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NOTE

THE BINARY NETWORK FLOW PROBLEM IS LOGSPACE COMPLETE FOR P*

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Abstract. It is shown that the problem of whether the maximum flow in a given network exceeds a given natural number is logspace many-one complete for P if the edge capacities are presented in binary (even if the problem is restricted to acyclic graphs). This improves a result by Goldschlager et al. (1982) that this problem is logspace Turing complete for P.

1. Introduction

The binary (unary, resp.) network flow problem asks whether the maximum flow in a given network exceeds a given natural number if the edge capacities of the network are presented in binary (unary, resp.). Recently, the interest in the unary network flow problem arose because its complexity lies inside the interesting area between NL and P. It turned out that the unary network flow problem is equivalent to the bipartite perfect matching problem with respect to several simple reducibilities. In [1] this was proved for a restricted kind of AC_0 truth-table reducibility, and in [2] this was proved for NC_1 many-one reducibility. (Note that these results seem to be incomparable because AC_0 is a subclass of NC_1 on the one hand and, on the other hand, many-one reducibility is a restricted kind of truth-table reducibility.) In [5] it was proved that the bipartite perfect matching problem is in randomized NC (RNC). Thus the unary network flow problem is also in RNC, and hence it is very unlikely to be complete for P.

It was proved in [4] that the problem of whether the maximum flow in a given network is *odd*, is logspace many-one complete (for short, logspace m-complete) for P, if the edge capacities are presented in binary. Since this problem is obviously

* This paper is the full version of a part of [7].

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logspace Turing-reducible to the binary network flow problem, the latter problem is logspace Turing-complete for P. In the present paper we prove that the binary network flow problem is logspace m-complete for P. We prove this even for the restriction of the problem to acyclic networks. Notice that the proof in [4] makes essential use of the fact that the networks contain cycles.

For the proof of our main result (Section 3) we need a lemma stating that a certain severely restricted version of the circuit value problem is logspace m-complete for P. This is proved in Section 2.

2. Restricted versions of the circuit value problem

A *circuit* $C=(V, E, v_0, f)$ consists of a directed acyclic graph (V, E) (whose vertices are calles gates), an output gate v_0, and a function f mapping every input gate (this is a gate with fan-in 0) into $\{0, 1\}$ and every non-input gate into {AND, OR, NOT, ID}. An AND (OR, NOT, ID, resp.) gate has fan-in two (two, one, one, resp.) and computes the conjunction (disjunction, negation, identity, resp.) of its inputs.

Circuit value problem (CVP)
Instance: A circuit C.
Question: Does the output gate compute 1?

Lemma 2.1 (Ladner [6]). *CVP is logspace m-complete for* P.

In the following we restrict the circuit value problem step by step. The *monotone circuit value problem* (MCVP) is the circuit value problem restricted to circuits that do not use negation.

Lemma 2.2 (Goldschlager [3]). *MCVP is logspace m-complete for* P.

The problem MCV2 is the problem MCVP restricted to circuits whose gates have fan-out at most two.

Lemma 2.3 (Goldschlager et al. [4]). *MCV2 is logspace m-complete for* P. (*In* [4] *MCV2 is defined as an even more severely restricted version of MCV.*)

A *layered circuit* is a circuit (V, C, v_0, f) such that

- V is a subset of $\{0, 1, \ldots, k\} \times \Sigma^*$ (for a suitable $k \geq 0$ and a suitable finite alphabet Σ). The vertices whose first component is i make up the ith *level* of the circuit.
- All edges are between adjacent levels, i.e., for every $e \in E$ there exist $i \in \{1, 2, \ldots, k\}$ and $x, y \in \Sigma^*$ such that $e=((i-1, x), (i, y))$.
- Level 0 consists exactly of the gates with fan-in zero (the input gates).

- Level k (the output level) consists exactly of the gates with fan-out zero.

The problem MLCV2 is the problem MCV2 restricted to layered circuits. The problem MLCVE2 is the problem MLCV2 restricted to circuits whose gates have fan-out two or zero.

Lemma 2.4. *MLCV2 is logspace m-complete for* P.

Proof. We give a logspace m-reduction from MCV2 to MLCV2. Consider a monotone circuit (V, E, v_0, f) whose gates have fan-out at most two. Without loss of generality we can assume that $V=\{0,1,\ldots,k\}$ for some $k \geqslant 0$, that $v_0=k$ and that $(i,j)\in E$ implies $i<j$. Furthermore, we can assume that every AND gate and every OR gate has two different predecessors (in the other case such a gate could be replaced with an ID gate). We convert C into a monotone layered circuit $C'=(V', E', v_0', f')$ as follows: V' is a subset of $\{0,1,\ldots,k\}\times(\{0,1,\ldots,k\}\cup\{0,1,\ldots,k\}^2)$. Gate i from C becomes gate (i,i) in C'. If $i=0$ or if i has non-zero fan-in then $f'((i,i))=f(i)$. An edge (i,j) becomes a chain with gates (i,i), $(i+1,(i,j))$, $(i+2,(i,j))$, $\ldots$, $(j-1,(i,j))$, (j,j) and $f'((m,(i,j)))=\mathrm{ID}$ for $m=i+1$, $i+2$, $\ldots$, $j-1$. A gate (i,i) with $i>0$ and fan-in zero is connected with the input level by a chain with gates $(0,i)$, $(1,i)$, $\ldots$, $(i-1,i)$, (i,i) and $f'((0,i))=f(i)$ and $f'((m,i))=\mathrm{ID}$ for $m=1,2,\ldots,i$. A gate (i,i) with $i<k$ and fan-out zero is connected with the output-level by a chain with gates (i,i), $(i+1,i)$, $\ldots$, $(k-1,i)$, (k,i) and $f'((m,i))=\mathrm{ID}$ for $m=i+1,\ldots,k$. Finally put $v_0'=(k,k)$.

It is obvious that this construction can be performed in logspace and that v_0 computes the same value in C as v_0' computes in C'. □

Lemma 2.5. *MLCVE2 is logspace m-complete for* P.

Proof. We give a logspace m-reduction from MLCV2 to MLCVE2. Consider a monotone layered circuit (V, E, v_0, f) with levels 0, 1, $\ldots$, k whose gates have fan-out

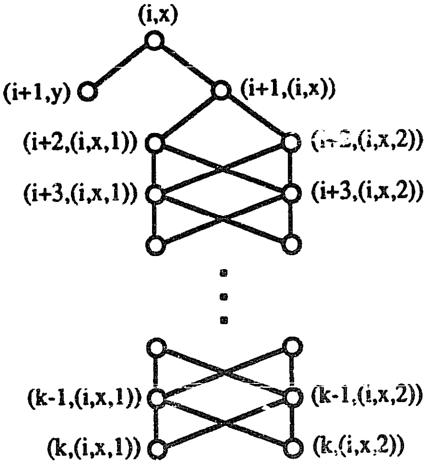

Fig. 1.

at most two. We simply attach to every gate with fan-out one an additional successor. Let (i, x) be such a gate having only one successor $(i+1, y)$. Then we add the new gates $(i+1, (i, x)), (i+2, (i, x, 1)), (i+2, (i, x, 2)), (i+3, (i, x, 1)), (i+3, (i, x, 2)), \ldots, (k-1, (i, x, 1)), (k-1, (i, x, 2)), (k, (i, x, 1)), (k, (i, x, 2))$ with edges as shown in Fig. 1. Obviously, this construction can be performed in logspace and does not influence the value of the output. □

3. The result

A *flow network* is a quadruple (G, S, T, c) where $G=(V, E)$ is a digraph, S is a set of vertices of G with indegree zero (called the *sources* of G), T is a set of vertices of G with outdegree zero (called the *sinks* of G), and $c: E \mapsto \mathbb{N}$ is the *capacity function* of G. A *flow* in (G, S, T, c) is any function $\phi: E \mapsto \mathbb{N}$ such that

- $\phi(e) \leq c(e)$ for every $e \in E$ and
- $\sum_{(u,v)\in E} \phi((u, v)) = \sum_{(v,w)\in E} \phi((v, w))$ for every $v \in V \backslash (S \cup T)$.

For a flow ϕ in (G, S, T, c) we define $\phi(v) = \sum_{(v,w)\in E} \phi((v, w))$ for all $v \in V \backslash T$ and $\phi(t) = \sum_{(u,t)\in E} \phi((u, t))$ for every $t \in T$. Obviously, $\sum_{s\in S} \phi(s) = \sum_{t\in T} \phi(t)$. The value $\max\{\sum_{t\in T} \phi(t)$: ϕ flow in $(G, S, T, c)\}$ is called the maximum flow in (G, S, T, c).

Now we are able to define the problem we are interested in.

Binary acyclic network flow (BANF)

Instance: A dag $G=(V, E)$ with the unique source s and the unique sink t, binary presentations of natural numbers $c(v)$ for all $v \in V$, and the binary presentation of a natural number m.

Question: Is the maximum flow in the flow network $(G, \{s\}, \{t\}, c)$ greater than or equal to m?

Theorem 3.1. *BANF is logspace m-complete for* P.

Proof. We give a logspace m-reduction from MLCVE2 to BANF. Let $C = (V, E, v_0, f)$ be a monotone layered circuit whose gates have fan-out two or zero (the latter gates make up the output level). Let V be a subset of $\{0, 1, \ldots, k\} \times \Sigma^*$ for some finite Σ, and let $v_0 = (k, x_0)$. We construct from C a flow network $N = ((V', E'), \{s\}, \{t\}, c)$ by local replacements. Every gate (i, x) of C is replaced with the vertices $(i, x, 1)$ and $(i, x, 2)$ in N. The edges between (i, x) and its predecessors (for $i > 0$) are replaced with small flow networks $N(i, x)$. In addition we have edges leaving the source s and edges entering the sink t. Specifically the construction is as follows.

(1) Edges from s to vertices of the form $(0, x, 1)$ and $(0, x, 2)$: If $f((0, x)) = 1$ then there is an edge with capacity 4^k from s to $(0, x, 1)$. Otherwise there is an edge with capacity 4^k from s to $(0, x, 2)$.

(2) Flow networks $N(i, x)$ for $i > 0$:

(2.1) Flow network $N(i, x)$ for $f((i, x)) = \mathrm{ID}$: If $f((i, x)) = \mathrm{ID}$ and $((i-1, y), (i, x)) \in E$ then $(i-1, y, 1)$ and $(i-1, y, 2)$ are connected with $(i, x, 1)$, $(i, x, 2)$ and t using a flow network that is shown in Fig. 2 where all edges have capacity 4^{k-i}.

(2.2) Flow network $N(i, x)$ for $f((i, x)) = \mathrm{AND}$: If $f((i, x)) = \mathrm{AND}$ and $((i-1, y), (i, x))$, $((i-1, z), (i, x)) \in E$ then $(i-1, y, 1)$, $(i-1, y, 2)$, $(i-1, z, 1)$ and $(i-1, z, 2)$ are connected with $(i, x, 1)$, $(i, x, 2)$ and t using a flow network that is shown in Fig. 3 where all edges have capacity 4^{k-i}.

(2.3) Flow network $N(i, x)$ for $f((i, x)) = \mathrm{OR}$: if $f((i, x)) = \mathrm{OR}$ and $((i-1, y), (i, x))$, $((i-1, z), (i, x)) \in E$ then $(i-1, y, 1)$, $(i-1, y, 2)$, $(i-1, z, 1)$ and $(i-1, z, 2)$ are connected with $(i, x, 1)$, $(i, x, 2)$ and t using a flow network that is shown in Fig. 4 where all edges have capacity 4^{k-i}.

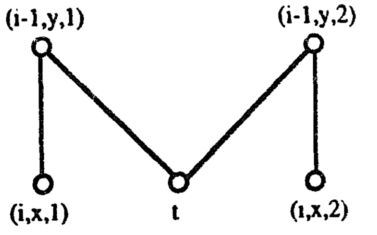

Fig. 2.

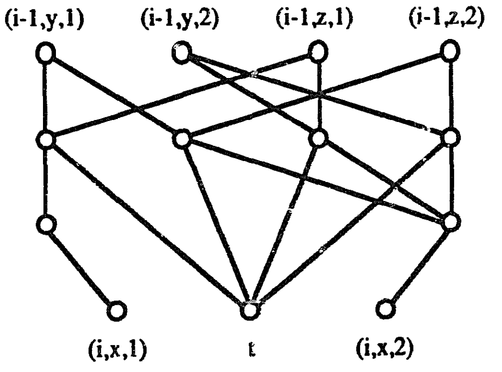

Fig. 3.

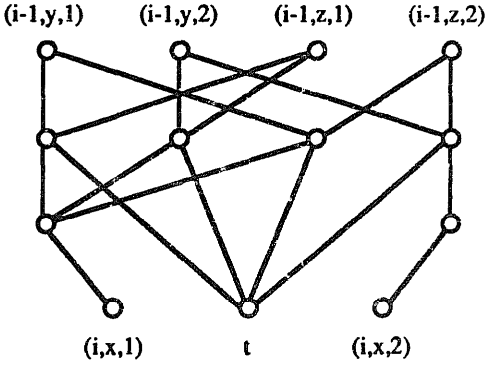

Fig. 4.

(3) Edges from vertices of the form $(k, x, 1)$, $(k, x, 2)$ to t: There are edges with capacity 1 from $(k, x, 1)$ to t and from $(k, x, 2)$ to t.

For the proof of the correctness of this construction we make some observations about N. Let D be the number of different gates of the form $(0, x)$ in C. Let $C(i, x)$ be the value computed by the gate (i, x) of C.

- For every flow ϕ in N we have $\phi(s) = \phi(t) \leq D4^k$.
- There exists a flow ϕ in N such that $\phi(s) = \phi(t) = D4^k$, $\phi((i, x, 1)) = 4^{k-i}C(i, x)$ and $\phi((i, x, 2)) = 4^{k-i}(1 - C(i, x))$. This can easily be proved by induction on i using the properties of the flow networks $N(i, x)$ and the fact that every gate (i, x) in C with $i < k$ has fan-out exactly two.
- For every flow ϕ in N such that $\phi(s) = \phi(t) = D4^k$ we have $4^{k-i}C(i, x) \leq \phi((i, x, 1)) \leq 4^{k-i}$ and $4^{k-i}(1 - C(i, x)) \leq \phi((i, x, 2)) \leq 4^{k-i}$. Again, this can be proved by induction on i.

This means: every maximum flow in N "simulates" the computations made in C. In particular, for $v_0 = (k, x_0)$ we have the following:

- There exists a flow ϕ in N such that $\phi(s) = \phi(t) = D4^k$ and if $C(v_0) = 1$ then $\phi((k, x_0, 2)) = 0$.
- For every flow ϕ in N such that $\phi(s) = \phi(t) = D4^k$ we have if $C(v_0) = 0$ then $\phi((k, x_0, 2)) = 1$.

Now we delete the edge from $(k, x_0, 2)$ to t. In such a way we obtain a new flow network N'. Obviously, we have:

- If $C(v_0) = 1$ then the maximum flow in N' is $D4^k$.
- If $C(v_0) = 0$ then the maximum flow in N' is $D4^k - 1$.

Consequently, C is in MLCVE2 if and only if $(N', D4^k)$ is in BANF. This reduction can easily be performed in logspace. □

The above reduction rests on the fact that edge capacities can be exponential in the size of the network. A reduction using only polynomial edge capacities would show that the unary network flow problem is logspace m-complete for P. Since this problem is in RNC such a reduction is not believed to exist. However, if the circuit C in the proof has logarithmic depth then the capacities can be presented in unary notation.

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