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Analytic Solutions of an Iterative Functional Differential Equation

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This paper is concerned with an iterative functional differential equation $x'(x^{[r]}(z)) = c_0 z + c_1 x(z) + c_2 x(x(z)) + \cdots + c_m x^{[m]}(z)$, where r and m are nonnegative integers, $x^{[0]}(z) = z$, $x^{[1]}(z) = x(z)$, $x^{[3]}(z) = x(x(x(z)))$, etc. are the iterates of the function $x(z)$, and $\sum_{j=0}^{m} c_j \neq 0$. By constructing a convergent power series solution $y(z)$ of a companion equation of the form $\alpha y'(\alpha^{r+1} z) = y'(\alpha^r z) \sum_{j=0}^{m} c_j y(\alpha^j z)$, analytic solutions of the form $y(\alpha y^{-1}(z))$ for the original differential equation are obtained. © 2001 Academic Press

Key Words: functional differential equation; analytic solution.

In the last few years there has been a growing interest in studying functional equations with state dependent delay. We refer the reader to the papers by Eder [1], Feckan [2], Wang [3], and Stanek [4]. In [5], the authors considered the equation

$$x'(z) = x^{[m]}(z)$$

and established sufficient conditions for the existence of analytic solutions. In [7, 8], Minsker studied the asymptotic property of the equation

$$a'(a(x)) = a(x)/x.$$

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In this paper, we will be concerned with a class of iterative functional differential equations of the form

$$x'(x^{[r]}(z)) = c_0 z + c_1 x(z) + c_2 x(x(z)) + \cdots + c_m x^{[m]}(z), \tag{1}$$

where r and m are nonnegative integers, $c_0, c_1, \ldots, c_m$ are complex constants, and $\sum_{j=0}^{m} c_j \neq 0$. By means of the method of majorant series, we construct analytic solutions for our equations in a neighborhood of the complex number $\alpha/(\sum_{j=0}^{m} c_j)$, where α satisfies one of the following conditions:

(H1) $|\alpha| > 1$.

(H2) $0 < |\alpha| < 1$.

(H3) $|\alpha| = 1$, α is not a root of unity, and

$$\log \frac{1}{|\alpha^n - 1|} \leq \mu \log n, \qquad n = 2, 3, \ldots,$$

for some positive constant μ.

The technique for obtaining such solutions is as follows. We first seek an analytic solution $y(z)$ for the initial value problem

$$\alpha y'(\alpha^{r+1} z) = y'(\alpha^r z) \sum_{j=0}^{m} c_j y(\alpha^j z), \tag{2}$$

$$y(0) = \frac{\alpha}{\sum_{j=0}^{m} c_j}. \tag{3}$$

Then we show that

$$x(z) = y(\alpha y^{-1}(z)) \tag{4}$$

is an analytic solution of (1) in a neighborhood of $\alpha/(\sum_{j=0}^{m} c_j)$. Here $y^{-1}(z)$ denotes the inverse function of $y(z)$. Finally, we make use of (4) to show how to derive an explicit power series solution.

First of all, we seek a solution of (2) in a power series of the form

$$y(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{5}$$

where $b_0 = \alpha/(\sum_{j=0}^{m} c_j)$. By substituting (5) into (2), we see that

$$\left(\alpha - b_0 \sum_{j=0}^{m} c_j\right) b_1 = 0, \tag{6}$$

$$(n+1)\left(\alpha^{(r+1)n+1} - \alpha^{rn+1}\right) b_{n+1} = \sum_{k=0}^{n-1} \left(\sum_{j=0}^{m} c_j \alpha^{j(n-k)+rk}\right)(k+1) \times b_{k+1} b_{n-k}, \; n = 1, 2, \ldots. \tag{7}$$

So $b_1 = \eta \in C$ and the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by (7) in a unique manner. This implies that for (2) there exists a formal power series solution

$$y(z) = \frac{\alpha}{\sum_{j=0}^{m} c_j} + \eta z + \sum_{n=2}^{\infty} b_n z^n. \tag{8}$$

Next, we show that such a power series solution is majorized by a convergent power series. We begin with the following preparatory lemma, the proof of which can be found in [6, Chap. 6].

LEMMA 1. *Assume that* $(H3)$ *holds. Then there is a positive number* δ *such that* $|\alpha^n - 1|^{-1} < (2n)^{\delta}$ *for* $n = 1, 2, \ldots$. *Furthermore, the sequence* $\{d_n\}_{n=1}^{\infty}$, *defined by* $d_1 = 1$ *and*

$$d_n = \frac{1}{|\alpha^{n-1} - 1|} \max_{\substack{n=n_1+\cdots+n_l, \\ 0<n_1\leq\cdots\leq n_l,\, t\geq 2}} \{d_{n_1} \cdots d_{n_l}\}, \qquad n = 2, 3, \ldots,$$

satisfies

$$d_n \leq (2^{5\delta+1})^{n-1} n^{-2\delta}, \qquad n = 1, 2, \ldots.$$

LEMMA 2. *Suppose* $(H3)$ *holds. Then for* $\eta = 1$, *Eq.* (2) *has an analytic solution of the form* (8) *in a neighborhood of the origin and there exists a positive constant* δ *such that*

$$|y(z)| \leq \left| \frac{\alpha}{\sum_{j=0}^{m} c_j} \right| + \frac{1}{2^{5\delta+1}} \sum_{n=1}^{\infty} \frac{1}{n^{2\delta}}.$$

Proof. In view of (8), we define $b_1 = \eta = 1$. Since $0 \leq k \leq n-1$ and

$$\left| \frac{\left(\sum_{j=0}^{m} c_j \alpha^{j(n-k)+rk}\right)(k+1)}{(n+1)(\alpha^{(r+1)n+1} - \alpha^{rn+1})} \right| = \left| \frac{\left(\sum_{j=0}^{m} c_j \alpha^{(j-r)(n-k)}\right)(k+1)}{(\alpha^n - 1)(n+1)} \right|$$
$$\leq \frac{\sum_{j=0}^{m} |c_j|}{|\alpha^n - 1|}, \qquad n \geq 2, \tag{9}$$

thus

$$|b_{n+1}| \leq \frac{\sum_{j=0}^{m} |c_j|}{|\alpha^n - 1|} \sum_{k=0}^{n-1} |b_{k+1}||b_{n-k}|$$
$$= \frac{\sum_{j=0}^{m} |c_j|}{|\alpha^n - 1|} \sum_{\substack{n_1+n_2=n+1; \\ 1\leq n_1,\, n_2\leq n}} |b_{n_1}||b_{n_2}|, \qquad n = 1, 2, \ldots.$$

Let us now consider the function

$$G(z) = \frac{1}{2(\sum_{j=0}^{m} |c_j|)} \left\{ 1 - \sqrt{1 - 4\left(\sum_{j=0}^{m} |c_j|\right) z} \right\},$$

which, in view of the binomial series expansion, can be written as

$$G(z) = z + \sum_{n=2}^{\infty} B_n z^n$$

for $|z| < 1/4(\sum_{j=0}^{m} |c_j|)$. Since $G(z)$ satisfies the equation

$$\left(\sum_{j=0}^{m} |c_j|\right) G^2(z) - G(z) + z = 0,$$

thus, by the method of undetermined coefficients, it is not difficult to see that the coefficient sequence $\{B_n\}_{n=1}^{\infty}$ satisfies $B_1 = 1$ and

$$\begin{aligned} B_{n+1} &= \left(\sum_{j=0}^{m} |c_j|\right) \sum_{k=0}^{n-1} B_{k+1} B_{n-k} \\ &= \left(\sum_{j=0}^{m} |c_j|\right) \sum_{\substack{n_1+n_2=n+1; \\ 1 \le n_1,\, n_2 \le n}} B_{k+1} B_{n-k}, \qquad n = 1, 2, \dots . \end{aligned}$$

Hence by induction, we easily see that

$$|b_n| \le B_n d_n, \qquad n = 1, 2, \dots,$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined in Lemma 1.

Since $G(z)$ converges in the open disc $|z| < 1/4(\sum_{j=0}^{m} |c_j|)$, there exists a positive T such that

$$B_n \le T^n$$

for $n = 1, 2, \dots$. In view of this and Lemma 1, we finally see that

$$|b_n| \le T^n Q^{n-1} n^{-2\delta}, \qquad n = 1, 2, \dots,$$

where $Q = 2^{5\delta+1}$, which shows that the series (8) converges for $|z| < (TQ)^{-1}$.

Next, note that $|z| \le (TQ)^{-1}$,

$$\begin{aligned} |y(z)| &\le \left| \frac{\alpha}{\sum_{j=0}^{m} c_j} \right| + \sum_{n=1}^{\infty} |b_n||z|^n \le \left| \frac{\alpha}{\sum_{j=0}^{m} c_j} \right| + \sum_{n=1}^{\infty} B_n d_n |z|^n \\ &\le \left| \frac{\alpha}{\sum_{j=0}^{m} c_j} \right| + \sum_{n=1}^{\infty} T^n Q^{n-1} n^{-2\delta} (TQ)^{-n} \\ &= \left| \frac{\alpha}{\sum_{j=0}^{m} c_j} \right| + \frac{1}{Q} \sum_{n=1}^{\infty} \frac{1}{n^{2\delta}} \end{aligned}$$

as required. The proof is complete.

LEMMA 3. *Suppose (H1) holds. Then for any $r \geq m$ and any complex number $\eta \neq 0$, Eq. (2) has an analytic solution of the form (8) in a neighborhood of the origin and there exists a positive constant M such that for z in this neighborhood,*

$$|y(z)| \leq \left|\frac{\alpha}{\sum_{j=0}^{m} c_j}\right| + \frac{1}{2M}.$$

Proof. As in the proof of Lemma 2, we seek a power series solution of the form (8). By defining $b_0 = \alpha/(\sum_{j=0}^{m} c_j)$ and $b_1 = \eta \neq 0$, we see that (7) again holds. Furthermore, since $r \geq m$, $0 \leq k \leq n-1$, $|\alpha| > 1$, and

$$\left|\frac{\left(\sum_{j=0}^{m} c_j \alpha^{j(n-k)+rk}\right)(k+1)}{(n+1)(\alpha^{(r+1)n+1} - \alpha^{rn+1})}\right| = \left|\frac{\left(\sum_{j=0}^{m} c_j \alpha^{(j-r)(n-k)}\right)(k+1)}{\alpha(\alpha^n - 1)(n+1)}\right|$$
$$\leq \frac{\sum_{j=0}^{m} |c_j|}{|\alpha^n - 1|} \leq M, \qquad n \geq 2,$$

for some positive number M, thus if we define a sequence $\{D_n\}_{n=1}^{\infty}$ by $D_1 = |\eta|$ and

$$D_{n+1} = M \sum_{k=0}^{n-1} D_{k+1} D_{n-k}, \qquad n = 1, 2, \ldots,$$

then $|b_n| \leq D_n$ for $n = 1, 2, \ldots$. Now if we define

$$G(z) = \sum_{n=1}^{\infty} D_n z^n,$$

then

$$G^2(z) = \sum_{n=2}^{\infty} (D_1 D_{n-1} + D_2 D_{n-2} + \cdots + D_{n-1} D_1) z^n$$
$$= \sum_{n=1}^{\infty} (D_1 D_n + D_2 D_{n-1} + \cdots + D_n D_1) z^{n+1}$$
$$= \frac{1}{M} \sum_{n=1}^{\infty} D_{n+1} z^{n+1} = \frac{1}{M} G(z) - \frac{1}{M} |\eta| z.$$

Hence

$$G(z) = \frac{1}{2M}\left\{1 \pm \sqrt{1 - 4M|\eta|z}\right\}.$$

However, since $G(0) = 0$, only the negative sign of the square root is possible, so that

$$G(z) = \frac{1}{2M}\left\{1 - \sqrt{1 - 4M|\eta|z}\right\}.$$

It follows that the power series $G(z)$ converges for $|z| < 1/(4M|\eta|)$; thus (8) is also convergent in the neighborhood of the origin.

Next, note that for $|z| \le 1/(4M|\eta|)$,

$$\frac{1}{G(|z|)} = \frac{2M}{1-\sqrt{1-4M|\eta||z|}} = \frac{1+\sqrt{1-4M|\eta||z|}}{2|\eta||z|} \ge \frac{1}{2|\eta||z|}$$

or

$$G(|z|) \le 2|\eta||z| \le 2|\eta|\frac{1}{4M|\eta|} = \frac{1}{2M}.$$

Thus

$$|y(z)| \le \left|\frac{\alpha}{\sum_{j=0} c_j}\right| + \sum_{n=1}^{\infty} |b_n||z|^n \le \left|\frac{\alpha}{\sum_{j=0}^{m} c_j}\right| + \sum_{n=1}^{\infty} B_n|z|^n$$
$$= \left|\frac{\alpha}{\sum_{j=0}^{m} c_j}\right| + G(|z|) \le \left|\frac{\alpha}{\sum_{j=0}^{m} c_j}\right| + \frac{1}{2M}$$

as required. The proof is complete.

LEMMA 4. *Suppose* (*H*2) *holds. Then for* $r \le m$, $c_0 = 0, \dots, c_{r-1} = 0$, *and any complex number* $\eta \ne 0$, *Eq.* (2) *has an analytic solution of the form* (8) *in a neighborhood of the origin and there exists a positive constant M such that for z in this neighborhood*,

$$|y(z)| \le \left|\frac{\alpha}{\sum_{j=r}^{m} c_j}\right| + \frac{1}{2M}.$$

Proof. As in the proof of Lemma 2, we seek a power series solution of the form (8). By defining $b_0 = \alpha/(\sum_{j=r}^{m} c_j)$ and $b_1 = \eta \ne 0$, we see that (7) again holds. Furthermore, since $r \le m$, $0 \le k \le n-1$, $0 < |\alpha| < 1$, and

$$\left|\frac{\left(\sum_{j=r}^{m} c_j \alpha^{j(n-k)+rk}\right)(k+1)}{(n+1)(\alpha^{(r+1)n+1} - \alpha^{rn+1})}\right| = \left|\frac{\left(\sum_{j=r}^{m} c_j \alpha^{(j-r)(n-k)}\right)(k+1)}{\alpha(\alpha^n - 1)(n+1)}\right|$$
$$\le \frac{\sum_{j=r}^{m} |c_j|}{|\alpha^n - 1|} \le M, \qquad n \ge 2,$$

for some positive number M, thus if we define a sequence $\{D_n\}_{n=1}^{\infty}$ by $D_1 = |\eta|$ and

$$D_{n+1} = M \sum_{k=0}^{n-1} D_{k+1} D_{n-k}, \qquad n = 1, 2, \dots,$$

then $|b_n| \le D_n$ for $n = 1, 2, \dots$. Moreover, we infer as in the proof of Lemma 3 that the power series (8) is convergent in the neighborhood of the origin and

$$|y(z)| \le \left|\frac{\alpha}{\sum_{j=r}^{m} c_j}\right| + \frac{1}{2M}.$$

We now state and prove our main result.

THEOREM 1. *Consider the following three hypotheses*:

(i) $(H3)$ *holds*.

(ii) $(H1)$ *holds and* $r \geq m$.

(iii) $(H2)$ *holds*, $r \leq m$, $c_0 = 0, \ldots, c_{r-1} = 0$.

Suppose one of conditions (i)–(iii) *is fulfilled. Then Eq.* (1) *has an analytic solution* $x(z) = y(\alpha y^{-1}(z))$ *in a neighborhood of* $\alpha/(\sum_{j=0}^m c_j)$, *where* $y(z)$ *is an analytic solution of Eq.* (2). *Furthermore, when* (i) *holds, there is a positive number* δ *such that*

$$|x(z)| \leq \left|\frac{\alpha}{\sum_{j=0}^m c_j}\right| + \frac{1}{Q}\sum_{n=1}^{\infty}\frac{1}{n^{2\delta}}$$

in a neighborhood of $\alpha/(\sum_{j=0}^m c_j)$. *When* (ii) *or* (iii) *holds, there is a positive constant* M *such that*

$$|x(z)| \leq \left|\frac{\alpha}{\sum_{j=0}^m c_j}\right| + \frac{1}{2M}$$

in a neighborhood of $\alpha/(\sum_{j=0}^m c_j)$.

Proof. In view of Lemmas 2–4, we may pick a complex number $\eta \neq 0$ and find a sequence $\{b_n\}_{n=2}^{\infty}$ such that the function $y(z)$ defined by (8) is an analytic solution of (2) in a neighborhood of the origin. Since $y'(0) = \eta \neq 0$, the inverse function $y^{-1}(z)$ is analytic in a neighborhood of the point $y(0) = \alpha/(\sum_{j=0}^m c_j)$. If we now define $x(z)$ by means of (4), then

$$x'(x^{[r]}(z)) = \frac{\alpha y'(\alpha^{r+1}y^{-1}(z))}{y'(\alpha^r y^{-1}(z))} = \sum_{j=0}^{m} c_j y(\alpha^j y^{-1}(z)) = \sum_{j=0}^{m} c_j x^{[j]}(z)$$

as required.

Next, if either (ii) or (iii) holds, then in view of Lemma 3 or Lemma 4,

$$|x(z)| = |y(\alpha y^{-1}(z))| \leq \left|\frac{\alpha}{\sum_{j=0}^m c_j}\right| + \frac{1}{2M};$$

if (i) holds, then in view of Lemma 2,

$$|x(z)| = |y(\alpha y^{-1}(z))| \leq \left|\frac{\alpha}{\sum_{j=0}^m c_j}\right| + \frac{1}{Q}\sum_{n=1}^{\infty}\frac{1}{n^{2\delta}}.$$

The proof is complete.

We now show how to explicitly construct an analytic solution of (1) by means of (4). By means of Lemmas 2–4, Eq. (2) has an analytic solution of the form

$$y(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where $b_0 = \alpha/(\sum_{j=0}^{m} c_j)$, $b_1 = \eta \neq 0$, and $\{b_n\}_{n=2}^{\infty}$ is determined by (7). We calculate the first few terms of the coefficients b_n by means of (7):

$$b_2 = \frac{y''(0)}{2!} = \frac{\sum_{j=0}^{m} c_j \alpha^j}{2\alpha^{r+1}(\alpha - 1)} \eta^2,$$

$$b_3 = \frac{y'''(0)}{3!} = \frac{\sum_{i=0}^{m} \sum_{j=0}^{m} c_i c_j \alpha^{i+j}(\alpha^j + 2\alpha^r)}{6\alpha^{3r+2}(\alpha + 1)(\alpha - 1)^2} \eta^3,$$

$$\vdots \quad .$$

Next, by calculating the derivatives of both sides of (4), we obtain successively

$$x'(z) = \frac{\alpha y'(\alpha y^{-1}(z))}{y'(y^{-1}(z))},$$

$$x''(z) = \frac{\alpha^2 y''(\alpha y^{-1}(z)) y'(y^{-1}(z)) - \alpha y'(\alpha y^{-1}(z)) y''(y^{-1}(z))}{[y'(y^{-1}(z))]^3},$$

$$\begin{aligned} x'''(z) = \Big\{ & \alpha^3 y'''(\alpha y^{-1}(z))[y'(y^{-1}(z))]^3 - \alpha y'(\alpha y^{-1}(z)) y'''(y^{-1}(z)) \\ & \times [y'(y^{-1}(z))]^2 - 3\alpha^2 y''(\alpha y^{-1}(z)) y''(y^{-1}(z))[y'(y^{-1}(z))]^2 \\ & + 3\alpha y'(\alpha y^{-1}(z))[y''(y^{-1}(z))]^2 y'(y^{-1}(z)) \Big\} \Big/ [y'(y^{-1}(z))]^6, \end{aligned}$$

so that

$$x\left(\frac{\alpha}{\sum_{j=0}^{m} c_j}\right) = \frac{\alpha}{\sum_{j=0}^{m} c_j},$$

$$x'\left(\frac{\alpha}{\sum_{j=0}^{m} c_j}\right) = \alpha,$$

$$x''\left(\frac{\alpha}{\sum_{j=0}^{m} c_j}\right) = \sum_{j=0}^{m} c_j \alpha^{j-r},$$

$$x'''\left(\frac{\alpha}{\sum_{j=0}^{m} c_j}\right) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{m} c_i c_j \alpha^{i+j}(\alpha^j + 2\alpha^r) - 3\alpha^r (\sum_{j=0}^{m} c_j \alpha^j)^2}{\alpha^{3r+1}(\alpha - 1)},$$

$$\vdots \quad .$$

Thus, the desired solution is

$$x(z) = \frac{\alpha}{\sum_{j=0}^{m} c_j} + \alpha\left(z - \frac{\alpha}{\sum_{j=0}^{m} c_j}\right) + \frac{\sum_{j=0}^{m} c_j \alpha^{j-r}}{2!}\left(z - \frac{\alpha}{\sum_{j=0}^{m} c_j}\right)^2 + \frac{\sum_{i=0}^{m}\sum_{j=0}^{m} c_i c_j \alpha^{i+j}(\alpha^j + 2\alpha^r) - 3\alpha^r(\sum_{j=0}^{m} c_j \alpha^j)^2}{3!\alpha^{3r+1}(\alpha - 1)} \times \left(z - \frac{\alpha}{\sum_{j=0}^{m} c_j}\right)^3 + \cdots.$$

REFERENCES

1. E. Eder, The functional differential equation $x'(t) = x(x(t))$, *J. Differential Equations* **54** (1984), 390–400.
2. E. Feckan, On certain type of functional differential equations, *Math. Slovaca* **43** (1993), 39–43.
3. K. Wang, On the equation $x'(t) = f(x(x(t)))$, *Funkcial. Ekvac.* **33** (1990), 405–425.
4. S. Stanek, On global properties of solutions of functional differential equation $x'(t) = x(x(t)) + x(t)$, *Dynam. Systems Appl.* **4** (1995), 263–278.
5. J. G. Si, W. R. Li, and S. S. Cheng, Analytic solutions of an iterative functional differential equation, *Comput. Math. Appl.* **33**(6) (1997), 47–51.
6. M. Kuczma, "Functional Equations in a Single Variable," Polish Scientific Publishers, Warsaw, 1968.
7. S. Minsker, On splitting the area under curves, *J. Differential Equations* **26** (1977), 443–457.
8. S. Minsker, On splitting the area under curves II, *J. Differential Equations* **45** (1982), 182–190.