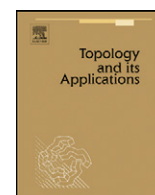Contents lists available at ScienceDirect

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Algebraic and geometric intersection numbers for free groups

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A R T I C L E   I N F O

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We show that the algebraic intersection number of Scott and Swarup for splittings of free groups coincides with the geometric intersection number for the sphere complex of the connected sum of copies of $S^2 \times S^1$.

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1. Introduction

The *geometric intersection number* of homotopy classes of (simple) closed curves on a surface is the minimum number of intersection points of curves in the homotopy classes. This is a much studied concept and has proved to be extremely useful in low-dimensional topology.

Scott and Swarup [18] introduced an algebraic analogue, called the *algebraic intersection number*, for a pair of *splittings* of groups. This is based on the associated partition of the ends of a group [19]. Splittings of groups are the natural analogue of simple closed curves on a surface F—splittings of $\pi_1(F)$ corresponding to homotopy classes of simple closed curves on F. Scott and Swarup showed that, in the case of surfaces, the algebraic and geometric intersection numbers coincide.

We show here that the analogous result holds for free groups, viewed as the fundamental group of the connected sum $M = \sharp_n S^2 \times S^1$ of n copies of $S^2 \times S^1$. Observe that this is a closed 3-manifold with fundamental group the free group on n generators. Thus, the manifold can be regarded as a model for studying the free group and its automorphisms.

Embedded spheres in M correspond to splittings of the free group. Hence, given a pair of embedded spheres in M, we can consider their *geometric intersection number* (defined below) as well as the *algebraic intersection number* of Scott and Swarup for the corresponding splittings. Our main result is that, for embedded spheres in M these two intersection numbers coincide. The principal method we use is the normal form for embedded spheres developed by Hatcher.

Before stating our result, we recall the definition of the intersection numbers.

Definition 1.1. Let A and B be two isotopy classes of embedded spheres S and T, respectively, in M. The *geometric intersection number $I(A, B)$* of A and B is defined as the minimum of the number of components $|S \cap T|$ of $S \cap T$ over embedded transversal spheres S and T representing the isotopy classes A and B, respectively.

This is clearly symmetric. Further, for an embedded sphere S, if $A = [S]$ then $I(A, A) = 0$.

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We consider next the algebraic intersection number. Let $\widetilde{M}$ be the universal cover of M. Observe that $\pi_2(M) = \pi_2(\widetilde{M}) = H_2(\widetilde{M})$. The fundamental group $\pi_1(M) = G$ of M, which is a free group of rank n, acts freely on the universal cover $\widetilde{M}$ of M by deck transformations. Homotopy classes of spheres in M correspond to equivalence classes of elements in $H_2(\widetilde{M})$ up to the action of deck transformations. For embedded spheres, we can consider isotopy classes instead of homotopy classes as the homotopy classes of embedded spheres are the same as isotopy classes of embedded spheres [15].

For an embedded sphere $S \in M$ with lift $\widetilde{S} \in \widetilde{M}$, all the translates of $\widetilde{S}$ are embedded and disjoint from $\widetilde{S}$. In particular, if $\widetilde{A} = [\widetilde{S}]$ is the isotopy class represented by $\widetilde{S}$, then $\widetilde{A}$ and $g\widetilde{A}$ can be represented by disjoint embedded spheres for each deck transformation $g \in G$.

Definition 1.2. Let $A = [S]$ and $B = [T]$ be two isotopy classes of embedded spheres S and T, respectively, in M. Let $\widetilde{A} = [\widetilde{S}]$ and $\widetilde{B} = [\widetilde{T}]$, where $\widetilde{S}$ and $\widetilde{T}$ are the lifts of S and T, respectively, to $\widetilde{M}$. The *algebraic intersection number* $\widetilde{I}(A, B)$ of A and B is defined as the number of translates $g\widetilde{B}$ of $\widetilde{B}$ such that $\widetilde{A}$ and $g\widetilde{B}$ can not be represented by disjoint embedded spheres in $\widetilde{M}$.

It was shown in [6] that this coincides with the algebraic intersection number of Scott and Swarup.

We say that two isotopy classes $\widetilde{A} = [\widetilde{S}]$ and $\widetilde{B} = [\widetilde{T}]$ of embedded spheres in $\widetilde{M}$ *cross* if they cannot be represented by disjoint embedded spheres. Thus, the algebraic intersection number is the number of elements $g \in \pi_1(M)$ such that $\widetilde{A}$ and $g\widetilde{B}$ cross. We shall also say that $\widetilde{S}$ and $\widetilde{T}$ cross if the classes they represent cross.

It is immediate that $\widetilde{A}$ and $g\widetilde{B}$ cross if and only if $g^{-1}\widetilde{A}$ and $\widetilde{B}$ cross. It follows that $\widetilde{I}(A, B) = \widetilde{I}(B, A)$. Thus the *algebraic intersection number* is symmetric.

Clearly, for all but finitely many translates $g\widetilde{B}$ of $\widetilde{B}$, $\widetilde{A}$ and $g\widetilde{B}$ can be represented by disjoint embedded spheres in $\widetilde{M}$. This is because, for any pair of embedded spheres S and T in M, all but finitely many translates of $\widetilde{T}$ are disjoint from $\widetilde{S}$ in $\widetilde{M}$. Hence $\widetilde{I}(A, B)$ is finite for all isotopy classes A and B of embedded spheres in M.

As was shown in [6], it follows from results of Scott and Swarup that if the algebraic intersection number between classes A and B as above vanishes, then they can be represented by disjoint embedded spheres, i.e., their geometric intersection number vanishes. The converse is an easy observation.

We prove here a much stronger result—that the algebraic and geometric intersection numbers are equal.

Theorem 1.3. *For isotopy classes A and B of embedded spheres in M, $\widetilde{I}(A, B) = I(A, B)$.*

Our proof is based on the normal form for spheres in M due to Hatcher [7], which we recall in Section 2. We extend a sphere Σ in the isotopy class B to a maximal system of spheres and consider a sphere S in the isotopy class of A in normal form with respect to this system. We then show in Section 3 that, when S is in normal form, the number of components of intersection between S and Σ is the algebraic intersection number between the isotopy classes $A = [S]$ and B.

Our methods also show that, if $A_1, \ldots, A_n$ is a collection of isotopy classes of embedded spheres, each pair of which can be represented by disjoint spheres, then all the classes A_i can be simultaneously represented by disjoint spheres. We prove this in Theorem 3.3.

The sphere complex associated to M is a simplicial complex whose vertices are the isotopy classes of embedded spheres in M. A set of isotopy classes of embedded spheres in M is deemed to span a simplex if they can be realized disjointly in M. This is an analogue of the curve complex associated to a surface. The topological properties of the sphere complex have been studied by Hatcher, Hatcher, Vogtmann and Hatcher and Wahl in [7–12]. Culler and Vogtmann have constructed a contractible complex *Outer space* on which the outer automorphism group $Out(F_n)$ of the free group F_n acts discretely and with finite stabilizers [3]. This is an analogue of Teichmüller space of surface on which the mapping class group of the surface acts. Culler and Morgan have constructed a compactification of Outer space much like Thurston's compactification of Teichmüller space [2]. The curve complex has proved to be fruitful in studying Teichumüller space (see, for instance, [13,14]).

The *geometric intersection number* of curves on a surface has been used to give constructions like the space of measured laminations whose projectivization is the boundary of Teichmüller space, [16], as well as to study geometric properties, including hyperbolicity of the curve complex in [1,17]. One may hope that the *geometric intersection number* of embedded spheres in M might be useful to give such constructions in case sphere complex and Outer space. The sphere complex is useful for studying the mapping class group of M, $Out(F_n)$ and Outer space.

An important ingredient of our proofs is the observation that if S and T are embedded spheres in M and S is in normal form with respect to a maximal system of spheres containing T, then S and T intersect minimally. This is somewhat analogous to results for geodesics and least-area surfaces [4,5]. Further the components of intersection correspond to *crossing*. This is very similar to the case of geodesics, where intersections correspond to linking of end points.

2. Normal spheres

We recall the notion of *normal sphere systems* from [7].

Definition 2.1. A smooth, embedded 2-sphere in M is said to be essential if it does not bound a 3-ball in M.

Definition 2.2. A *system of* 2-*spheres* in M is defined as a finite collection of disjointly embedded, pair-wise non-isotopic, essential smooth 2-spheres $S_i \subset M$.

Let $\Sigma = \bigcup_j \Sigma_j$ be a maximal system of 2-sphere in M. Splitting M along Σ, then produces a finite collection of 3-punctured 3-spheres P_k. Here a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere.

Definition 2.3. A system of 2-spheres $S = \bigcup_i S_i$ in M is said to be in *normal form with respect to* Σ if each S_i either coincides with a sphere Σ_j or meets Σ transversely in a non-empty finite collection of circles splitting S_i into components called pieces, such that the following two conditions hold in each P_k:

1. Each piece in P_k meets each component of ∂P_k in at most one circle.
2. No piece in P_k is a disk which is isotopic, fixing its boundary, to a disk in ∂P_k.

Thus each piece is a disk, a cylinder or a pair of pants. A disk piece has its boundary on one component of ∂P_k and separates the other two components of ∂P_K.

Recall the following result from [7].

Proposition 2.4 *(Hatcher). Every system $S \subset M$ can be isotoped to be in normal form with respect to Σ. In particular, every embedded sphere S which does not bound a ball in M can be isotoped to be in normal form with respect to Σ.*

We recall some constructions from [7]. First, we associate a tree T to $\widetilde{M}$ corresponding to the decomposition of M by Σ. Let $\widetilde{\Sigma}$ be the pre-image of Σ in $\widetilde{M}$. The closure of each component of $\widetilde{M} - \widetilde{\Sigma}$ is a 3-punctured 3-sphere. The vertices of the tree are of two types, with one vertex corresponding to the closure of each component of $\widetilde{M} - \widetilde{\Sigma}$ and one vertex for each component of $\widetilde{\Sigma}$. An edge of T joins a pair of vertices if one of the vertices corresponds to the closure of a component X of $\widetilde{M} - \widetilde{\Sigma}$ and the other vertex corresponds to a component of $\widetilde{\Sigma}$ that is in the boundary of X. Thus, we have an Y-shaped subtree corresponding to each complementary component. We pick an embedding of T in $\widetilde{M}$ respecting the correspondences.

Given a sphere S in normal form with respect to Σ and a lift $\widetilde{S}$ of S to $\widetilde{M}$, we associate a tree $T(\widetilde{S})$ corresponding to the decomposition of $\widetilde{S}$ into pieces. The tree has two types of vertices, vertices corresponding to closures of components of $\widetilde{S} - \widetilde{\Sigma}$ (i.e., pieces) and vertices corresponding to each component of $\widetilde{S} \cap \widetilde{\Sigma}$. Edges join a pair of vertices if one of the vertices corresponds to a piece and the other to a boundary component of the piece.

In [7], it is shown that $T(\widetilde{S})$ is a tree. Moreover, the inclusion $\widetilde{S} \hookrightarrow \widetilde{M}$ induces a natural inclusion map $T(\widetilde{S}) \hookrightarrow T$. So we can view $T(\widetilde{S})$ as a subtree of T. The bivalent vertices of T correspond to spheres components in $\widetilde{\Sigma}$, i.e., lifts of the spheres Σ_j and their translates.

3. Algebraic and geometric intersection numbers

Consider now two isotopy classes A and B of embedded spheres in M. Choose an embedded sphere Σ_1 in the isotopy class B and extend this to a maximal collection Σ of spheres. Let S be a representative for A in normal form with respect to Σ. Theorem 1.3 is equivalent to showing that $\widetilde{I}(A, [\Sigma_j]) = I(A, [\Sigma_j])$ for $j = 1$. We begin by showing the non-trivial inequality here.

Lemma 3.1. *If $A = [S]$ is the isotopy class of the embedded sphere S in M, then for the isotopy class $[\Sigma_j]$ of Σ_j in M, $\widetilde{I}(A, [\Sigma_j]) \geqslant I(A, [\Sigma_j])$.*

Proof. The sphere S, which is in normal form with respect to Σ, represents the class A. We shall show that the number of components of intersection of S with Σ_j is $\widetilde{I}(A, [\Sigma_j])$. As the geometric intersection number is the minimum of the number of components of intersection of spheres in the isotopy classes, the lemma is an immediate consequence of this claim.

Fix a lift $\widetilde{S}$ of S. The components of $S \cap \Sigma_j$ are homotopically trivial circles in M. These lift to circles of intersection between $\widetilde{S}$ and components of the pre-image of Σ_j. These correspond to vertices of $T(\widetilde{S})$. As $T(\widetilde{S})$ is a tree which embeds in T, different circles of intersection of S and Σ_j correspond to intersections of $\widetilde{S}$ with different components of the pre-image of Σ_j. It follows that the number of components of intersection of S with Σ_j is the number of components of the pre-image of Σ_j that intersect $\widetilde{S}$.

The main observation needed is the following lemma.

Lemma 3.2. *If $\widetilde{S}$ intersects a component $\widetilde{\Sigma_j}$ of the pre-image of Σ_j, then the spheres $\widetilde{S}$ and $\widetilde{\Sigma_j}$ cross.*

Proof. Assume that $\widetilde{S}$ intersects the component $\widetilde{\Sigma_j}$ of the pre-image of Σ_j. The sphere $\widetilde{\Sigma_j}$ corresponds to a vertex v_0 of T. As $\widetilde{S}$ intersects $\widetilde{\Sigma_j}$ and S is in normal form, the vertex v_0 is an interior vertex of $T(\widetilde{S})$.

We recall the notion of crossing due to Scott and Swarup, which by [6] is equivalent to the notion we use. The spheres $\widetilde{S}$ and $\widetilde{\Sigma}_j$ partition the ends of $\widetilde{M}$ into pairs of complementary subsets $E_S^\pm$ and $E_\Sigma^\pm$, corresponding to the components of the complement of the respective spheres in $\widetilde{M}$. The spheres $\widetilde{S}$ and $\widetilde{\Sigma}_j$ cross if all the four intersections $E_S^\pm \cap E_\Sigma^\pm$ are non-empty.

A properly embedded path $c : \mathbb{R} \to \widetilde{M}$ induces a map from the ends $\pm\infty$ of $\mathbb{R}$ to the ends of $\widetilde{M}$. Thus, we can associate to c a pair of ends $c_\pm$. We say that the path c is a path from c_- to c_+. Poincaré duality gives a useful criterion for when two ends E and E' of $\widetilde{M}$ are in different equivalence classes with respect to the partition corresponding to $\widetilde{S}$. Namely, E and E' are in different equivalence classes if and only if there is a proper path c from E to E' so that $c \cdot \widetilde{S} = \pm 1$, with $c \cdot \widetilde{S}$ the intersection pairing obtained from the cup product using the duality between homology and cohomology with compact support.

The ends of $\widetilde{M}$ can be naturally identified with the ends of the tree T. The sets $E_\Sigma^\pm$ correspond to the ends of the two components of $T - \{v_0\}$. It is easy to see that $\widetilde{\Sigma}$ and $\widetilde{S}$ cross if and only if each of the sets $E_\Sigma^\pm$ contain pairs of ends E_1 and E_2 which are in different equivalence classes with respect to the partition corresponding to $\widetilde{S}$. By symmetry, it suffices to consider the case of E_Σ^+. Let X denote the closure of the component of $\widetilde{M} - \widetilde{\Sigma}_j$ with $\mathrm{ends}(X) = E_\Sigma^+$.

As v_0 is an internal vertex of the tree $T(\widetilde{S})$, there is a terminal vertex w of $T(\widetilde{S})$ contained in X. A terminal vertex of $T(\widetilde{S})$ corresponds to a piece which is a disc D in a 3-punctured sphere P, with P the closure of a component of $\widetilde{M} - \widetilde{\Sigma}$. Let Q_1 and Q_2 denote the boundary components of P disjoint from D (hence from S). Then D separates Q_1 and Q_2.

For $i = 1, 2$, let W_i denote the closure of the component of $\widetilde{M} - Q_i$ which does not contain S. As Q_i is the lift of an essential sphere, and $\widetilde{M}$ is simply-connected, Q_i is non-trivial as an element of $H_2(\widetilde{M})$. Hence W_i is non-compact. By construction $W_i \subset X$, hence the ends of W_i are contained in E_Σ^+.

As D separates Q_1 and Q_2, (after possibly interchanging Q_1 and Q_2) there is a path $c : [0, 1] \to P$ intersecting S transversely in one point (with the sign of the intersection $+1$) so that $c(0) \in Q_1$ and $c(1) \in Q_2$. As W_1 and W_2 are non-compact, we can extend c to a proper function $c : \mathbb{R} \to \widetilde{M}$ with $c((-\infty, 0)) \subset W_1$ and $c((1, \infty)) \subset W_2$.

The ends E_1 and E_2 of c are ends of X (as $W_i \subset X$ for $i = 1, 2$). Further, by construction $c \cdot S = 1$. It follows that the ends $E_1, E_2 \subset E_\Sigma^+$ are in different components with respect to the partition corresponding to S. By symmetry, we can find a similar pair of ends in E_Σ^-. It follows that $\widetilde{S}$ and $\widetilde{\Sigma}$ cross. $\square$

We now complete the proof of Lemma 3.1. We have seen that the number of components of $S \cap \Sigma_j$ is the number of components of the pre-image of Σ_j which intersect $\widetilde{S}$. For a fixed lift $\widetilde{\Sigma}_j$ of Σ_j, the components of the pre-images of Σ_j are the translates $g\widetilde{\Sigma}_j$ of $\widetilde{\Sigma}_j$.

By Lemma 3.2, it follows that if $\widetilde{S}$ intersects $g\widetilde{\Sigma}_j$, then $\widetilde{S}$ crosses $g\widetilde{\Sigma}_j$. The converse of this is obvious. By the definition of algebraic intersection number, Lemma 3.1 follows. $\square$

Proof of Theorem 1.3. We have seen that it suffices to consider the case when $A = [S]$, $B = [\Sigma_1]$ and S is in normal form with respect to Σ. By Lemma 3.1, $\widetilde{I}(A, B) \geqslant I(A, B)$.

Conversely, let S and Σ_1 be embedded spheres with $A = [S]$, $B = [\Sigma_1]$ and $I(A, B) = |S \cap \Sigma_1|$. Let $\widetilde{S}$ and $\widetilde{\Sigma}_1$ be lifts of S and Σ_1, respectively, to $\widetilde{M}$. Observe that (distinct) components of intersection of S with Σ_1 lift to (distinct) components of intersection of $\widetilde{S}$ with translates of $\widetilde{\Sigma}_1$. Hence the number of translates of $\widetilde{\Sigma}_1$ that intersect $\widetilde{S}$ is at most $I(A, B)$. As $\widetilde{I}(A, B)$ is the number of translates of $\widetilde{\Sigma}_1$ that *cross* $\widetilde{S}$, and components that cross must intersect, it follows that $\widetilde{I}(A, B) \leqslant I(A, B)$.

This completes the proof of the theorem. $\square$

Our methods also yield the following result. This also follows from the work of Scott and Swarup, see [18].

Theorem 3.3. *If $A_1, \ldots, A_n$ are isotopy classes of embedded spheres in M such that, for $1 \leqslant i, j \leqslant n$, A_i and A_j can be represented by disjoint spheres, then there exist disjointly embedded spheres S_i, $1 \leqslant i \leqslant n$, such that $A_i = [S_i]$.*

Proof. We prove this by induction on n. For $n = 1, 2$, the conclusion is immediate from the hypothesis. Assume that the result holds for $n = k$ and consider a collection A_i as in the hypothesis with $n = k + 1$.

Suppose one of the spheres, which we can assume without loss of generality is A_n, is not essential. By the induction hypothesis, there are disjoint embedded spheres S_i, $1 \leqslant i < n$, with $[S_i] = A_i$. Choose a 3-ball disjoint from the spheres S_i, $1 \leqslant i < n$, and let S_n be its boundary. Then the spheres S_i, $1 \leqslant i \leqslant n$, give the required collection.

Thus we may assume that all the isotopy classes A_i of spheres are essential. By induction hypothesis, there are disjoint embedded spheres S_i, $1 \leqslant i < n$, with $[S_i] = A_i$. As these are essential by our assumption, we can extend the collection S_i to a maximal system of spheres. We let S_n be a sphere in normal form with respect to this collection. By hypothesis, $I(S_n, S_i) = 0$ for $1 \leqslant i \leqslant n$. By the proof of Lemma 3.1, it follows that S_n is disjoint from S_i. Thus, S_i, $1 \leqslant i \leqslant n$, is a collection of disjoint embedded spheres with $A_i = [S_i]$. $\square$

Remark 3.4. The above theorem shows that the sphere complex associated to M is a full complex in the sense that if $V_1, V_2, \ldots, V_k$ are the vertices of the sphere complex and if there is an edge between every pair V_i, V_j of vertices, where $1 \leqslant i, j \leqslant k$, then these vertices bound a simplex in the sphere complex.

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