2} \tag{3.2}$$

and thus

$$|\Omega_0|^{1/2}\inf_{\Omega_0}u\leqslant\|u\|_{2,\Omega_0}\leqslant\lambda|\Omega|^{1/2}.$$

Therefore,

$$\inf_{\Omega_0}u\leqslant\lambda\left(\frac{|\Omega|}{|\Omega_0|}\right)^{1/2}. \tag{3.3}$$

Similarly, we have

$$\int_{\Omega_1}v^2\,dx=\mu\int_{\Omega_1}v\,dx+c\int_{\Omega_1}uv\,dx\leqslant\mu_+|\Omega_1|^{1/2}\|v\|_{2,\Omega_1}+c\|u\|_{2,\Omega_1}\|v\|_{2,\Omega_1},$$

where $\mu_+:=\max\{\mu,0\}$. It follows from (3.2) that

$$\|v\|_{2,\Omega_1}\leqslant\mu_+|\Omega_1|^{1/2}+c\|u\|_{2,\Omega_1}\leqslant\mu_+|\Omega_1|^{1/2}+c\lambda|\Omega|^{1/2}, \tag{3.4}$$

which, in particular, gives

$$\min_{\overline{\Omega}_1}v\leqslant\mu_++c\lambda\left(\frac{|\Omega|}{|\Omega_1|}\right)^{1/2}. \tag{3.5}$$

By (3.2), (3.4) and the assumption that the spatial dimension N satisfies $N \leqslant 3$, we can apply Lemma 3.2 with $p = 2$ to (EP). Consequently, we can find two positive constants C_1 and C_2 independent of k such that

$$\max_{\overline{\Omega}} U \leqslant C_1 \min_{\overline{\Omega}} U \leqslant C_1 \inf_{\Omega_0} u \leqslant C_1 \lambda \left(\frac{|\Omega|}{|\Omega_0|} \right)^{1/2}$$

and

$$\max_{\overline{\Omega}_1} v \leqslant C_2 \min_{\overline{\Omega}_1} v \leqslant C_2 \left(\mu_+ + c\lambda \left(\frac{|\Omega|}{|\Omega_1|} \right)^{1/2} \right),$$

where we have used (3.3) and (3.5). Therefore, we get the conclusion by elliptic regularity theory and the Sobolev embedding theorem. $\square$

4. Local bifurcation from semitrivial solutions

In this section, we regard λ as a bifurcation parameter. We will apply the local bifurcation theorem of Crandall and Rabinowitz [3] to (EP) in order to obtain a branch of positive solutions which bifurcates from the semitrivial solution curve

$$\Gamma_U = \{(\lambda, U, v) = (\lambda, \lambda, 0)\colon \lambda > 0\} \quad \text{or} \quad \Gamma_v = \{(\lambda, U, v) = (\lambda, 0, \mu)\colon \lambda > 0\}.$$

For $p > N$, we define

$$X_1 = W_n^{2,p}(\Omega) \times W_n^{2,p}(\Omega_1) \quad \text{and} \quad X_2 = L^p(\Omega) \times L^p(\Omega_1),$$

where $W_n^{2,p}(O) = \{w \in W^{2,p}(O)\colon \partial_n w = 0 \text{ on } \partial O\}$. We note that $X_1 \subset E$ by the Sobolev embedding theorem, where E is the Banach space defined by (2.3).

We first study the local bifurcation from Γ_v for any fixed $\mu > 0$. Let $\lambda^* = \lambda^*(\mu)$ be the positive number defined in Lemma 2.1 and let ϕ^* be a positive solution of

$$-\Delta\phi^* + \frac{b(x)\mu - \lambda^*}{1 + k\rho(x)\mu}\phi^* = 0 \quad \text{in } \Omega, \qquad \partial_n\phi^* = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

We also define

$$\psi^* = (-\Delta + \mu I)^{-1}_{\Omega_1}\left[\frac{c\mu}{1 + k\mu}\phi^*\right], \tag{4.2}$$

where I is the identity mapping and $(-\Delta + \mu I)^{-1}_{\Omega_1}$ is the inverse operator of $-\Delta + \mu I$ over Ω_1 subject to the homogeneous Neumann boundary condition. Then the following local bifurcation property holds true.

Proposition 4.1. *Assume that $\mu > 0$. Positive solutions of* (EP) *bifurcate from Γ_v if and only if $\lambda = \lambda^*$. Precisely, all positive solutions of* (EP) *near $(\lambda^*, 0, \mu) \in \mathbb{R} \times X_1$ can be expressed as*

$$\hat{\Gamma}_\delta = \{(\lambda, U, v) = (\lambda(s), s(\phi^* + U(s)), \mu + s(\psi^* + v(s)))\colon s \in (0, \delta)\}$$

for some $\delta > 0$. Here $(\lambda(s), U(s), v(s))$ is a smooth function with respect to s and satisfies $(\lambda(0), U(0), v(0)) = (\lambda^*, 0, 0)$ and $\int_\Omega U(s)\phi^*\,dx = 0$.

Proof. Let $V := v - \mu$ in (EP) and define a mapping $F : \mathbb{R} \times X_1 \to X_2$ by

$$F(\lambda, U, V) = \begin{pmatrix} \Delta U + f_1(\lambda, U, V + \mu) \\ \Delta V + f_2(U, V + \mu) \end{pmatrix},$$

where f_1 and f_2 are functions defined by (2.2). Then $F(\lambda, U, V) = 0$ if and only if $(U, V + \mu)$ is a solution of (EP). We note that $F(\lambda, 0, 0) = 0$ for any λ. By a simple calculation, the Fréchet derivative of F at $(U, V) = (0, 0)$ is given by

$$F_{(U,V)}(\lambda, 0, 0)[\phi, \psi] = \begin{pmatrix} \Delta\phi + \frac{\lambda - b(x)\mu}{1 + k\rho(x)\mu}\phi \\ \Delta\psi - \mu\psi + \frac{c\mu}{1 + k\mu}\phi \end{pmatrix}. \tag{4.3}$$

By Lemma 2.1 and the Krein–Rutman theorem, $F_{(U,V)}(\lambda, 0, 0)[\phi, \psi] = (0, 0)$ has a solution with $\phi > 0$ if and only if $\lambda = \lambda^*$; thus λ^* is the only possible bifurcation point where positive solutions of (EP) bifurcate from Γ_V. It follows from (4.1)–(4.3) that the kernel of $F_{(U,V)}(\lambda^*, 0, 0)$ is given by

$$\operatorname{Ker} F_{(U,V)}\big(\lambda^*, 0, 0\big) = \operatorname{span}\big\{\big(\phi^*, \psi^*\big)\big\}, \tag{4.4}$$

and hence $\dim \operatorname{Ker} F_{(U,V)}(\lambda^*, 0, 0) = 1$. By the Fredholm alternative theorem, the range of $F_{(U,V)}(\lambda^*, 0, 0)$ is given by

$$\operatorname{Range} F_{(U,V)}\big(\lambda^*, 0, 0\big) = \left\{(\phi, \psi) \in X_2 \colon \int\limits_{\Omega} \phi\phi^* \, dx = 0\right\}. \tag{4.5}$$

Thus it holds that $\operatorname{codim} \operatorname{Range} F_{(U,V)}(\lambda^*, 0, 0) = 1$. Moreover, we see from (4.5) that

$$F_{\lambda(U,V)}\big(\lambda^*, 0, 0\big)\big[\phi^*, \psi^*\big] = \begin{pmatrix} \frac{\phi^*}{1 + k\rho(x)\mu} \\ 0 \end{pmatrix} \notin \operatorname{Range} F_{(U,V)}\big(\lambda^*, 0, 0\big).$$

Consequently, we can apply the local bifurcation theorem [3] to F at $(\lambda^*, 0, 0)$. Therefore, the proof of Proposition 4.1 is complete. $\square$

Next we study the local bifurcation from Γ_U for any fixed $\mu < 0$. We define

$$\phi_* = \left(-\Delta + \frac{-\mu}{c} I\right)_{\Omega}^{-1} \left[-\frac{\mu}{c}\left(-\frac{k\rho(x)\mu}{c} - b(x)\right)\right].$$

Then the following local bifurcation property holds true.

Proposition 4.2. *Assume that $\mu < 0$. Positive solutions of* (EP) *bifurcate from Γ_U if and only if $\lambda = -\mu/c$. Precisely, all positive solutions of* (EP) *near $(-\mu/c, \lambda, 0) \in \mathbb{R} \times X_1$ can be expressed as*

$$\big\{(\lambda, U, v) = \big(\tilde{\lambda}(s), \lambda + s\big(\phi_* + \tilde{U}(s)\big), s\big(1 + \tilde{v}(s)\big)\big) \colon s \in (0, \tilde{\delta})\big\}$$

for some $\tilde{\delta} > 0$. Here $(\tilde{\lambda}(s), \tilde{U}(s), \tilde{v}(s))$ is a smooth function with respect to s and satisfies $(\tilde{\lambda}(0), \tilde{U}(0), \tilde{v}(0)) = (-\mu/c, 0, 0)$ and $\int_{\Omega_1} \tilde{v}(s) \, dx = 0$.

Proof. Let $z := U - \lambda$ in (EP) and define a mapping $\Phi : \mathbb{R} \times X_1 \to X_2$ by

$$\Phi(\lambda, z, v) = \begin{pmatrix} \Delta U + f_1(\lambda, z+\lambda, v) \\ \Delta v + f_2(z+\lambda, v) \end{pmatrix}.$$

Then $\Phi(\lambda, z, v) = 0$ if and only if $(z+\lambda, v)$ is a solution of (EP). We note that $\Phi(\lambda, 0, 0) = 0$ for any λ. By a simple calculation,

$$\Phi_{(z,v)}(\lambda, 0, 0)[\phi, \psi] = \begin{pmatrix} \Delta\phi - \lambda\phi + \lambda(k\rho(x)\lambda - b(x))\psi \\ \Delta\psi + (\mu + c\lambda)\psi \end{pmatrix}.$$

Thus we can easily verify that $\lambda = -\mu/c$ is the only possible bifurcation point where positive solutions of (EP) bifurcate from Γ_U. In addition, we see

$$\operatorname{Ker} \Phi_{(z,v)}(-\mu/c, 0, 0) = \operatorname{span}\{(\phi_*, 1)\}$$

and

$$\operatorname{Range} \Phi_{(z,v)}(-\mu/c, 0, 0) = \left\{(\phi, \psi) \in X_2 \colon \int_{\Omega_1} \psi \, dx = 0\right\}. \tag{4.6}$$

Hence

$$\dim \operatorname{Ker} \Phi_{(z,v)}(-\mu/c, 0, 0) = \operatorname{codim} \operatorname{Range} \Phi_{(z,v)}(-\mu/c, 0, 0) = 1.$$

Moreover, (4.6) yields

$$\Phi_{\lambda(z,v)}(-\mu/c, 0, 0)[\phi_*, 1] = \begin{pmatrix} -\phi_* - \frac{2k\rho(x)\mu}{c} - b(x) \\ c \end{pmatrix} \notin \operatorname{Range} \Phi_{(z,v)}(-\mu/c, 0, 0).$$

Therefore, we can apply the local bifurcation theorem [3] to Φ at $(-\mu/c, 0, 0)$. Thus we have completed the proof of Proposition 4.2. □

5. Proof of main results

5.1. Proof of Theorem 2.2

In this subsection, we will prove Theorem 2.2 by combining the results of the previous sections with the global bifurcation theory.

Proof of Theorem 2.2. We first consider the case $\mu > 0$. In order to apply the global bifurcation theorem, we define a mapping $\tilde{F} : \mathbb{R} \times E \to E$ by

$$\tilde{F}(\lambda, U, v) = \begin{pmatrix} U \\ v - \mu \end{pmatrix} - \begin{pmatrix} (-\Delta + I)_{\Omega}^{-1}[U + f_1(\lambda, U, v)] \\ (-\Delta + I)_{\Omega_1}^{-1}[v - \mu + f_2(U, v)] \end{pmatrix}.$$

By elliptic regularity theory and the Sobolev embedding theorem, the second term of $\tilde{F}$ is a compact operator for any fixed λ. Moreover, (EP) is equivalent to $\tilde{F}(\lambda, U, v) = 0$. Let $\hat{\Gamma}_\delta$ be the local bifurcation branch in Proposition 4.1 and let $\hat{\Gamma}_M \subset \mathbb{R} \times E$ denote the maximal connected set satisfying

$$\hat{\Gamma}_\delta \subset \hat{\Gamma}_M \subset \left\{(\lambda, U, v) \in (\mathbb{R} \times E) \setminus \left\{(\lambda^*, 0, \mu)\right\} \colon \tilde{F}(\lambda, U, v) = 0\right\}. \tag{5.1}$$

Define $P_O = \{w \in C_n^1(\overline{O}):\ w > 0 \text{ in } \overline{O}\}$. We first show

$$\hat{\Gamma}_M \subset \mathbb{R} \times P_\Omega \times P_{\Omega_1} \tag{5.2}$$

by contradiction. Suppose that $\hat{\Gamma}_M \not\subset \mathbb{R} \times P_\Omega \times P_{\Omega_1}$. Then there exist

$$(\lambda_\infty, U_\infty, v_\infty) \in \hat{\Gamma}_M \cap \big(\mathbb{R} \times \partial(P_\Omega \times P_{\Omega_1})\big) \tag{5.3}$$

and a sequence $\{(\lambda_i, U_i, v_i)\}_{i=1}^\infty \subset \hat{\Gamma}_M \cap (\mathbb{R} \times P_\Omega \times P_{\Omega_1})$ such that

$$\lim_{i\to\infty} (\lambda_i, U_i, v_i) = (\lambda_\infty, U_\infty, v_\infty) \quad \text{in } \mathbb{R} \times E.$$

In addition, (U_∞, v_∞) is a non-negative solution of (EP) with $\lambda = \lambda_\infty$. It follows from the strong maximum principle that one of the following (a)–(c) must occur:

(a) $U_\infty \equiv 0$ in $\overline{\Omega}$, $v_\infty \equiv 0$ in $\overline{\Omega}_1$.
(b) $U_\infty > 0$ in $\overline{\Omega}$, $v_\infty \equiv 0$ in $\overline{\Omega}_1$.
(c) $U_\infty \equiv 0$ in $\overline{\Omega}$, $v_\infty > 0$ in $\overline{\Omega}_1$.

Integrating the second equation of (EP) with $(U, v) = (U_i, v_i)$ over Ω_1, we find that

$$\int_{\Omega_1} v_i\left(\mu + \frac{cU_i}{1+kv_i} - v_i\right) dx = 0 \quad \text{for any } i \in \mathbb{N}. \tag{5.4}$$

If (a) or (b) holds, then

$$\mu + \frac{cU_i}{1+kv_i} - v_i > 0 \quad \text{in } \Omega_1$$

for sufficiently large $i \in \mathbb{N}$ because of $\mu > 0$. Hence the integrand in (5.4) is positive for sufficiently large $i \in \mathbb{N}$ since $v_i > 0$ in $\overline{\Omega}_1$ for any $i \in \mathbb{N}$. This contradicts (5.4). If (c) holds, then

$$\Delta v_\infty + v_\infty(\mu - v_\infty) = 0 \quad \text{in } \Omega_1, \qquad \partial_n v_\infty = 0 \quad \text{on } \partial\Omega_1, \qquad v_\infty > 0 \quad \text{in } \overline{\Omega}_1$$

and thus $v_\infty \equiv \mu$ in $\overline{\Omega}_1$. Hence Proposition 4.1 implies that $(\lambda_\infty, U_\infty, v_\infty) = (\lambda^*, 0, \mu)$. This contradicts (5.1) and (5.3). Therefore, the assertion (5.2) holds true. We define

$$Y = \left\{(\phi, \psi) \in E:\ \int_\Omega \phi\phi^*\, dx = 0\right\}, \tag{5.5}$$

that is, Y is the supplement of span $\{(\phi^*, \psi^*)\}$ (which appeared in (4.4)) in E. According to the global bifurcation theorem based on the global bifurcation theory of Rabinowitz [19], one of the following non-excluding properties holds (see Theorem 6.4.3 in López-Gómez [14]):

(1) $\hat{\Gamma}_M$ is unbounded in $\mathbb{R} \times E$.
(2) There exists a constant $\bar{\lambda} \neq \lambda^*$ such that $(\bar{\lambda}, 0, \mu) \in \hat{\Gamma}_M$.
(3) There exists $(\hat{\lambda}, \hat{\phi}, \hat{\psi}) \in \mathbb{R} \times (Y \setminus \{(0, \mu)\})$ such that $(\hat{\lambda}, \hat{\phi}, \hat{\psi}) \in \hat{\Gamma}_M$.

Case (2) cannot occur because of (5.2). On account of (5.2), (5.5) and $\phi^* > 0$, case (3) is also impossible. Therefore, case (1) must hold. It follows from (5.2) and Lemmas 3.3 and 3.4 that (EP) has at least one positive solution if and only if $\lambda > \lambda^*$. Thus the proof for the case $\mu > 0$ is complete.

We can discuss the case $\mu < 0$ in a similar manner and so omit the proof. Hence it only remains to discuss the case $\mu = 0$. Fix any $\lambda > 0$. By virtue of the above result, we can take a sequence $\{(\mu_i, U_i, v_i)\}_{i=1}^{\infty}$ such that (U_i, v_i) is a positive solution of (EP) with $\mu = \mu_i$ and $\lim_{i\to\infty} \mu_i = 0$. Since $\{\mu_i\}_{i=1}^{\infty}$ is a bounded sequence, it follows from Lemma 3.4 that there exists a subsequence, still denoted by $\{(\mu_i, U_i, v_i)\}_{i=1}^{\infty}$, such that

$$\lim_{i\to\infty} (U_i, v_i) = (U_\infty, v_\infty) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$$

for a pair of non-negative functions $(U_\infty, v_\infty) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$. On account of $\lim_{i\to\infty} \mu_i = 0$, (U_∞, v_∞) is a non-negative solution of (EP) with $\mu = 0$. Then the strong maximum principle implies that (U_∞, v_∞) satisfies either $U_\infty > 0$ in $\overline{\Omega}$ and $v_\infty > 0$ in $\overline{\Omega}_1$ or one of (a)–(c) which appeared in the proof for the case $\mu > 0$. Since

$$\int_{\Omega_1} v_i\left(\mu_i + \frac{cU_i}{1+kv_i} - v_i\right) dx = 0$$

for any $i \in \mathbb{N}$, neither (b) nor (c) occurs because of $\lim_{i\to\infty} \mu_i = 0$. Integrating the first equation of (EP) with $(U, v) = (U_i, v_i)$, we have

$$\int_{\Omega} \frac{U_i}{1+k\rho(x)v_i}\left(\lambda - \frac{U_i}{1+k\rho(x)v_i} - b(x)v_i\right) dx = 0$$

for any $i \in \mathbb{N}$. Thus (a) is also excluded since $\lambda > 0$. Therefore, $U_\infty > 0$ in $\overline{\Omega}$ and $v_\infty > 0$ in $\overline{\Omega}_1$. This means the existence of a positive solution of (EP) with $\mu = 0$ for any fixed $\lambda > 0$. We have thus proved Theorem 2.2. □

5.2. Proof of Theorem 2.3

Proof of Theorem 2.3. We first prove part (i) for fixed $\mu > 0$. Fix any $k_1 \geqslant 0$ and note that

$$\lambda_1^N\left(\frac{b(x)\mu - \lambda^*(\mu, k_1, \Omega_0)}{1+k_1\rho(x)\mu}, \Omega\right) = 0.$$

It follows from the assumption (1.1) that

$$\frac{b(x)\mu - \lambda^*(\mu, k_1, \Omega_0)}{1+k_1\rho(x)\mu} = \begin{cases} \frac{\beta\mu - \lambda^*(\mu, k_1, \Omega_0)}{1+k_1\mu} & \text{in } \Omega \setminus \Omega_0, \\ -\lambda^*(\mu, k_1, \Omega_0) < 0 & \text{in } \Omega_0. \end{cases} \tag{5.6}$$

Then the monotone increasing property of $\lambda_1^N(q)$ with respect to $q \in L^\infty(\Omega)$ and the fact $\lambda_1^N(0) = 0$ imply that the constant $\beta\mu - \lambda^*(\mu, k_1, \Omega_0)$ must be positive in (5.6). Thus, if $k_2 > k_1$, then

$$\frac{\beta\mu - \lambda^*(\mu, k_1, \Omega_0)}{1+k_2\mu} < \frac{\beta\mu - \lambda^*(\mu, k_1, \Omega_0)}{1+k_1\mu}$$

and hence

$$\lambda_1^N\left(\frac{b(x)\mu-\lambda^*(\mu,k_1,\Omega_0)}{1+k_2\rho(x)\mu},\Omega\right)<0.$$

Therefore, if $k_2>k_1$, then $\lambda^*(\mu,k_2,\Omega_0)<\lambda^*(\mu,k_1,\Omega_0)$. This completes the proof of part (i).

Next we prove part (ii). For any $\mu\geqslant 0$, let ϕ_μ be a unique positive solution of

$$-\Delta\phi_\mu+\frac{b(x)\mu-\lambda^*(\mu,k,\Omega_0)}{1+k\rho(x)\mu}\phi_\mu=0 \quad \text{in } \Omega, \qquad \partial_n\phi_\mu=0 \quad \text{on } \partial\Omega, \qquad \int_\Omega \phi_\mu^2\,dx=1. \tag{5.7}$$

Multiplying the above differential equation by ϕ_μ and integrating the resulting equation over Ω, we see from Lemma 2.1 that

$$\int_\Omega |\nabla\phi_\mu|^2\,dx=\int_\Omega \frac{\lambda^*(\mu,k,\Omega_0)-b(x)\mu}{1+k\rho(x)\mu}\phi_\mu^2\,dx\leqslant\lambda_1^D(\Omega_0).$$

Thus $\{\phi_\mu\}_{\mu\geqslant 0}$ is bounded in $H^1(\Omega)$ and so there exists a sequence $\{\mu_i\}_{i=1}^\infty$ with $\lim_{i\to\infty}\mu_i=\infty$ such that $\lim_{i\to\infty}\phi_{\mu_i}=\phi_\infty$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ for some non-negative function $\phi_\infty\in H^1(\Omega)$ satisfying $\int_\Omega\phi_\infty^2\,dx=1$. Moreover, we find from (5.7) that

$$\int_\Omega\left(\nabla\phi_{\mu_i}\cdot\nabla\psi+\frac{b(x)\mu_i-\lambda^*(\mu_i,k,\Omega_0)}{1+k\rho(x)\mu_i}\phi_{\mu_i}\psi\right)dx=0$$

for any $\psi\in H^1(\Omega)$. Letting $i\to\infty$ in the above equation, we have

$$\int_\Omega\nabla\phi_\infty\cdot\nabla\psi\,dx+\frac{\beta}{k}\int_{\Omega\setminus\Omega_0}\phi_\infty\psi\,dx-\lambda_\infty^*(k,\Omega_0)\int_{\Omega_0}\phi_\infty\psi\,dx=0$$

for any $\psi\in H^1(\Omega)$, where $\lambda_\infty^*(k,\Omega_0)=\lim_{\mu\to\infty}\lambda^*(\mu,k,\Omega_0)$. Namely, ϕ_∞ is a weak solution of

$$-\Delta\phi_\infty+\frac{\beta}{k}\chi_{\Omega\setminus\Omega_0}\phi_\infty-\lambda_\infty^*(k,\Omega_0)\chi_{\Omega_0}\phi_\infty=0 \quad \text{in } \Omega, \qquad \partial_n\phi_\infty=0 \quad \text{on } \partial\Omega.$$

Since $\phi_\infty\geqslant 0$ in Ω and $\int_\Omega\phi_\infty^2\,dx=1$, we see $\phi_\infty>0$ in $\overline{\Omega}$ by the strong maximum principle. This means that $\eta=\lambda_\infty^*(k,\Omega_0)$ is the first eigenvalue of

$$-\Delta\phi+\frac{\beta}{k}\chi_{\Omega\setminus\Omega_0}\phi=\eta\chi_{\Omega_0}\phi \quad \text{in } \Omega, \qquad \partial_n\phi=0 \quad \text{on } \partial\Omega.$$

Therefore, by the variational characterization of the first eigenvalue, we have

$$\lambda_\infty^*(k,\Omega_0)=\inf_{\phi\in S}\frac{\int_\Omega|\nabla\phi|^2\,dx+\frac{\beta}{k}\int_{\Omega\setminus\Omega_0}\phi^2\,dx}{\int_{\Omega_0}\phi^2\,dx}\leqslant\frac{\beta|\Omega\setminus\Omega_0|}{k|\Omega_0|}$$

for $S=\{\phi\in H^1(\Omega)\colon \int_{\Omega_0}\phi^2\,dx>0\}$, where the last inequality is obtained by setting $\phi\equiv 1$ in Ω. $\square$

5.3. Proof of Theorem 2.4

In order to prove Theorem 2.4, we first derive the following three lemmas.

Lemma 5.1. *Let $\{(k_i, u_{k_i}, v_{k_i})\}_{i=1}^{\infty}$ be any sequence such that (u_{k_i}, v_{k_i}) is a positive solution of* (SP) *with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$, and set $U_{k_i} := (1 + k_i\rho(x)v_{k_i})u_{k_i}$. Then, by passing to a subsequence if necessary,*

$$\lim_{i\to\infty}(U_{k_i}, v_{k_i}) = \big(\bar{U}, \max\{\mu, 0\}\big) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$$

for some non-negative function $\bar{U} \in C^1(\overline{\Omega})$.

Proof. With the aid of Lemma 3.4, by passing to a subsequence if necessary,

$$\lim_{i\to\infty}(U_{k_i}, v_{k_i}) = (\bar{U}, \bar{v}) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$$

for a pair of non-negative functions $(\bar{U}, \bar{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$. Since $v_{k_i} > \max\{\mu, 0\}$ in $\overline{\Omega}_1$ for any $i \in \mathbb{N}$ by Lemma 3.1, we have

$$\bar{v} \geqslant \max\{\mu, 0\} \quad \text{in } \overline{\Omega}_1. \tag{5.8}$$

Note that

$$\lim_{i\to\infty} u_{k_i}(x)v_{k_i}(x) = \lim_{i\to\infty} \frac{U_{k_i}(x)}{1 + k_i v_{k_i}(x)} v_{k_i}(x) = 0$$

for each $x \in \Omega_1$. Then the Lebesgue dominated convergence theorem yields

$$0 = \lim_{i\to\infty} \int_{\Omega_1} v_{k_i}(\mu + cu_{k_i} - v_{k_i})\,dx = \int_{\Omega_1} \bar{v}(\mu - \bar{v})\,dx. \tag{5.9}$$

By (5.8) and (5.9), we obtain $\bar{v} \equiv \max\{\mu, 0\}$ in $\overline{\Omega}_1$. This completes the proof of Lemma 5.1. □

Lemma 5.2. *Suppose $\lambda > -\mu/c \geqslant 0$ and let $\{(k_i, u_{k_i}, v_{k_i})\}_{i=1}^{\infty}$ be any sequence such that (u_{k_i}, v_{k_i}) is a positive solution of* (SP) *with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$. If $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded, then $\mu < 0$ and by passing to a subsequence if necessary,*

$$\lim_{i\to\infty} u_{k_i} = \bar{u} \quad \text{uniformly in } \overline{\Omega} \quad \text{and} \quad \lim_{i\to\infty} k_i v_{k_i} = \bar{w} \quad \text{in } C^1(\overline{\Omega}_1),$$

where $(\bar{u}, \bar{w})$ is a positive solution of (2.4).

Proof. Set $w_{k_i} := k_i v_{k_i}$ and then (u_{k_i}, w_{k_i}) satisfies

$$\begin{cases} \Delta\big[\big(1 + \rho(x)w_{k_i}\big)u_{k_i}\big] + u_{k_i}\big(\lambda - u_{k_i} - b(x)v_{k_i}\big) = 0 & \text{in } \Omega, \\ \Delta w_{k_i} + w_{k_i}(\mu + cu_{k_i} - v_{k_i}) = 0 & \text{in } \Omega_1, \\ \partial_n u_{k_i} = 0 & \text{on } \partial\Omega, \\ \partial_n w_{k_i} = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{5.10}$$

Assume that $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded. On account of Lemma 5.1, elliptic regularity theory and the Sobolev embedding theorem, by passing to a subsequence if necessary,

$$\lim_{i\to\infty}\big(\big(1+\rho(x)w_{k_i}\big)u_{k_i}, v_{k_i}, w_{k_i}\big) = (\bar{U}, 0, \bar{w}) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1) \times C^1(\overline{\Omega}_1) \tag{5.11}$$

for a pair of non-negative functions $(\bar{U}, \bar{w}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$. Therefore, we obtain

$$\lim_{i\to\infty} u_{k_i} = \frac{\bar{U}}{1+\rho(x)\bar{w}} =: \bar{u} \geqslant 0 \quad \text{uniformly in } \overline{\Omega}. \tag{5.12}$$

It follows from (5.10)–(5.12) that $(\bar{u}, \bar{w})$ is a non-negative solution of (2.4).

We finally show that $\bar{u} > 0$ in $\overline{\Omega}$, $\bar{w} > 0$ in $\overline{\Omega}_1$ and $\mu < 0$. We note that $\bar{U}$ is a non-negative solution of

$$\Delta\bar{U} + \frac{\bar{U}}{1+\rho(x)\bar{w}}(\lambda - \bar{u}) = 0 \quad \text{in } \Omega, \qquad \partial_n\bar{U} = 0 \quad \text{on } \partial\Omega.$$

Using the strong maximum principle, we see either $\bar{U} > 0$ or $\bar{U} \equiv 0$ in $\overline{\Omega}$. Suppose that $\bar{U} \equiv 0$ in $\overline{\Omega}$. Then we find from (5.12) that $\lim_{i\to\infty} u_{k_i} = 0$ uniformly in $\overline{\Omega}$. This, together with (5.11), leads to a contradiction:

$$0 = \int_{\Omega} u_{k_i}\big(\lambda - u_{k_i} - b(x)v_{k_i}\big)\,dx > 0$$

for large i since $\lambda > 0$. Thus $\bar{U} > 0$ in $\overline{\Omega}$ and hence $\bar{u} > 0$ in $\overline{\Omega}$. By using the strong maximum principle again, either $\bar{w} > 0$ or $\bar{w} \equiv 0$ in $\overline{\Omega}_1$ holds. Suppose that $\bar{w} \equiv 0$ in $\overline{\Omega}_1$. Then we have

$$\Delta\bar{u} + \bar{u}(\lambda - \bar{u}) = 0 \quad \text{in } \Omega, \qquad \partial_n\bar{u} = 0 \quad \text{on } \partial\Omega, \qquad \bar{u} > 0 \quad \text{in } \overline{\Omega},$$

so that $\bar{u} \equiv \lambda$ in $\overline{\Omega}$. Hence

$$\lim_{i\to\infty}(\mu + cu_{k_i} - v_{k_i}) = \mu + c\lambda > 0$$

uniformly in $\overline{\Omega}_1$ by assumption. This leads to a contradiction:

$$0 = \int_{\Omega_1} v_{k_i}(\mu + cu_{k_i} - v_{k_i})\,dx > 0$$

for large i. Therefore, $\bar{w}$ must be positive in $\overline{\Omega}_1$. Since $\bar{w}$ satisfies (2.4), we have $\lambda_1^N(-c\bar{u}) = \mu$ and $-c\bar{u} < 0$ in $\overline{\Omega}_1$. By the properties of $\lambda_1^N(\cdot)$, this is impossible when $\mu = 0$. Therefore, $\mu < 0$ and $\bar{w} > 0$ in $\overline{\Omega}_1$. This completes the proof of Lemma 5.2. $\square$

Lemma 5.3. *Assume that $\mu = 0$ and let $\{(k_i, u_{k_i}, v_{k_i})\}_{i=1}^{\infty}$ be any sequence such that (u_{k_i}, v_{k_i}) is a positive solution of* (SP) *with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$. Then $\{\min_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is unbounded.*

Proof. By the assumption $\mu = 0$ and Lemma 5.2, $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is unbounded. Since $k_i v_{k_i}$ satisfies

$$\Delta(k_i v_{k_i}) + k_i v_{k_i}(\mu + cu_{k_i} - v_{k_i}) = 0 \quad \text{in } \Omega_1, \qquad \partial_n(k_i v_{k_i}) = 0 \quad \text{on } \partial\Omega_1,$$

it follows from Lemmas 3.2 and 3.4 that

$$\max_{\overline{\Omega}_1} k_i v_{k_i} \leqslant K \min_{\overline{\Omega}_1} k_i v_{k_i}$$

for some positive constant K independent of i. Therefore, $\{\min_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is also unbounded. $\square$

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Let $\{(k_i, u_{k_i}, v_{k_i})\}_{i=1}^{\infty}$ be any sequence such that (u_{k_i}, v_{k_i}) is a positive solution of (SP) with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$, and set $U_{k_i} := (1 + k_i\rho(x)v_{k_i})u_{k_i}$. We first prove part (i) for fixed $\mu \geqslant 0$. It follows from Lemmas 5.1 and 5.3 that there exists a subsequence of $\{k_i\}_{i=1}^{\infty}$, still denoted by $\{k_i\}_{i=1}^{\infty}$, such that

$$\lim_{i\to\infty} (U_{k_i}, v_{k_i}) = (\bar{U}, \mu) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$$

for some non-negative function $\bar{U} \in C^1(\overline{\Omega})$ and

$$\lim_{i\to\infty} k_i v_{k_i} = \infty \quad \text{uniformly in } \overline{\Omega}_1. \tag{5.13}$$

Thus we have

$$\lim_{i\to\infty} u_{k_i} = \lim_{i\to\infty} \frac{U_{k_i}}{1 + k_i v_{k_i}} = 0 \quad \text{in } C^1(\overline{\Omega}_1). \tag{5.14}$$

We next show that $\lim_{i\to\infty} u_{k_i} = \lambda$ in $C^1(\Omega_0)$. Since $u_{k_i} = U_{k_i}$ in Ω_0, it suffices to show that $\bar{U} \equiv \lambda$ in Ω_0. Dividing the first equation of (EP) with $(U, v) = (U_{k_i}, v_{k_i})$ by U_{k_i} and integrating the resulting equation over Ω, we have

$$\int_{\Omega} \frac{\lambda - u_{k_i} - b(x)v_{k_i}}{1 + k_i\rho(x)v_{k_i}}\,dx = -\int_{\Omega} \frac{|\nabla U_{k_i}|^2}{U_{k_i}^2}\,dx \leqslant 0,$$

that is,

$$\int_{\Omega_0} (\lambda - u_{k_i})\,dx \leqslant -\int_{\Omega\setminus\Omega_0} \frac{\lambda - u_{k_i} - \beta v_{k_i}}{1 + k_i v_{k_i}}\,dx.$$

Letting $i \to \infty$ in the above inequality, we see from (5.13) that

$$\int_{\Omega_0} (\lambda - \bar{U})\,dx \leqslant 0. \tag{5.15}$$

On the other hand, it holds that

$$\int_{\Omega_0} u_{k_i}(\lambda - u_{k_i})\,dx + \int_{\Omega\setminus\Omega_0} u_{k_i}(\lambda - u_{k_i} - \beta v_{k_i})\,dx = 0.$$

Letting $i \to \infty$ in the above equation, we see from (5.14) that

$$\int_{\Omega_0} \bar{U}(\lambda - \bar{U})\,dx = 0. \tag{5.16}$$

Hence, (5.15) and (5.16) yield

$$\int_{\Omega_0} (\lambda - \bar{U})^2\,dx = \lambda \int_{\Omega_0} (\lambda - \bar{U})\,dx - \int_{\Omega_0} \bar{U}(\lambda - \bar{U})\,dx \leqslant 0.$$

Therefore, $\bar{U} \equiv \lambda$ in Ω_0. Thus we have obtained

$$\lim_{i\to\infty} (u_{k_i}, u_{k_i}, v_{k_i}) = (\lambda, 0, \mu) \quad \text{in } C^1(\Omega_0) \times C^1(\overline{\Omega}_1) \times C^1(\overline{\Omega}_1) \tag{5.17}$$

and $\lim_{i\to\infty} k_i v_{k_i} = \infty$ uniformly in $\overline{\Omega}_1$. Since the limit of (u_{k_i}, v_{k_i}) is uniquely determined in (5.17), the conclusion of part (i) holds true.

We prove part (ii) by assuming $\lambda > -\mu/c > 0$. On account of Lemma 5.2, it suffices to show that $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded. We prove this by contradiction. Suppose that $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is unbounded. Then, by the same argument as in the proof of Lemma 5.3, $\{\min_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is also unbounded. Hence, by passing to a subsequence if necessary,

$$\lim_{i\to\infty} \min_{\overline{\Omega}_1} k_i v_{k_i} = \infty. \tag{5.18}$$

Combining Lemma 3.4 and (5.18), we have

$$\lim_{i\to\infty} u_{k_i} = \lim_{i\to\infty} \frac{U_{k_i}}{1 + k_i v_{k_i}} = 0 \quad \text{uniformly in } \overline{\Omega}_1. \tag{5.19}$$

We define $\tilde{v}_{k_i} = v_{k_i} / \max_{\overline{\Omega}_1} v_{k_i}$. Then

$$\Delta \tilde{v}_{k_i} + \tilde{v}_{k_i}(\mu + c u_{k_i} - v_{k_i}) = 0 \quad \text{in } \Omega_1, \qquad \partial_n \tilde{v}_{k_i} = 0 \quad \text{on } \partial\Omega_1, \qquad \max_{\overline{\Omega}_1} \tilde{v}_{k_i} = 1. \tag{5.20}$$

It follows from elliptic regularity theory and the Sobolev embedding theorem that there exists a subsequence of $\{k_i\}_{i=1}^{\infty}$, still denoted by $\{k_i\}_{i=1}^{\infty}$, such that

$$\lim_{i\to\infty} \tilde{v}_{k_i} = \tilde{v} \quad \text{in } C^1(\overline{\Omega}_1), \qquad \max_{\overline{\Omega}_1} \tilde{v} = 1 \tag{5.21}$$

for some non-negative function $\tilde{v} \in C^1(\overline{\Omega}_1)$. Thus by virtue of Lemma 5.1 and (5.19)–(5.21), $\tilde{v}$ is a positive solution of

$$\Delta \tilde{v} + \mu \tilde{v} = 0 \quad \text{in } \Omega_1, \qquad \partial_n \tilde{v} = 0 \quad \text{on } \partial\Omega_1.$$

This is impossible since $\mu < 0$. Hence $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded. Therefore, we get the conclusion by Lemma 5.2. $\square$

5.4. Proof of Theorem 2.5

In this subsection, we fix λ and regard μ as a bifurcation parameter. Set $\bar{U} := (1+\rho(x)\bar{w})\bar{u}$. Then (2.4) is rewritten in the following form:

$$\begin{cases} \Delta\bar{U} + g_1(\bar{U},\bar{w}) = 0 & \text{in } \Omega, \\ \Delta\bar{w} + g_2(\mu,\bar{U},\bar{w}) = 0 & \text{in } \Omega_1, \\ \partial_n\bar{U} = 0 & \text{on } \partial\Omega, \\ \partial_n\bar{w} = 0 & \text{on } \partial\Omega_1, \end{cases} \tag{5.22}$$

where

$$\begin{cases} g_1(\bar{U},\bar{w}) = \dfrac{\bar{U}}{1+\rho(x)\bar{w}}\left(\lambda - \dfrac{\bar{U}}{1+\rho(x)\bar{w}}\right), \\ g_2(\mu,\bar{U},\bar{w}) = \bar{w}\left(\mu + \dfrac{c\bar{U}}{1+\bar{w}}\right). \end{cases}$$

Define

$$\bar{\phi} = (-\Delta + \lambda I)_\Omega^{-1}\left[\rho(x)\lambda^2\right].$$

We first prove the following local bifurcation result.

Proposition 5.4. *Positive solutions of* (5.22) *bifurcate from* $\{(\mu,\lambda,0)\colon \mu\in\mathbb{R}\}$ *if and only if* $\mu = -c\lambda$. *Precisely, all positive solutions of* (5.22) *near* $(-c\lambda,\lambda,0)\in\mathbb{R}\times X_1$ *can be expressed as*

$$\Gamma_{\bar{\delta}} = \left\{(\mu,\bar{U},\bar{w}) = \left(\mu(s), \lambda + s\left(\bar{\phi}+\bar{U}(s)\right), s\left(1+\bar{w}(s)\right)\right)\colon s\in(0,\bar{\delta})\right\}$$

for some $\bar{\delta} > 0$. *Here* $(\mu(s),\bar{U}(s),\bar{w}(s))$ *is a smooth function with respect to s and satisfies* $(\mu(0),\bar{U}(0),\bar{w}(0)) = (-c\lambda,0,0)$ *and* $\int_{\Omega_1}\bar{w}(s)\,dx = 0$.

Proof. Let $\bar{z} := \bar{U} - \lambda$ in (5.22) and define a mapping $G:\mathbb{R}\times X_1 \to X_2$ by

$$G(\mu,\bar{z},\bar{w}) = \begin{pmatrix} \Delta\bar{z} + g_1(\bar{z}+\lambda,\bar{w}) \\ \Delta\bar{w} + g_2(\mu,\bar{z}+\lambda,\bar{w}) \end{pmatrix}.$$

Then $G(\mu,\bar{z},\bar{w}) = 0$ if and only if $(\bar{z}+\lambda,\bar{w})$ is a solution of (5.22). We note that $G(\mu,0,0) = 0$ for any μ. Since

$$G_{(\bar{z},\bar{w})}(\mu,0,0)[\phi,\psi] = \begin{pmatrix} \Delta\phi - \lambda\phi + \rho(x)\lambda^2\psi \\ \Delta\psi + (\mu+c\lambda)\psi \end{pmatrix},$$

we can easily see that $\mu = -c\lambda$ is the only possible bifurcation point where positive solutions of (5.22) bifurcate from $\{(\mu,\lambda,0)\colon \mu\in\mathbb{R}\}$. Moreover, we find

$$\operatorname{Ker} G_{(\bar{z},\bar{w})}(-c\lambda,0,0) = \operatorname{span}\left\{(\bar{\phi},1)\right\}$$

and

$$\operatorname{Range} G_{(\bar{z},\bar{w})}(-c\lambda,0,0) = \left\{(\phi,\psi)\in X_2\colon \int_{\Omega_1}\psi\,dx = 0\right\}. \tag{5.23}$$

Consequently, we have

$$\dim \operatorname{Ker} G_{(\bar{z},\bar{w})}(-c\lambda, 0, 0) = \operatorname{codim} \operatorname{Range} G_{(\bar{z},\bar{w})}(-c\lambda, 0, 0) = 1.$$

Furthermore, (5.23) yields

$$G_{\mu(\bar{z},\bar{w})}(-c\lambda, 0, 0)[\bar{\phi}, 1] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin \operatorname{Range} G_{(\bar{z},\bar{w})}(-c\lambda, 0, 0).$$

Therefore, we can apply the local bifurcation theorem [3] to G at $(-c\lambda, 0, 0)$. Hence the proof of Proposition 5.4 is complete. □

Combining Proposition 5.4 with the global bifurcation theory, we can prove the following proposition.

Proposition 5.5. *The set of positive solutions of* (5.22) *with bifurcation parameter* μ *contains an unbounded connected set* Γ_M *in* $\mathbb{R} \times E$ *satisfying the following properties*:

(i) Γ_M *bifurcates from* $\{(\mu, \bar{U}, \bar{w}) = (\mu, \lambda, 0)\colon\ \mu \in \mathbb{R}\}$ *at* $\mu = -c\lambda$,
(ii) $(-c\lambda, 0) \subset \{\mu\colon\ (\mu, \bar{U}, \bar{w}) \in \Gamma_M\} \subset (\tilde{\mu}, 0)$ *for some* $\tilde{\mu} \in (-\infty, -c\lambda]$,
(iii) $\{\max_{\overline{\Omega}} \bar{U}_\mu\}_{(\mu,\bar{U}_\mu,\bar{w}_\mu)\in\Gamma_M}$ *is bounded*,
(iv) $\lim_{\mu\to 0} \bar{w}_\mu = \infty$ *uniformly in* $\overline{\Omega}_1$, *where* $(\mu, \bar{U}_\mu, \bar{w}_\mu) \in \Gamma_M$.

Proof. We define a mapping $\tilde{G}\colon \mathbb{R} \times E \to E$ by

$$\tilde{G}(\mu, \bar{U}, \bar{w}) = \begin{pmatrix} \bar{U} - \lambda \\ \bar{w} \end{pmatrix} - \begin{pmatrix} (-\Delta + I)_{\Omega}^{-1}[\bar{U} - \lambda + g_1(\bar{U}, \bar{w})] \\ (-\Delta + I)_{\Omega_1}^{-1}[\bar{w} + g_2(\mu, \bar{U}, \bar{w})] \end{pmatrix}.$$

Then (5.22) is equivalent to $\tilde{G}(\mu, \bar{U}, \bar{w}) = 0$. Let $\Gamma_{\bar{\delta}}$ be the local bifurcation branch in Proposition 5.4 and let $\Gamma_M \subset \mathbb{R} \times E$ denote the maximal connected set satisfying

$$\Gamma_{\bar{\delta}} \subset \Gamma_M \subset \bigl\{(\mu, \bar{U}, \bar{w}) \in (\mathbb{R} \times E) \setminus \bigl\{(-c\lambda, \lambda, 0)\bigr\}\colon\ \tilde{G}(\mu, \bar{U}, \bar{w}) = 0\bigr\}.$$

Thus the assertion (i) is satisfied for this Γ_M. By the same argument as in the proof for the case $\mu > 0$ in Theorem 2.2, we can verify that Γ_M is contained in the set of positive solutions of (5.22) with bifurcation parameter μ and that Γ_M is unbounded in $\mathbb{R} \times E$.

We next show the assertion (iii). Let $(\mu, \bar{U}_\mu, \bar{w}_\mu) \in \Gamma_M$. Integrating the first equation of (5.22) over Ω, we have

$$\int_\Omega \left(\frac{\bar{U}_\mu}{1 + \rho(x)\bar{w}_\mu}\right)^2 dx = \lambda \int_\Omega \frac{\bar{U}_\mu}{1 + \rho(x)\bar{w}_\mu}\, dx \leqslant \lambda |\Omega|^{1/2} \left\| \frac{\bar{U}_\mu}{1 + \rho(x)\bar{w}_\mu} \right\|_{2,\Omega}$$

and thus

$$\left\| \frac{\bar{U}_\mu}{1 + \rho(x)\bar{w}_\mu} \right\|_{2,\Omega} \leqslant \lambda |\Omega|^{1/2}. \tag{5.24}$$

Hence applying Lemma 3.2 with $p = 2$ to the first equation of (5.22), we obtain

$$\max_{\overline{\Omega}} \bar{U}_\mu \leqslant C^* \min_{\overline{\Omega}} \bar{U}_\mu \tag{5.25}$$

for some positive constant C^* independent of μ. Moreover, we see that

$$|\Omega_0|^{1/2}\min_{\overline{\Omega}}\bar{U}_\mu \leqslant \|\bar{U}_\mu\|_{2,\Omega_0} \leqslant \left\|\frac{\bar{U}_\mu}{1+\rho(x)\bar{w}_\mu}\right\|_{2,\Omega} \tag{5.26}$$

because of $\bar{U}_\mu/(1+\rho(x)\bar{w}_\mu)=\bar{U}_\mu$ in Ω_0. Combining (5.24)–(5.26), we have

$$\max_{\overline{\Omega}}\bar{U}_\mu \leqslant C^*\lambda\left(\frac{|\Omega|}{|\Omega_0|}\right)^{1/2}.$$

Therefore, we get the assertion (iii).

We finally prove the assertions (ii) and (iv). Let $(\mu,\bar{U}_\mu,\bar{w}_\mu)\in\Gamma_M$. Then $\bar{U}_\mu>0$ in $\overline{\Omega}$ and $\bar{w}_\mu$ is a positive solution of

$$-\Delta\bar{w}_\mu-\frac{c\bar{U}_\mu}{1+\bar{w}_\mu}\bar{w}_\mu=\mu\bar{w}_\mu \quad \text{in } \Omega_1, \qquad \partial_n\bar{w}_\mu=0 \quad \text{on } \partial\Omega_1.$$

Hence we have

$$\mu=\lambda_1^N\left(-\frac{c\bar{U}_\mu}{1+\bar{w}_\mu},\Omega_1\right)<0.$$

Since we have already shown the assertion (iii), we obtain

$$\left\{\mu\colon (\mu,\bar{U},\bar{w})\in\Gamma_M\right\}\subset(\tilde{\mu},0) \quad \text{for some } \tilde{\mu}\in(-\infty,-c\lambda].$$

Moreover, in view of elliptic regularity theory and the Sobolev embedding theorem, the assertion (iii) yields the boundedness of $\{\|\bar{U}_\mu\|_{C^1(\overline{\Omega})}\}$. Thus $\{\|\bar{w}_\mu\|_{C^1(\overline{\Omega}_1)}\}$ is unbounded because of the unboundedness of Γ_M in $\mathbb{R}\times E$. It follows from the boundedness of $\{\|\bar{U}_\mu\|_{C^1(\overline{\Omega})}\}$, elliptic regularity theory and the Sobolev embedding theorem that $\{\max_{\overline{\Omega}_1}\bar{w}_\mu\}$ is also unbounded. Furthermore, applying Lemma 3.2 to the second equation of (5.22), we see that $\{\min_{\overline{\Omega}_1}\bar{w}_\mu\}$ is also unbounded. Consequently, there exists a sequence $\{\mu_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty}\mu_i=\mu_\infty$ for some $\mu_\infty\subset[\tilde{\mu},0]$ and $\lim_{i\to\infty}\min_{\overline{\Omega}_1}\bar{w}_{\mu_i}=\infty$. We define $\tilde{w}_{\mu_i}=\bar{w}_{\mu_i}/\max_{\overline{\Omega}_1}\bar{w}_{\mu_i}$. Then

$$\Delta\tilde{w}_{\mu_i}+\tilde{w}_{\mu_i}\left(\mu_i+\frac{c\bar{U}_{\mu_i}}{1+\bar{w}_{\mu_i}}\right)=0 \quad \text{in } \Omega_1, \qquad \partial_n\tilde{w}_{\mu_i}=0 \quad \text{on } \partial\Omega_1, \qquad \max_{\overline{\Omega}_1}\tilde{w}_{\mu_i}=1.$$

Hence, by passing to a subsequence if necessary,

$$\lim_{i\to\infty}\tilde{w}_{\mu_i}=\tilde{w} \quad \text{in } C^1(\overline{\Omega}_1), \qquad \max_{\overline{\Omega}_1}\tilde{w}=1$$

for some non-negative function $\tilde{w}\in C^1(\overline{\Omega}_1)$. In addition, $\tilde{w}$ is a positive solution of

$$\Delta\tilde{w}+\mu_\infty\tilde{w}=0 \quad \text{in } \Omega_1, \qquad \partial_n\tilde{w}=0 \quad \text{on } \partial\Omega_1.$$

Thus $\mu_\infty=0$. Therefore, we get the assertion (ii). Moreover, by applying Lemma 3.1 to (5.22), we have

$$\min_{\overline{\Omega}}\bar{U}_\mu \geqslant \lambda \quad \text{and} \quad \mu+\frac{c\bar{U}_\mu(x_0)}{1+\min_{\overline{\Omega}_1}\bar{w}_\mu}\leqslant 0,$$

where $\bar{w}_\mu(x_0) = \min_{\overline{\Omega}_1} \bar{w}_\mu$. Namely,

$$\min_{\overline{\Omega}_1} \bar{w}_\mu \geqslant \frac{c\bar{U}_\mu(x_0)}{-\mu} - 1 \geqslant \frac{c\lambda}{-\mu} - 1 \to \infty \quad \text{as } \mu \to 0-.$$

Hence we have obtained the assertion (iv). □

We are now in a position to prove Theorem 2.5.

Proof of Theorem 2.5. The conclusions of Theorem 2.5 except for the convergence result of $\bar{u}_\mu$ immediately follow from (i), (ii) and (iv) of Proposition 5.5. Furthermore, owing to (iii) and (iv) of Proposition 5.5, we can prove the convergence result of $\bar{u}_\mu$ by the same argument as in the proof of (i) of Theorem 2.4. Therefore, the proof of Theorem 2.5 is complete. □

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