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Group Representations Arising from Lorentz Conformal Geometry

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It is shown that there exist conformally covariant differential operators $D_{2l,k}$ of all even orders $2l$, on differential forms of all orders k, in the double cover $\bar{\bar{M}}^n$ of the n-dimensional compactified Minkowski space $\bar{M}^n$. These act as intertwining differential operators for natural representations of $O(2, n)$, the conformal group of $\bar{M}^n$. For even n, the resulting decompositions of differential form representations of $O^\uparrow(2, n)$, the *orthochronous* conformal group, produce infinite families of unitary representations, the most interesting of which are carried by "positive mass-squared, positive frequency" quotients for $2l \geqslant |n-2k|$. Physically, these generalize unitary representations of the conformal group associated with the modified wave operator $D_{2,0} = \square + ((n-2)/2)^2$, and the Maxwell operator on vector potentials $D_{2,(n-2)/2} = \delta d$. All the representation spaces produced, unitary and nonunitary, may be viewed as infinite systems of harmonic oscillators. As a by-product of the spectral resolution of the $D_{2l,k}$, one gets some striking wave propagative properties for all of the equations $D_{2l,k}\ \Phi = 0$, including Huygens' principle in the curved spacetime $\bar{\bar{M}}^n$. The operators $D_{2l,k}$ have not been seen before except in the special cases $k=0$ or n, and $k=(n\pm 2)/2$, $l=1$ (the Maxwell operator). Thus much new information is obtained even in the physical case $n=4$.

0. INTRODUCTION

In classical physics, particles are identified with *field equations* like the wave, Maxwell, and Dirac equations. A field equation should be invariant

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under some fundamental spacetime symmetry group G; for example, the Lorentz, scale-extended Poincaré, or conformal group. One then gets a representation u of $G^{\uparrow}$, the *orthochronous* (preserving the forward direction of time) subgroup of G, in a space X of positive frequency solutions to the field equation. Once realized as a *unitary* representation, i.e., once X is supplied with a positive definite invariant complex inner product and completed to a complex Hilbert space $\mathscr{H}$, the pair $(\mathscr{H}, u)$ gives the quantum mechanical picture of the particle: classical observables like the energy and angular momenta go over to operators given by the infinitesimal generators of G in the representation. Historically, this construction was made explicit for G the Lorentz group in, for example, the work of Bargmann and Wigner [2, 41]. Since that time, the group representation aspect of a particle has come to be regarded as more fundamental than the field equation aspect. This point of view has provided at least some of the motivation for the central problem in group representation theory, the classification of irreducible unitary representations.

In the present paper, we adopt an approach directly motivated by physics to produce decompositions of natural differential form representations of $O^{\uparrow}(2, n)$, the orthochronous conformal group of its homogeneous space $\bar{\bar{M}}^n$, the double cover of the n-dimensional compactified Minkowski space. The methods are primarily those of differential geometry: *intertwining*, or *covariant* differential operators are treated as the fundamental objects. They arise as higher-order generalizations of the wave and Maxwell operators, and of certain wave-Maxwell hybrids, with representation spaces and formulas for nondegenerate complex inner products naturally attached. These inner products are then easily tested for definiteness.

Some remarks are in order concerning the role of the *conformal group* in physics. One of the earliest-noted [7, 3, 8] aspects of the standard *massless* field equations was their invariance under the 15-parameter conformal group (locally $O(2, 4)$) of 4-dimensional Minkowski space. Conformal transformations are those which carry the Lorentz metric tensor to a multiple of itself by a positive function. Conformal transformations not only preserve the null hyperfaces (light conoids) but also permute the null geodesics, i.e., the paths of massless particles. Thus one should have conformal invariance of any reasonable massless particle model. *Massive* field equations, on the other hand, result from eigenvalue problems for differential operators D for which $D\Phi = 0$ is conformally invariant. These eigenvalue problems are not themselves conformally invariant, but exhibit invariance only under some smaller group like the Poincaré group.

An important mathematical device in the present work is the replacement of n-dimensional Minkowski space M^n by its *conformal compactification* $\bar{M}^n = (S^1 \times S^{n-1})/\mathbb{Z}_2$, obtained by adding a light cone at infinity, and by various covering spaces of $\bar{M}^n$. $\bar{M}^n$ is the "compact picture" of M^n in much the same way as the Riemann sphere S^2 is the compact picture of the complex plane. $\bar{M}^n$ was used by Dirac in [8], is central to the Penrose Twistor program [53], and is a basic ingredient in the far-reaching and highly predictive Chronometric Theory of Segal [38]. One major effect of looking at the compact picture is that all our conformally covariant differential operators have discrete spectra. In the Chronometric Theory, this circumstance for the wave operator $\square + ((n-2)/2)^2$ implies a discrete set of admissible particle masses [25, 17]. As a by-product of the spectral resolution of the operators considered here, we get some striking wave propagative results.

The main results of the present paper are as follows:

(1) Let $\bar{\bar{M}}^n$ be the double cover of $\bar{M}^n$, i.e., $S^1 \times S^{n-1}$ with the Lorentz metric $g = -g_{S^1} + g_{S^{n-1}}$. Let d be the exterior derivative on forms in $\bar{\bar{M}}^n$ and δ its formal adjoint. Then for n even and $l = 1, 2, 3, \ldots$, there exist differential operators $D_{2l,k}$ on k-forms in $\bar{\bar{M}}^n$, $0 \leqslant k \leqslant n$, with leading term

$$\frac{n-2k+2l}{2}(\delta d)^l + \frac{n-2k-2l}{2}(d\delta)^l$$
$$= \left(\frac{n-2k+2l}{2}\delta d + \frac{n-2k-2l}{2} d\delta\right)\square^{l-1},$$

where $\square = \delta d + d\delta$, which are covariant under conformal transformations in the same sense as the modified wave operator $D_{2,0} = \square + ((n-2)/2)^2$ on functions and the Maxwell operator $D_{2,(n-2)/2} = \delta d$. ($D_{2,0}$ is the realization in $\bar{\bar{M}}^n$ of the general conformally covariant operator $\square + (n-2)R/4(n-1)$, $R =$ scalar curvature [26]. $D_{2,(n-2)/2}$ is the Maxwell operator on "vector potentials.") Conformal covariance on $\bar{\bar{M}}^n$ is an interesting property because the conformal group $O(2, n)$ of $\bar{\bar{M}}^n$ has the maximal dimension (Sect. 1.b). The $D_{2l,k}$ can actually be defined for any n, and on any covering space of $\bar{M}^n$; and the proof of covariance given extends to these situations after slight modifications. In particular, the $D_{2l,k}$ are defined and conformally covariant on the universal cover $\tilde{M}^n$ of $\bar{\bar{M}}^n$, i.e., $\mathbb{R} \times S^{n-1}$ with the metric $g = -g_{\mathbb{R}} + g_{S^{n-1}}$.

(2) Suppose n is even, $l \neq \pm(n-2k)/2$, and either

$$l < (n-2)/2 \tag{0.1}$$

and

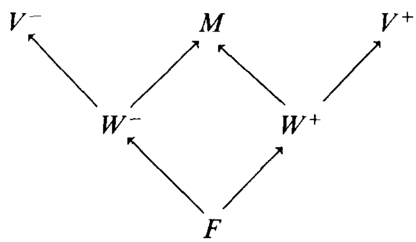
$$l < n/2, \qquad k \neq 0, 1, n-1, n. \tag{0.2}$$

Then all solutions of $D_{2l,k}\Phi=0$ on $\tilde{M}^n$ are 2π-periodic in t, the parameter on $\mathbb{R}$, i.e., they "live on" $\bar{\bar{M}}^n$; solutions are either odd or even under the product of S^1 and S^{n-1} antipodal maps, according to whether $(n-2k-2l)/2$ is odd or even; and the equation $D_{2l,k}\Phi$ satisfies Huygens' principle. If neither (0.1) nor (0.2) is satisfied, all solutions are of the form $\Phi+t\Psi$, where Φ and Ψ are 2π-periodic solutions (though not all fields of this form are solutions); but all *periodic* solutions still have parity $(-1)^{(n-2k-2l)/2}$ under the product of antipodal maps. In the case $l=(n-2k)/2$, $D_{2l,k}$ has the form $\delta\mathscr{D}d$, where $\mathscr{D}$ is a differential operator with leading term $(d\delta)^{l-1}$, and the higher-order "Maxwell" system

$$\begin{aligned} dF&=0\\ \delta\mathscr{D}F&=0 \end{aligned} \tag{0.3}$$

on $(k+1)$-forms F is conformally invariant. Solutions of (0.3) are automatically 2π-periodic in t, and even under the product of antipodal maps in $\bar{\bar{M}}^n$; the system (0.3) satisfies Huygens' principle. (The case $l=-(n-2k)/2$ carries the same information via the Hodge $*$ operator.)

(3) Suppose n is even, and $l\neq(n-2k)/2$. For a certain multiplier representation u of the orthochronous conformal transformation group $G=O^{\uparrow}(2,n)$ of $\bar{\bar{M}}^n$, there exist $u(G)$-invariant subspaces



(0.4)

of the space $\mathscr{D}_k$ of C^∞ k-forms in $\bar{\bar{M}}^n$, where the arrows represent inclusions; with F finite dimensional, $W^++W^-=\mathscr{N}(D_{2l,k})$, and $V^++V^-+M=\mathscr{D}_k$. The resulting representations on W^+/F, W^-/F, V^+/W^+, V^-/W^-, and $M/(W^++W^-)$ all admit $u(G)$-invariant nondegenerate complex inner products. The inner products on $W^\pm/F$ are definite exactly when $k=0$ or n and $l=1$; those on $M/(W^++W^-)$ exactly when $k=0$ or n. The inner products on $V^\pm/W^\pm$ are definite exactly when $k=0$ or n, *or*

$$2l>|n-2k|. \tag{0.5}$$

All definite inner product spaces result, after completion, in continuous unitary representations of G. If $l=(n-2k)/2$, the subspaces of (0.4) are $u(G)$-invariant, as is $\mathscr{G}=\mathscr{N}(d)$. Invariant nondegenerate complex inner products are carried by $(W^+ + \mathscr{G})/\mathscr{G}$, $(W^- + \mathscr{G})/\mathscr{G}$, $(V^+ + \mathscr{G})/(W^+ + \mathscr{G})$, $(V^- + \mathscr{G})/(W^- + \mathscr{G})$, and $(M+\mathscr{G})/(W^+ + W^- + \mathscr{G})$. The inner product on $(W^\pm + \mathscr{G})/\mathscr{G}$ is definite only for $l=1$; that on $(M+\mathscr{G})/(W^+ + W^- + \mathscr{G})$ only for $k=0$. The inner product on $(V^\pm + \mathscr{G})/(W^\pm + \mathscr{G})$ is *always* definite, and again, all definite inner products result in continuous unitary representations. (The case $l= -(n-2k)/2$ is dual under the Hodge $*$.)

In the exceptional case $l=(n-2k)/2$, the system (0.3) may be regarded as a higher-order generalization of the Maxwell field equations, with $\mathscr{G}$ the space of "pure gauges." In the case of $D_{2,0}$, W^+ and W^- are the spaces of positive and negative frequency wave functions, while V^+/W^+ is the positive mass-squared, positive frequency quotient. An interesting feature of the representations determined by the $D_{2l,k}$ is that the *massive* representation (on V^+/W^+) is sometimes unitary even when the *massless* representation (on W^+/F) is not.

The representation theoretic part of present work is motivated partly by the following recent work:

(1) In [18], Jakobsen and Vergne showed that powers $\Box^l$ of the ordinary d'Alembertian on functions in 4-dimensional Minkowski space are conformally covariant, and used these operators to decompose representations of $SU(2,2)$ (locally $O(2,4)$). (2) In [40], Speh determined the full composition series for scalar field representations of $SO(2,4)$. Also working in the scalar case, Molčanov [52] constructed the intertwining operators $D_{2l,0}$, realizing them as integral intertwining operators (without proving that they are *differential* operators). He also tested the resulting composition factors for unitarity. (3) in [25, 26], Ørsted used the wave and Maxwell operators to decompose representations of $O(2,n)$ and its subgroups, and obtained information along the same lines for $O(p,q)$. (4) In [31–33], Paneitz and Segal found full composition series for all the relevant scalar and spinor representations in $\tilde{M}^4$ and its covering spaces, and for the differential form representations associated to the Maxwell equations. This work in the 4-dimensional case used to great advantage the group structure of $S^3=SU(2)$, which results in the triviality of the bundles involved. The present work in the arbitrary even-dimensional case may be regarded as, among other things, a step toward determining which results can be freed from reliance on this special circumstance, but we also obtain much new information in dimension 4. (5) In [4], in the setting of a general pseudo-Riemannian manifold of dimension $n\neq 1,2$, the present author introduced a second-order conformally covariant operator $D_{2,k}$ on forms of any order k. The main idea was to form a "Laplacian" in which

the δd and $d\delta$ terms are weighted differently; the covariant operator is then obtained by "correcting" with a zeroth-order operator depending on the Ricci tensor

$$D_{2,k} = \frac{n-2k+2}{2}\,\delta d + \frac{n-2k-2}{2}\,d\delta + Z_{\text{Ricci}}. \tag{0.6}$$

(In $\bar{\bar{M}}^n$, this operator and its natural higher-order generalizations are the keys to the composition structure of the differential form representations.) Later, Paneitz [34] found a general fourth-order operator $D_{4,0}$ on functions which is conformally covariant in pseudo-Riemannian manifolds of dimension $n \neq 1, 2$, and the present author [6] generalized this to a $D_{4,k}$ on forms for $n \neq 1, 2, 4$. The dimension constraints $n \neq 2, 4$ turn out not to be a factor for very symmetric manifolds like covers of $\bar{M}^n$.

The wave propagative results in this paper (automatic periodicity, oddness/evenness in $\bar{\bar{M}}^n$, and Huygens' principle) generalize the results of Lax and Phillips [24] on the wave equation $D_{2,0}\Phi = 0$, and of Ørsted [27] on the wave and Maxwell equations. The proofs are adaptations of the Lax–Phillips approach. It is interesting to note that $D_{2,k}$ seems to be *exactly* the correct second order differential operator on forms to produce the automatic periodicity results.

The present representation theoretic work is at once more general and less complete than the recent works cited above in (1)–(4). No general approach to the decomposition of the differential form representations seems to have existed previously. The construction of the intertwining differential operators $D_{2l,k}$ in the case of forms $(k \neq 0, n)$ is considerably more involved than in the scalar case, because one must keep track of four different Hodge-theoretic "sectors." Previous attempts to construct such operators have been hampered by the natural tendency to expect $\square = \delta d + d\delta$ or $\square^l$ as the leading term. The idea of weighting the δd and $d\delta$ parts of the leading term, inspired by (0.6), turns out to be the right one. However, though we get composition factors roughly analogous to those in the full composition series for scalar representations, we do not prove that the resulting decomposition is complete.

It is possible to use the methods of this paper to study differential form representations of all the $O(p, q)$ for $p, q \geqslant 1$. The results for the Lorentz groups $O(1, n)$ are included here as remarks; the case $p, q \geqslant 3$ will be treated in a later paper [43]. Molčanov's work [52] on the scalar case, in fact, is carried out for general $p, q \geqslant 2$. The theory is richest, however, in the case where p or q is 2, because of the "positive–negative frequency" decomposition, which is ultimately traceable to the existence of a complex structure for $O(2, n)$.

It should be mentioned that there is an alternate approach to the construction of unitary representations of $O^{\uparrow}(2, n)$ which has been the subject of intensive recent work, viz. the classification of unitary highest weight modules of groups with a Hermitian symmetric space. This classification was carried out by Jakobsen in [48], and independently by Enright, Howe, and Wallach in [46]. In principle, all the *unitary* representations constructed here are visible in this classification. After this paper was written, Jakobsen [49] classified "conformal covariants" in dimension 4; in the language of this paper, intertwining differential operators for $O(2, 4)$ on tensor and spinor bundles over $\bar{\bar{M}}^4$. Thus in the special case $n = 4$, the intertwining operators of this paper (and others, on higher spin bundles) can be seen anew. Finally, Angelopoulos [1] has given an inductive classification scheme for irreducible unitary representations of $\widetilde{SO_0}(2, n)$, in which the answer for $\widetilde{SO_0}(2, n)$ depends on that for $\widetilde{SO_0}(2, n-2)$. However, in the special case $n = 4$ [42] (which was also handled by Knapp and Speh in [50]), direct comparison with the unitary representations constructed here is possible.

This paper is organized as follows. In Section 1, we standardize differential geometric notation, introduce the natural multiplier representations of the conformal transformation group of a pseudo-Riemannian manifold, and review the conformal compactification. Section 2 contains the results on $D_{2l,0}$ and the scalar representations; some of these results are already known [52, 40]. Section 3 contains the results on the generic $(l \neq \pm(n-2k)/2)\, D_{2l,k}$, and the differential form representations. The exceptional "Maxwell-like" case $l = \pm(n-2k)/2$ is treated in Section 4. Problems and prospects opened up by the new results are discussed in Section 5.

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1. Differential Geometric Preliminaries

a. *Notation and Conventions*

Let (M, g) be a pseudo-Riemannian manifold, i.e., a C^∞ manifold M equipped with a C^∞ metric tensor $g = (g_{\alpha\beta})$ which is nondegenerate, but not necessarily positive definite. n will always denote the dimension of M. g determines a distinguished metric g^k on the kth exterior bundle $\Lambda^k(M)$, and this gives an L^2-formal adjoint δ for the exterior derivative d. The *d'Alembertian* (*Laplacian* if (M, g) is Riemannian) $\square$ is the operator $\delta d + d\delta$ on differential forms. Our convention is such that $\square = -(\partial/\partial x)^2 - (\partial/\partial y)^2$ on Riemannian $\mathbb{R}^2$.

Exterior multiplication of forms by a one-form η will be denoted $\varepsilon(\eta)$; exterior multiplication by a vector field X is just exterior multiplication by the g-associated one-form $(X_\alpha) = (g_{\alpha\beta} X^\beta)$ (summation convention). *Interior multiplication* of forms by a vector field X will be denoted $\iota(X)$; if η is a one-form, $\iota(\eta)$ is interior multiplication by $(\eta^\alpha) = (g^{\alpha\beta}\eta_\beta)$. $\iota(\eta)$ is the *pointwise* adjoint of $\varepsilon(\eta)$: if φ is a k-form and ψ a $(k-1)$-form, then $g^k(\varphi, \varepsilon(\eta)\psi) = g^{k-1}(\iota(\eta)\varphi, \psi)$.

The following notational convention will be helpful in a number of places.

DEFINITION 1.1. If $A = (A^\alpha{}_\beta)$ is a type (1, 1) tensor field on M, we denote by $A\#$ the unique type-preserving derivation on the mixed tensor field algebra of M which annihilates functions, commutes with contractions, and has $(A\# X)^\alpha = -A^\alpha{}_\beta X^\beta$ on vector fields.

Remark 1.2. If $\varphi = (\varphi^{\lambda_1 \cdots \lambda_s}{}_{\mu_1 \cdots \mu_t})$ is a tensor field, then

$$(A\#\varphi)^{\lambda_1 \cdots \lambda_s}{}_{\mu_1 \cdots \mu_t} = -\sum_{i=1}^{s} A^{\lambda_i}{}_\beta \varphi^{\lambda_1 \cdots \overset{\lambda_i \text{place}}{\beta} \cdots \lambda_s}{}_{\mu_1 \cdots \mu_t} + \sum_{j=1}^{t} A^\alpha{}_{\mu_j} \varphi^{\lambda_1 \cdots \lambda_s}{}_{\mu_1 \cdots \underset{\mu_j \text{place}}{\alpha} \cdots \mu_t}.$$

On forms, in a local frame (X_α) and dual coframe (η^β), $A\# = A^\alpha{}_\beta \varepsilon(\eta^\beta)\, i(X_\alpha)$.

If X is a vector field and $\mathscr{L}(X)$ the Lie derivative,

$$\mathscr{L}(X) = i(X)\, d + di(X) \qquad \text{on forms,}$$

and thus

$$\mathscr{L}(X)^* = \delta\varepsilon(X) + \varepsilon(X)\,\delta \qquad \text{on forms.}$$

An easy local calculation [6, Sect. 1.d] now gives

$$\mathscr{L}(X) + \mathscr{L}(X)^* = -\nabla_\alpha X^\alpha + (\nabla X)\# + (\nabla X)^t\# \qquad \text{on forms,} \qquad (1.1)$$

where ∇ is the covariant derivative in the unique symmetric pseudo-Riemannian connection, and in general, $(A^t)^\alpha{}_\beta = A_\beta{}^\alpha$.

If f is a (possibly local) diffeomorphism on M, denote by $f\cdot$ the natural action of f on tensor fields as in [14, p. 90]. On vector fields, $f \cdot X = (df)\, X$; on covariant tensors, including differential forms and the metric tensor, $f\cdot$ acts as $(f^{-1})^*$.

b. *Representations of the Conformal Group*

A *conformal vector field* T on (M, g) is one for which $\mathscr{L}(T)\, g = 2\omega g$, $\omega \in C^\infty(M)$; in classical notation,

$$\nabla_\alpha T_\beta + \nabla_\beta T_\alpha = 2\omega g_{\alpha\beta}. \tag{1.2}$$

Equation (1.2), its contraction $\nabla_\alpha T^\alpha = n\omega$, and (1.1) give

$$\mathscr{L}(T) + \mathscr{L}(T)^* = (2k - n)\, \omega \qquad \text{on } k\text{-forms}. \tag{1.3}$$

If T_1 and T_2 are conformal, $\mathscr{L}(T_i)\, g = 2\omega_i g$, then $[T_1, T_2]$ is conformal with

$$\omega_{[T_1, T_2]} = T_1 \omega_2 - T_2 \omega_1. \tag{1.4}$$

Thus the conformal vector fields form a Lie algebra $\mathfrak{c}(M, g)$. For M connected and $n > 2$, it is shown in [9] that $\dim \mathfrak{c}(M, g) \leqslant (n+1)(n+2)/2$. (The reference works in the Riemannian case, but the proof applies equally well to the pseudo-Riemannian case. See also [20, Notes 9, 11, 13]).

A *conformal transformation* on (M, g) is a (possibly local) diffeomorphism h for which $h \cdot g = \Omega^2 g$, $\Omega > 0 \in C^\infty(M)$. If h_1 and h_2 are conformal, so are $h_1 \circ h_2$ and h_1^{-1},

$$\Omega_{h_1 \circ h_2} = (h_1 \cdot \Omega_2)\, \Omega_1, \; \Omega_{h_1^{-1}} = (\Omega_1 \circ h_1)^{-1}. \tag{1.5}$$

Thus the global conformal transformations $\mathscr{C}(M, g)$ form a group. If M is connected and $n > 2$, $\mathscr{C}(M, g)$ is a Lie transformation group on M, of dimension at most $(n+1)(n+2)/2$ [20, Note 11].

If X is a vector field and $\{f_\varepsilon\}$ is the local one-parameter group of local diffeomorphisms generated by X, then

$$(\mathscr{L}(X)\, \varphi)(x) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (f_{-\varepsilon} \cdot \varphi)(x) \tag{1.6}$$

for $x \in M$ and φ a tensor field on M. Equation (1.6) implies that if each f_ε is conformal, so is X; conversely, (1.6) and the group law $f_\varepsilon \circ f_{\varepsilon'} = f_{\varepsilon+\varepsilon'}$ show that if X is conformal, then each f_ε is, and [25]

$$\omega_X(x) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Omega_{-\varepsilon}(x). \tag{1.7}$$

c. *Covariance*

Let $\mathscr{T}$ be a space of C^∞ tensor fields on M obtained by specifying some covariant–contravariant type, and (possibly) some list of pointwise symmetry and trace conditions invariant under the orthogonal group of g. (For example, we could consider the three-forms or the trace-free symmetric covariant two-tensors.) By (1.4) and (1.5), the maps

$$\mathfrak{c}(M, g) \xrightarrow{U_a} \text{End}\,\mathscr{T}, \qquad U_a(T) = \mathscr{L}(T) + a\omega;$$

$$\mathscr{C}(M, g) \xrightarrow{u_a} \text{Aut}\,\mathscr{T}, \qquad u_a(h) = \Omega^a h\cdot,$$

for $a \in \mathbb{C}$ are homomorphisms. We shall refer to these maps as *representations*, even though $\mathscr{T}$ has not been topologized. By (1.6) and (1.7), U_a is the infinitesimal representation corresponding to u_a in the following sense: if a conformal T integrates to a one-parameter group of *global* conformal transformations $\{h_\varepsilon\}$, then

$$\{U_a(T)\,\varphi\}(x) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \{u_a(h_{-\varepsilon})\,\varphi\}(x). \tag{1.8}$$

If V is a U_a-invariant (resp. u_a-invariant) subspace of $\mathscr{T}$, we shall also denote by U_a (resp. u_a) the corresponding representations on V and $\mathscr{T}/V$. The same notation will be used for spaces obtained by iterated passage to and quotient by invariant subspaces: if V is already considered to carry U_a or u_a and W is an invariant subspace of V, then W and V/W will be considered to carry U_a or u_a. Similar conventions will apply to the restrictions of the u_a to Lie subalgebras $\mathfrak{g}$ of $\mathfrak{c}(M, g)$ and subgroups G of $\mathfrak{c}(M, g)$. *Every representation studied in this paper will arise in this way*, or will result by completion in a pre-Hilbert space inner product from a representation arising in this way.

DEFINITION 1.3. Let $\mathscr{T}$ be as above, and suppose $a, b \in \mathbb{R}$.

(i) A differential operator D: $\mathscr{T} \to \mathscr{T}$ is *infinitesimally conformally covariant of bidegree* (a, b) if

$$DU_a(T)\,\varphi = U_b(T)\,D\varphi$$

for all $T \in \mathfrak{c}(M, g)$, $\varphi \in \mathscr{T}$.

(ii) D is *conformally covariant of bidegree* (a, b) if

$$Du_a(h)\,\varphi = u_b(h)\,D\varphi$$

for all $h \in \mathscr{C}(M, g)$, $\varphi \in \mathscr{T}$.

If D is conformally covariant of bidegree (a, b), then its null space $\mathcal{N}(D)$ carries the representation u_a, and the range $\mathcal{R}(D)$ carries u_b. This latter representation is equivalent to the representation u_a on $\mathcal{T}/\mathcal{N}(D)$. Similar statements hold for infinitesimal covariance. Covariance and the related concept of quasi-invariance on homogeneous spaces, and the consequences for representation theory were first discussed in [23]; other fundamental references are [21, 22, 18].

Remark 1.4 [25]. Suppose D is conformally covariant,

$$[Du_a(h)\,\varphi](x) = [u_b(h)\,D\varphi](x), \qquad \varphi \in \mathcal{T}, x \in M.$$

Letting h run over local one-parameter groups $\{h_\varepsilon\}$, differentiating, and using (1.8), we find that D is infinitesimal conformally covariant:

$$[DU_a(T)\,\varphi](x) = [U_b(T)\,D\varphi](x), \qquad \varphi \in \mathcal{T}, x \in M, \tag{1.9}$$

for all $T \in \mathfrak{c}(M, g)$.

Conversely, suppose D is infinitesimally conformally covariant as in (1.9). If $T \in (M, g)$ generates $\{h_\varepsilon\}$, then [25]

$$\begin{aligned}\frac{d}{d\varepsilon}&[u_b(h_\varepsilon)\,Du_a(h_{-\varepsilon})\,\varphi](x)\\ &= [u_b(h_\varepsilon)\,DU_a(T)\,u_a(h_{-\varepsilon})\,\varphi + u_b(h_\varepsilon)\,U_b(-T)\,Du_a(h_{-\varepsilon})\,\varphi](x)\\ &= [u_b(h_\varepsilon)\{DU_a(T) - U_b(T)\,D\}\,u_a(h_{-\varepsilon})\,\varphi](x).\end{aligned}$$

Since $u_b(h_\varepsilon)\,Du_a(h_{-\varepsilon})\,\varphi|_{\varepsilon=0} = D\varphi$, this shows that

$$Du_a(h_{-\varepsilon})\,\varphi = u_b(h_{-\varepsilon})\,D\varphi.$$

The $\{h_\varepsilon\}$ generate the identity component of $\mathscr{C}(M, g)$, so we can get conformal covariance if we have infinitesimal covariance *and* covariance under one element of each connected component of $\mathscr{C}(M, g)$.

d. *Conformal Compactification*

Let $\mathbb{R}^{(p,q)}$ be the standard signature (p, q) flat space, $p+q=n$, $q \geqslant 1$; i.e., $\mathbb{R}^n$ equipped with coordinate functions $x_{-(p-1)}, \ldots, x_q$ and the metric tensor

$$g = -dx^2_{-(p-1)} - \cdots - dx_0^2 + dx_1^2 + \cdots + dx_q^2.$$

$\mathbb{R}^{(1,n-1)} \equiv M^n$ is the n-dimensional *Minkowski space*. (The rather strange assignment of indices is chosen so that M^4 will have coordinates x_0, x_1, x_2, x_3.) All local conformal transformations on $\mathbb{R}^{(p,q)}$ are generated by the linear isometries $O(p, q)$; the translations $x \mapsto x+b$, $b \in \mathbb{R}^n$; the uniform dilations $x \mapsto ax$, $a>0 \in \mathbb{R}$; and the *inversion* in the unit hyperboloid (or sphere if $p=0$), $\mathscr{I}$: $x \mapsto x/g(x, x)$. (See [38] and the references therein.) Since $\mathscr{I}$ is undefined on $\{g(x, x)=0\}$ (the light cone if $p=1$), this "group" acts with singularities, which can be resolved by passing to the *conformal compactification*, obtained essentially by adding a copy of $\{g(x, x)=0\}$ at infinity.

Consider, as in [26], the real projective space $\mathbb{P}(\mathbb{R}^{n+2})$ with homogeneous coordinate functions $\xi_{-p}, \ldots, \xi_{q+1}$. The map

$$J: \quad \mathbb{R}^{(p,q)} \to \mathbb{P}(\mathbb{R}^{n+2}),$$

$$J(x)=\left[\left(1+\frac{g(x, x)}{4}, x, 1-\frac{g(x, x)}{4}\right)\right],$$

where [ ] denotes equivalence class in projective space, has as its range the projective quadric $Q=\{\xi_{-p}^2+\cdots+\xi_0^2=\xi_1^2+\cdots+\xi_{q+1}^2\}$, minus the set $\{\xi_{-p}+\xi_{q+1}=0\}$. Q is naturally diffeomorphic to $\bar{\mathbb{R}}^{(p,q)}=(S^p \times S^q)/\mathbb{Z}_2$, the $\mathbb{Z}_2$ action coming from the product of antipodal maps. Giving $\bar{\mathbb{R}}^{(p,q)}$ the signature (p, q) metric built from the standard sphere metrics,

$$\bar{g}=-g_{S^p}+g_{S^q},$$

we have J conformal

$$J^* \bar{g}=\left\{\left(1+\frac{g(x, x)}{4}\right)^2+\sum_{i=-(p-1)}^{0} x_i^2\right\}^{-1} g. \tag{1.10}$$

The conformal "group" of $\mathbb{R}^{(p,q)}$, transplanted to $\bar{\mathbb{R}}^{(p,q)}$ by J, now acts as an "honest" group of global transformations. Double covering both the space and the group, we get the action

$$(A, \xi) \mapsto\left\{\sum_{i=-p}^{0}(A \xi)_i^2\right\}^{-1/2} A \xi, \qquad A \in O(p+1, q+1),\ \xi=(\xi_{-p}, \ldots, \xi_{q+1}) \tag{1.11}$$

of $O(p+1, q+1)$ on $\bar{\bar{\mathbb{R}}}^{(p,q)}=(S^p \times S^q, \bar{g})$. It was shown by Klein [19] that this gives the *full* conformal group of $\bar{\bar{\mathbb{R}}}^{(p,q)}$ (note that the dimension is the maximal $(n+1)(n+2)/2$). $O(p+1, q+1)$ has 4 connected components, each containing one element of the Klein-4 subgroup consisting of the identity I, $T=\operatorname{diag}(-1, 1, \ldots, 1, 1)$, $P=\operatorname{diag}(1, 1, \ldots, 1, -1)$, and PT. $SO(p+1, q+1)$ is the union of the I and PT components; the I component, as usual, is denoted $SO_0(p+1, q+1)$.

A basis of the conformal vector fields on $\bar{\bar{\mathbb{R}}}^{(p,q)}$, expressed in homogeneous coordinates, is (summation convention not in force)

$$L_{\alpha\beta} = \varepsilon_\alpha \xi_\alpha \partial_\beta - \varepsilon_\beta \xi_\beta \partial_\alpha, \qquad \alpha, \beta = -p, ..., q+1,$$

where $\partial_\alpha = \partial/\partial\xi_\alpha$, and $-\varepsilon_{-p} = \cdots = -\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{q+1} = 1$. The $L_{\alpha\beta}$ for $-p \leqslant \alpha, \beta \leqslant 0$ generate the $SO(p)$ group of isometries; likewise, the $L_{\alpha\beta}$ for $1 \leqslant \alpha, \beta \leqslant q+1$ generate the $SO(q)$ group of isometries. The "mixed" $L_{\alpha\beta}$, $-p \leqslant \alpha \leqslant 0 < \beta \leqslant q+1$, are *proper* conformal vector fields: $\mathscr{L}(L_{\alpha\beta})\bar{g} = 2\omega_{\alpha\beta}\bar{g}$ with $\omega_{\alpha\beta} \neq 0$.

In this paper, we shall be concerned only with the Minkowski ($p=1$) case, and (to a lesser extent) with the Euclidean ($p=0$) case. The conformal compactification of Minkowski space was first studied in connection with physical field equations by Dirac [8], and is a basic ingredient in the far-reaching Chronometric Theory of Segal [38]. From the points of view of both physics and representation theory, it will be important to distinguish between the conformal group $O(2, n)$ of $\bar{\bar{M}}^n = \bar{\bar{\mathbb{R}}}^{(1,n-1)}$, and the *orthochronous* (preserving the direction of time) conformal group $O^\uparrow(2, n)$, which consists of just the I and P components. The difference is that for representations of $O(2, n)$ on, say, wave and Maxwell fields (Sects. 2.b, 4.a), the time-reversal (essentially T) turns out to be *anti*-unitary; the restriction to $O^\uparrow(2, n)$ gives us a chance at unitarity once positive and negative frequency fields are distinguished.

Remark 1.5. The "typical" proper basic conformal vector field $L_{-1,n}$ can be expressed in intrinsic coordinates on $S^1 \times S^{n-1}$ as follows. Let t be the angular parameter on S^1 ($\xi_{-1} = \cos t$, $\xi_0 = \sin t$), and set

$$\xi_n = \cos\rho, \qquad 0 \leqslant \rho \leqslant \pi.$$

Complete ρ to a set of spherical angular coordinates $(\rho, \theta_1, ..., \theta_{n-2})$ on S^{n-1}, so that ∂_ρ is $g_{S^{n-1}}$-orthogonal to the ∂_{θ_i}. Then

$$L_{-1,n} = \sin t \cos\rho\, \partial_t + \cos t \sin\rho\, \partial_\rho,$$

$$\omega_{-1,n} = \cos t \cos\rho.$$

Note that

$$L_{-1,n} = (\sin t)\, \omega_Y \partial_t + (\cos t)\, Y,$$

where $Y = \sin\rho\, \partial_\rho$ is conformal on Riemannian S^{n-1} with $\omega_Y = \cos\rho$. In fact, Y is the L_{0n} of compactified Euclidean space $\bar{\mathbb{R}}^{(0,n-1)} = (S^{n-1}, g_{S^{n-1}})$.

2. Scalar Representations

a. *Wave Propagation in* $\bar{\bar{M}}^n$

Return for a moment to the setting of an arbitrary n-dimensional pseudo-Riemannian manifold (M, g). It is known [26] that the modified d'Alembertian on functions,

$$D_2 = \square + \frac{n-2}{4(n-1)} R, \qquad R = \text{scalar curvature},$$

is conformally covariant of bidegree $((n-2)/2, (n+2)/2)$,

$$\begin{aligned} D_2 u_{(n-2)/2}(h)\, \varphi &= u_{(n+2)/2}(h)\, D_2 \varphi, \\ D_2 U_{(n-2)/2}(T)\, \varphi &= U_{(n+2)/2}(T)\, D_2 \varphi \end{aligned} \tag{2.1}$$

for $\varphi \in C^\infty(M)$, $h \in \mathscr{C}(M, g)$, and $T \in \mathfrak{c}(M, g)$. The scalar curvature of Riemannian S^{n-1}, and thus of Lorentzian $S^1 \times S^{n-1}$ or $\mathbb{R} \times S^{n-1}$, is $(n-2)(n-1)$. Hence on M^n or its universal cover $\tilde{M}^n = (\mathbb{R} \times S^{n-1}, \bar{g})$,

$$D_2 = \square + \left(\frac{n-2}{2}\right)^2 = \partial_t^2 + \Delta + \left(\frac{n-2}{2}\right)^2, \qquad \Delta = \Delta_{S^{n-1}}.$$

We can diagonize D_2 as follows: let $0 = \lambda_0 < \lambda_1 < \cdots$ be the eigenvalues of Δ (ignoring multiplicity); by, e.g., [11], $\lambda_j = j(n-2+j)$. Let E_j be the λ_j eigenspace of Δ, and for $f = 0, \pm 1, \pm 2, \ldots$, let $\mathscr{E}_{f,j} = \{e^{ift}\varphi(x) \mid \varphi \in E_j\}$, where x is now the variable on S^{n-1}. The $\mathscr{E}_{f,j}$ are $L^2(\bar{\bar{M}}^n)$-orthogonal, and D_2 acts as $-f^2 + \lambda_j + ((n-2)/2)^2 = -f^2 + ((n-2)/2 + j)^2$ on $\mathscr{E}_{f,j}$.

Pictorially, we can represent the $\mathscr{E}_{f,j}$ as lattice points in an upper half-plane diagram as in Fig. 2.1. Physically, each point is a system of finitely many harmonic oscillators of the same frequency; group theoretically, each carries a representation of the maximal compact subgroup $K = SO(2) \times O(n)$ of $G = O^\uparrow(2, n)$. (The restriction of u_a to K is independent of a.) The $\times$'s represent fields which are even under the antipodal map η:

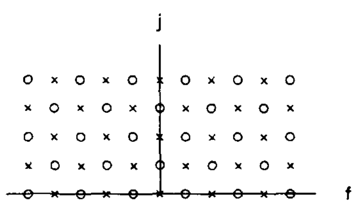


Figure 2.1

$(t, x) \mapsto (t+\pi, -x)$, and the ∘'s represent odd fields. Evenness and oddness is calculated using the fact that E_j consists of restrictions to S^{n-1} of j-homogeneous harmonic polynomials on $\mathbb{R}^n$, the so-called jth-*order spherical harmonics.*

According to (2.1), $\mathcal{N}(D_2)$ carries the representation $u_{(n-2)/2}$ of $O^{\uparrow}(2, n)$. But D_2 is a hyperbolic operator; solutions to $D_2 \Phi = 0$ are gotten by solving the Cauchy problem; i.e., by propagating initial ($t=0$) data. This we can easily do in $\mathbb{R} \times S^{n-1}$, but unless quite a few of the solutions are 2π-periodic in t, the null space representation will not be very interesting. It is remarkable that *all* solutions are 2π-periodic for even n. Note that

$$D_2 = [\partial_t + iB][\partial_t - iB], \qquad B = \sqrt{\Delta + ((n-2)/2)^2},$$

and that the nonlocal operator B acts as multiplication by $(n-2)/2 + j$ on E_j. Thus we can solve the Cauchy problem, and show uniqueness of our solution, by looking at second-order ODEs valued in the E_j. Using the fact that functions which are jointly C^∞ in t and x can be viewed as C^∞ functions in t with values in $L^2(S^{n-1})$, we have:

THEOREM 2.1 [24] (Automatic periodicity). *For $n \geqslant 3$, the general C^∞ solution of $D_2 \Phi = 0$,*

$$e^{itB}\varphi(x) + e^{-itB}\psi(x), \qquad \varphi, \psi \in C^\infty(S^{n-1}), \tag{2.2}$$

is 2π-periodic in t for even n, 4π-periodic for odd n. For $n=2$, every solution is a sum of a solution of the form (2.2) *and one of the form αt, $\alpha \in \mathbb{C}$.*

This "automatic periodicity" will turn out to be a recurring theme for all the conformally covariant operators studied here.

For even $n \geqslant 4$, the wave equation subspace $\mathcal{N}(D_2)$ occupies the lines $f = \pm((n-2)/2 + j)$ in the upper half plane diagram (Fig. 2.2). In particular, all wave functions have the same parity under the antipodal map.

THEOREM 2.2 [24] (Oddness–evenness). *If $n \geqslant 4$ is even, all C^∞ solutions of $D_2 \Phi = 0$ on $\mathbb{R} \times S^{n-1}$ have $\Phi(t+\pi, -x) = (-1)^{(n-2)/2}\, \Phi(t, x)$. In particular, if n is of the form $4p+2$, Φ "lives on" (covers a field on) $\tilde{M}^n = (S^1 \times S^{n-1})/\mathbb{Z}_2$. If $n=2$, this is true of periodic solutions.*

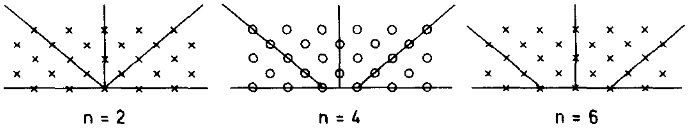


FIGURE 2.2

Remark 2.3. D_2 is related to the d'Alembertian $\square$ on Minkowski space M^n by the conformal compactification J,

$$\square(\Omega^{(n-2)/2}\Phi \circ J) = \Omega^{(n+2)/2}(D_2\Phi) \circ J, \tag{2.3}$$

where Ω is the conformal factor in (1.10). (See [27].) Thus D_2 fields on $\mathbb{R} \times S^{4p+1}$ determine wave functions on Minkowski space M^{4p+2}.

By an argument of Lax and Phillips based on Theorem 2.2, or by an argument of Ørsted based on (2.3), one immediately has Huygens' principle for the equation $D_2\Phi = 0$ in the curved spacetime $\bar{\bar{M}}^n$.

THEOREM 2.4 [24, 27] (Huygens' principle). *Suppose $n \geqslant 4$ is even, and $\Phi \in C^\infty$ with $D_2\Phi = 0$. If the support of the Cauchy data $\Phi(0, x)$, $\Phi_t(0, x)$ contains no point at a distance t_0 from $x_0 \in S^{n-1}$, $0 \leqslant t_0 \leqslant \pi$, then $\Phi(t_0, x_0) = 0$.*

Proof [24]. First note that the formulas given in, e.g., [10, Sect. 5] for the unique local solution to the Cauchy problem (say, within a neighborhood of the form $N = \{|t| < \varepsilon\} \times \{\mathrm{dist}(x, x_0) < \pi\}$), together with the uniqueness of the above global solution to the Cauchy problem, imply a global finite propagation speed principle: If the time 0 Cauchy data for Φ are supported in $\{\mathrm{dist}(x, x_0) < t_0\}$, then the time t data are supported in $\{\mathrm{dist}(x, x_0) < t_0 + |t|\}$. For by superposition and time translation, it is enough to prove this for small t_0 and $|t|$. In this case, the local formula for the solution to the Cauchy problem, extended by zero outside N, must give the unique global solution to the Cauchy problem. Since the local formula exhibits finite propagation speed, the principle follows.

Looking at the statement of the theorem, we may, by superposition, assume that the Cauchy data are supported either in $\{\mathrm{dist}(x, x_0) > t_0\}$ or $\{\mathrm{dist}(x, x_0) < t_0\}$. In the first case, finite propagation speed gives the conclusion. In the second case, we propagate backward from time π. The oddness–evenness result implies that the $t = \pi$ Cauchy data are supported in $\{\mathrm{dist}(x, -x_0) < t_0\}$; finite propagation speed then says that the $t = t_0$ data are supported in $\{\mathrm{dist}(x, -x_0) < \pi\}$, as desired. ▮

Remark 2.5. For $n = 2$, suppose the $t = 0$ Cauchy data of a $D_2 = \square$ field Φ are supported in $\{\mathrm{dist}(x, x_0) < t_0\}$, and let

$$b = \frac{1}{2\pi} \int_{S^1} \Phi_t(0, x)\, dx.$$

Then the $t = \pi$ data for $\Phi - \pi b$ are supported in $\{\mathrm{dist}(x, -x_0) < t_0\}$, and more generally, the $t = N\pi$ data of $\Phi - N\pi b$ are supported in

$\{\text{dist}(x, (-1)^N x_0) < t_0\}$, $N \in \mathbb{Z}$. Thus in the "lacuna" $\{\text{dist}(x, (-1)^N x_0) \leqslant t - N\pi - t_0\}$ for $N\pi \leqslant t \leqslant (N+1)\pi$, one has $\Phi \equiv \pi N b$.

The oddness–evenness result and Huygens' principle will also recur for all the operators we study. Note that Theorem 2.4 does not just follow from hyperbolicity, nor does Theorem 2.1 follow from the fact from the fact that geodesics in S^{n-1} have length 2π. The generic linear hyperbolic equation in curved spacetime, in particular $\square\Phi = 0$ in $\mathbb{R} \times S^{n-1}$, does *not* propagate all energy at characteristic speed; among all equations $(\square + \text{constant})\, \Phi = 0$, only $D_2 \Phi = 0$ has this property. It is worth mentioning here that Helgason [15] has proved Huygens' principle on $\mathbb{R} \times K$ for K a simply connected compact semisimple Lie group, for the operator $\partial_t^2 + \Delta_K + (\dim K)/24$. For $K = SU(2) = S^3$, this is $\partial_t^2 + (\Delta_{S^3} + 1)/8$, since Δ_K is calculated with respect to $-$ (the Killing form), which for $SU(2)$ is 8 times the usual S^3 metric. An argument for Huygens' principle for this operator on $\mathbb{R} \times SU(2)$ is an easy adaptation of the above.

b. *Representation Theoretic Content of* D_2

Restrict now to even n. Let $G = O^{\uparrow}(2, n)$, $G_0 = SO_0(2, n)$, and $K = SO(2) \times O(n)$; let $\mathfrak{g}$ be the Lie algebra of G (or G_0), and $\mathfrak{k}$ the Lie algebra of K. The antipodal map η: $(t, x) \mapsto (t + \pi, -x)$ is an isometry given by the action of the central element $-I$ of G. Thus each $u_a(G)$ acts on even and odd fields separately. Within the fields $\mathscr{F}$ of parity $(-1)^{(n-2)/2}$, D_2 decomposes $u_{(n-2)/2}(G)$ into representations on $\mathscr{N}(D_2)$ and $\mathscr{F}/\mathscr{N}(D_2)$. But much more can be said about decomposability, and about unitarity of the resulting pieces. First, a few functional analytic observations are needed.

Consider the L^2-Sobolev spaces $H^m = H^m(\bar{\tilde{M}}^n)$, $m \in \mathbb{R}$. Each $\Phi \in H^m$ can be identified with a series $\Sigma \Phi_{f,j}$, where $\Phi_{f,j} \in \mathscr{E}_{f,j}$ and

$$\Sigma (1 + f^2 + j^2)^m \|\Phi_{f,j}\|_{L^2}^2 < \infty.$$

Thus each H^m can be thought of as a weighted l^2 space. By the Sobolev Lemma, the usual Fréchet topology on $C^\infty = H^\infty = \bigcap H^m$ is equivalent to that gotten by using the H^m norms as seminorms. $H^{-\infty} = \bigcup H^m$ is the dual space of distributions.

Within any H^m, the distributions represented by series $\sum_{(f,j) \in \mathscr{A}} \Phi_{f,j}$ for any subset $\mathscr{A}$ of $\mathbb{Z} \times (\mathbb{Z}^+ \cup \{0\})$ form a closed subspace. It will be understood that a subspace gotten in this way will be denoted by its corresponding $\mathscr{A}$; for example, the space $\mathscr{F}$ above is $\{f \equiv (n-2)/2 + j \pmod 2\}$, while $\mathscr{N}(D_2) = \{|f| = (n-2)/2 + j\}$.

Remark 2.6. By an argument in [26, Sect. 4] (see also [12, 35]), the u_a are continuous representations on H^∞ and $H^{-\infty}$, and thus on each H^m.

The same can be said of the representations u_a on differential forms. Though a priori there is some ambiguity as to what L^2 and H^m mean for sections of a bundle (like the exterior bundles) with an indefinite metric, the compactness of $\bar{\tilde{M}}^n$ implies [30, Sect. 8] that any "artificial" Riemannian metric G (and corresponding form metrics G^k) we might impose yield the same L^2 and H^m classes.

Consider now the space $\mathscr{F}$ of fields of parity $(-1)^{(n-2)/2}$. We shall show that the closed subspaces

$$\begin{aligned} W^+ &= \left\{ f = \frac{n-2}{2} + j \right\}, \\ W^- &= \left\{ -f = \frac{n-2}{2} + j \right\}, \\ V^+ &= \left\{ f \geqslant \frac{n-2}{2} + j \right\}, \\ V^- &= \left\{ -f \geqslant \frac{n-2}{2} + j \right\}, \\ M &= \left\{ |f| \leqslant \frac{n-2}{2} + j \right\} \end{aligned} \tag{2.4}$$

are $u_{(n-2)/2}(G)$-invariant. By symmetry, the action of all the proper conformal vector fields in $\mathfrak{g}$ can be derived from that of $S = L_{-1,n}$. Let Y be as in Remark 1.5.

LEMMA 2.7. *Suppose $n \geqslant 2$ (n need not be even here), and let $\varphi \in E_j$. Then*

$$\begin{aligned} \varphi^+ &\equiv (\mathscr{L}(Y) + (n-2+j)\,\omega)\,\varphi \in E_{j+1}, \\ \varphi^- &\equiv (\mathscr{L}(Y) - j\omega)\,\varphi \in E_{j-1}, \end{aligned} \tag{2.5}$$

where $\omega = \omega_Y = \cos\rho$, and E_{-1} is $\{0\}$ by convention.

Proof. By (1.3),

$$\mathscr{L}(Y) + \mathscr{L}(Y)^* = -(n-1)\,\omega \qquad \text{on functions.}$$

Note that

$$(d\omega)_\alpha = -Y_\alpha,$$

i.e., $d\omega$ is the one-form corresponding to $-Y$ through $g_{S^{n-1}}$. Thus for $\varphi \in E_j$, if $\lambda = \lambda_j = j(n-2+j)$,

$$\begin{aligned}\Delta(\omega\varphi) &= \delta(\omega d\varphi + \varepsilon(d\omega)\,\varphi)\\ &= \omega\Delta\varphi - \iota(d\omega)\,d\varphi + \delta\varepsilon(d\omega)\,\varphi\\ &= \lambda\omega\varphi + (\mathscr{L}(Y) - \mathscr{L}(Y)^*)\,\varphi\\ &= \lambda\omega\varphi + (2\mathscr{L}(Y) + (n-1)\,\omega)\,\varphi\\ &= \{2\mathscr{L}(Y) + (\lambda + n - 1)\,\omega\}\,\varphi. \end{aligned} \tag{2.6}$$

On the other hand, the content of (2.1) for the conformal vector field Y on Riemannian S^{n-1} is

$$\begin{aligned}&\left(\Delta + \frac{(n-1)(n-3)}{4}\right)\left(\mathscr{L}(Y) + \frac{n-3}{2}\,\omega\right)\varphi\\ &\quad = \left(\mathscr{L}(Y) + \frac{n+1}{2}\,\omega\right)\left(\Delta + \frac{(n-1)(n-3)}{4}\right)\varphi,\end{aligned}$$

or

$$\Delta\left(\mathscr{L}(Y) + \frac{n-3}{2}\,\omega\right)\varphi = \left\{\lambda\mathscr{L}(Y) + \frac{1}{2}\left((n+1)\,\lambda + (n-1)(n-3)\right)\omega\right\}\varphi. \tag{2.7}$$

Putting (2.6) and (2.7) together gives

$$\begin{aligned}\Delta(\mathscr{L}(Y) + (n-2+j)\,\omega)\,\varphi &= (j+1)(n-1+j)(\mathscr{L}(Y) + (n-2+j)\,\omega)\,\varphi\\ &= \lambda_{j+1}(\mathscr{L}(Y) + (n-2+j)\,\omega)\,\varphi,\\ \Delta(\mathscr{L}(Y) - j\omega)\,\varphi &= (j-1)(n-3+j)(\mathscr{L}(Y) - j\omega)\,\varphi\\ &= \lambda_{j-1}(\mathscr{L}(Y) - j\omega)\,\varphi,\end{aligned}$$

as desired. ∎

Now look at $e^{ift}\varphi(x) \in \mathscr{E}_{f,j}$ for $\varphi \in E_j$. By (1.9) and $\omega_S = (\cos t)\,\omega$,

$$\begin{aligned}\mathscr{L}(S)\,e^{ift}\varphi &= (\sin t)(\cos\rho)\,ife^{ift}\varphi + (\cos t)\,e^{ift}\mathscr{L}(Y)\,\varphi\\ &= \tfrac{1}{2}\{e^{i(f+1)t}(\mathscr{L}(Y) + f\omega)\,\varphi + e^{i(f-1)t}(\mathscr{L}(Y) - f\omega)\,\varphi\},\\ \omega_S e^{ift}\varphi &= \tfrac{1}{2}\{e^{i(f+1)t}\omega\varphi + e^{i(f-1)t}\omega\varphi\}.\end{aligned}$$

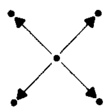


FIGURE 2.3

By (2.5),

$$\mathscr{L}(Y)\,\varphi = \frac{1}{n-2+2j}\,(j\varphi^+ + (n-2j)\,\varphi^-),$$
$$\omega\varphi = \frac{1}{n-2+2j}\,(\varphi^+ - \varphi^-),$$

unless $n=2$ and $j=0$ (in which case $\mathscr{L}(Y)\,\varphi = 0$, $\omega\varphi \in E_1$). This means

$$U_a(S)\,e^{ift}\varphi = \frac{1}{2(n-2+2j)}\,\{e^{i(f+1)t}[(j+f+a)\varphi^+ + (n-2+j-f-a)\,\varphi^-] + e^{i(f-1)t}[(j-f+a)\,\varphi^+ + (n-2+j+f-a)\,\varphi^-]\},$$
$$(n, j) \neq (2, 0),$$
$$U_a(S)\,e^{ift}\cdot 1 = \frac{1}{2}\,\{e^{i(f+1)t}(f+a)\omega - e^{i(f-1)t}(-f+a)\,\omega\}, \qquad n=2. \tag{2.8}$$

Thus any $U_a(S)$ carries a field at a point (f, j) of Fig. 2.1 to a linear combination of fields at the 4 "nearest neighbors" of the same parity, $(f \pm 1, j \pm' 1)$ (Fig. 2.3). It is immediate from (2.8) that moves off the wave equation subspace are forbidden for $a = (n-2)/2$, as expected (Fig. 2.4). But we can also show that the 5 subspaces of (2.4) are invariant:

LEMMA 2.8. *If $n \geqslant 2$ is even, the subspaces $W^\pm$, $V^\pm$, M of* (2.4), *within the fields of parity $(-1)^{(n-2)/2}$, are $U_{(n-2)/2}$ (G)-invariant. If $n=2$, the one-dimensional space of constants, $F = \{f = j = 0\} = W^+ \cap W^-$, is also invariant.*

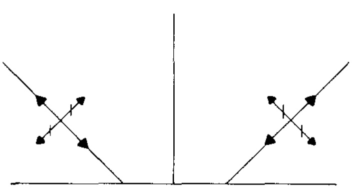


FIGURE 2.4

Proof. Let $u = u_{(n-2)/2}$ and $U = U_{(n-2)/2}$. To show invariance of $X = W^{\pm}, V^{\pm}, M$, we need only show that if $\Phi \in X \cap \mathscr{E}_{f,j}$ for some (f, j), then $u(h)\Phi \in X$ for all $h \in G$. Clearly each $\mathscr{E}_{f,j}$ is P-invariant, so it suffices to prove $u(G_0)$-invariance. Each X has a closed L^2-orthogonal direct complement $X^{\perp}$ in each H^m; if X corresponds to $\mathscr{A} \subseteq \mathbb{Z} \times (\mathbb{Z}^+ \cup \{0\})$, $X^{\perp}$ corresponds to the complement of $\mathscr{A}$. (Here we use the inner product $\langle \Phi, \Psi \rangle_{H^m} = \sum (1 + f^2 + j^2)^m \langle \Phi_{f,j}, \Psi_{f,j} \rangle_{L^2}$.) Thus it suffices to show that

$$(u(h_{-\varepsilon})\Phi, \Psi)_{L^2} = 0$$

for all $\Phi \in X \cap \mathscr{E}_{f,j}$, $\Psi \in X^{\perp} \cap \mathscr{E}_{f',j'}$, and each one-parameter group $\{h_\varepsilon\}$ in G_0. But such Φ and Ψ are real-analytic, and by (1.11), $u(h_{-\varepsilon})\Phi$ is jointly analytic in ε and (t, x) (note that Ω_h is the Jacobian determinant of h to the $1/n$ power). Since M is compact, $(u(h_{-\varepsilon})\Phi, \Psi)_{L^2}$ is analytic in ε, but

$$\left(\frac{d}{d\varepsilon}\right)^N \Big|_{\varepsilon = 0} (u(h_{-\varepsilon})\Phi, \Psi)_{L^2} = (U(T)^N \Phi, \Psi)_{L^2},$$

where T generates $\{h_\varepsilon\}$. This reduces the problem to $U(\mathfrak{g})$-invariance. For the isometric basic conformal vector fields $\mathscr{k}$, invariance (in fact, invariance of each $\mathscr{E}_{f,j}$) is obvious. By symmetry, we are reduced to the $U(S)$-invariance shown above. ∎

We now have 5 (or 6 if $n = 2$) representations to work with, namely $u_{(n-2)/2}(G)$ on W^+, W^- (*the positive* or *negative frequency solutions*); $V^+/W^+, V^-/W^-$ (the *positive* and *negative frequency quotients*); and $M/(W^+ + W^-)$ (the *mixed quotient*). (F and $W^{\pm}/F$ replace $W^{\pm}$ for $n = 2$.) The following theorem asserts that all of these composition factors are unitary.

THEOREM 2.9 (see [40, 31] for $n = 4$). *Suppose $n \geqslant 2$ is even. The Hermitian inner products*

$$\langle \Phi, \Psi \rangle_M = (\Phi, D_2 \Psi)_{L^2} \qquad on \quad M/(W^+ + W^-),$$

$$\langle \Phi, \Psi \rangle_{V^{\pm}} = -(\Phi, D_2 \Psi)_{L^2} \qquad on \quad V^{\pm}/W^{\pm},$$

$$\langle \Phi, \Psi \rangle_{W^{\pm}} = \pm i(\Phi, \partial_t \Psi)_{L^2} \qquad on \quad W^{\pm} \ (or\ W^{\pm}/F\ if\ n = 2)$$

are positive definite and invariant for $u_{(n-2)/2}(G)$. Completion (of, say, C^{∞}) in these inner products results in continuous unitary representations.

Proof. The $\langle\ ,\ \rangle_X$ for $X = M, V^{\pm}, W^{\pm}$ are Hermitian because D_2 and $(1/i)\,\partial_t$ are formally self-adjoint. Let $u_{\pm} = u_{(n \pm 2)/2}$, $U_{\pm} = U_{(n \pm 2)/2}$. The $\langle\ ,\ \rangle_X$ are clearly $u_-(P)$-invariant, so it suffices in each case to show $u_-(G_0)$-invariance. That $(\Phi, D_2\Psi)_{L^2}$ is invariant follows from the covariance relation for D_2, together with the fact that an orientation-preserving conformal transformation h, $h \cdot g = \Omega^2 g$, has the effect $h \cdot \mathcal{O} = \Omega^n \mathcal{O}$ on the normalized volume element $\mathcal{O}$. Indeed,

$$\begin{aligned}(u_-(h)\,\Phi, D_2 u_-(h)\,\Psi)_{L^2} &= (u_-(h)\,\Phi, u_+(h)\,D_2\Psi)_{L^2} \\ &= \int_{\bar{M}^n} \Omega^n (h\cdot\Phi)(h\cdot D_2\bar{\Psi})\,\mathcal{O} \\ &= \int_{\bar{M}^n} h\cdot(\Phi[D_2\bar{\Psi}]\,\mathcal{O}) \\ &= \int_{\bar{M}^n} \Phi[D_2\bar{\Psi}]\,\mathcal{O} = (\Phi, D_2\Psi)_{L^2}.\end{aligned}$$

The positive definiteness of this inner product on $M/(W^+ + W^-)$, and negative definiteness on $V^\pm/W^\pm$, follow from the fact that D_2 is multiplication by the scalar $-f^2 + ((n-2)/2 + j)^2$ at (f, j).

For the invariance of $\langle\ ,\ \rangle_X$, $X = W^\pm$, note that for smooth $\Phi, \Psi \in X$,

$$\begin{aligned}&\frac{d}{d\varepsilon}(u_-(h_{-\varepsilon})\,\Phi, \partial_t u_-(h_{-\varepsilon})\,\Psi)_{L^2} \\ &\quad = (u_-(h_{-\varepsilon})\,\Phi, \partial_t U_-(T)\,u_-(h_{-\varepsilon})\,\Psi)_{L^2} \\ &\qquad + (U_-(T)\,u_-(h_{-\varepsilon})\,\Phi, \partial_t u_-(h_{-\varepsilon})\,\Psi)_{L^2}\end{aligned}$$

for a conformal vector field T generating a one-parameter group $\{h_\varepsilon\}$. This means it is enough to show $U_-(\mathfrak{g})$-skewness for $\langle\ ,\ \rangle_X$. $U_-(\mathfrak{k})$-skewness is clear, so symmetry reduces the problem to that of $U_-(S)$-skewness.

For this, note first that $\partial_t = \frac{1}{2}[D_2, t]$. (Here t is an abbreviation for multiplication by t.) Taking $[\cdot, t]$ of both sides in the covariance relation $D_2 U_-(S) = U_+(S)\,D_2$, we get

$$D_2[U_-(S), t] + [D_2, t]\,U_-(S) = U_+(S)[D_2, t] + [U_+(S), t]\,D_2.$$

(Note that $[U_\pm(S), t]$ has periodic coefficients.) Because D_2 is formally self-adjoint, this shows that

$$(\Phi, \partial_t U_-(S)\,\Psi)_{L^2} = (\Phi, U_+(S)\,\partial_t\Psi)_{L^2}, \qquad \Phi, \Psi \in \mathcal{N}(D_2). \tag{2.9}$$

By (1.3), $U_+(S)$ and $U_-(S)$ are formal adjoints, so (2.9) establishes the $\langle\ ,\ \rangle_X$-skewness of $U_-(S)$.

For positive definiteness of $\langle\ ,\ \rangle_{W^\pm}$, note that ∂_t acts as multiplication by $\pm i((n-2)/2 + j)$ on $\mathscr{E}_{\pm((n-2)/2+j),j}$.

Looking at the asymptotics of the eigenvalues of D_2 and $(1/i)\,\partial_t$ on the

$\mathscr{E}_{f,j}$, it is clear that the $\langle\ ,\ \rangle_{W\pm}$ uniform structures "come from" (under passage to and quotient by closed subspaces) the $H^{1/2}$ uniform structure while the $\langle\ ,\ \rangle_{V\pm}$ and $\langle\ ,\ \rangle_M$ uniform structures are no weaker than that of $H^{1/2}$, and no stronger than that of H^1. This (recall Remark 2.6) shows that completion in each $\langle\ ,\ \rangle_X$ results in a *continuous* unitary representation. ▮

Remark 2.10 (Polarization of symplectic structure). We can also unitarize the wave equation subspace by polarization as in [37]; this gives the same result. Consider *real* solutions of $D_2\Phi=0$, represented by their Cauchy data Φ, Φ_t (with the usual abuse of notation), and the $u_-(G)$-invariant symplectic form

$$\mathscr{A}\left(\begin{pmatrix}\Phi\\ \Phi_t\end{pmatrix}, \begin{pmatrix}\Phi'\\ \Phi'_t\end{pmatrix}\right)=\int_{S^{n-1}}(\Phi\Phi'_t-\Phi'\Phi_t)\,dx.$$

(Invariance implies in particular that the integral can be taken at any fixed t.) The conserved energy $\mathscr{E}=\frac{1}{2}\int_{S^{n-1}}(|B\Phi|^2+|\Phi_t|^2)\,dx$ is the Hamiltonian function corresponding to the formal vector field

$$\mathscr{T}_{\left(\begin{smallmatrix}\Phi\\ \Phi_t\end{smallmatrix}\right)}=\begin{pmatrix}\Phi_t\\ -B^2\Phi\end{pmatrix}$$

of temporal evolution on phase space. Taking the Hessian of $\mathscr{E}$ in a flat formal connection yields the symmetric bilinear form

$$\mathscr{H}\left(\begin{pmatrix}\Phi\\ \Phi_t\end{pmatrix}, \begin{pmatrix}\Phi'\\ \Phi'_t\end{pmatrix}\right)=\int_{S^{n-1}}[(B\Phi)(B\Phi')+\Phi_t\Phi'_t]\,dx,$$

but $\mathscr{H}$ is not $u_-(G)$-invariant. However, writing

$$\mathscr{H}(X, Y)=\mathscr{A}(X, \mathscr{K}Y)$$

and polar decomposing $\mathscr{K}$ with respect to $\mathscr{H}$ as

$$\mathscr{K}=\mathscr{J}\mathscr{P},\qquad \mathscr{J}=\frac{\mathscr{K}}{\sqrt{-\mathscr{K}^2}},\qquad \mathscr{P}=\sqrt{-\mathscr{K}^2},$$

where $\mathscr{J}$ if $\mathscr{H}$-orthogonal and $\mathscr{P}$ is $\mathscr{H}$-self-adjoint and positive, we get the $u_-(G)$-invariant symmetric bilinear form

$$\mathscr{R}(X, Y)=\mathscr{A}(X, \mathscr{J}Y).$$

Note that $\mathscr{J}$ carries Cauchy data $\left(\begin{smallmatrix}\Phi\\ \Phi_t\end{smallmatrix}\right)$ to

$$\begin{pmatrix}0 & -B^{-1}\\ B & 0\end{pmatrix}\begin{pmatrix}\Phi\\ \Phi_t\end{pmatrix}.$$

$\mathscr{J}$ is a complex structure on the real solutions and $\mathscr{R} + i\mathscr{A}$ is a $u_-(G)$-invariant Hermitian inner product.

The relation of this to $\langle\ ,\ \rangle_{W^+}$ is as follows. Each real solution Φ is the real part of a unique positive frequency solution $\tilde{\Phi}$, and it is easily shown (identifying a solution with its Cauchy data) that

$$u_-(h)\,\tilde{\Phi} = (u_-(h)\,\Phi)^\sim,$$

$$(\mathscr{R} + i\mathscr{A})(\Phi, \Psi) = \frac{1}{\pi}\langle \tilde{\Phi}, \tilde{\Psi}\rangle_{W^+},$$

$$i\tilde{\Phi} = (\mathscr{J}\Phi)^\sim.$$

c. *Higher Order Operators and Wave Propagation*

According to (2.8), one direction of escape from each of the 4 lines

$$\pm f = \frac{n-2}{2} + j \pm'(l-1)$$

via $U_{(n-2l)/2}(\mathfrak{g})$ is cut off (Fig. 2.5). This suggest looking at the operator

$$D_{2l} = \begin{cases} D_2 \prod_{p=1}^{(l-1)/2} [\partial_t + i(B+2p)][\partial_t - i(B+2p)] \\ \qquad \times [\partial_t + i(B-2p)][\partial_t - i(B-2p)], \qquad l \text{ odd} \\ \prod_{p=1}^{l/2} [\partial_t + i(B+(2p-1))][\partial_t - i(B+(2p-1))] \\ \qquad \times [\partial_t + i(B-(2p-1))][\partial_t - i(B-(2p-1))], \qquad l \text{ even.} \end{cases} \tag{2.10}$$

It turns out that D_{2l} is a conformally covariant differential operator, and that one has full analogues of Theorems 2.1, 2.2, 2.4, and 2.9 involving D_{2l}, except for the unitarity of the representation spaces corresponding to $W^\pm$ or $W^\pm/F$. D_4 was studied by Paneitz [34], who showed that it is a special case of a general fourth-order operator which is well defined and conformally covariant in an arbitrary pseudo-Riemannian manifold of dimension 3 or higher; he also showed that D_4 plays an interesting role in the

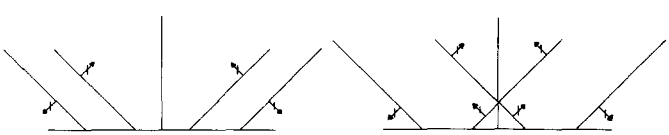


FIGURE 2.5

gauge theory of the Maxwell equations ([34] and Remark 4.11). D_6 is also a special case of an operator that exists somewhat more generally; the general version was introduced by the present author in [6].

LEMMA 2.11. *D_{2l} is a differential operator with leading term $\Box^l$.*

Proof. D_2 is a differential operator with leading term $\Box$, and the "block of 4,"

$$[\partial_t + i(B+m)][\partial_t - i(B+m)][\partial_t + i(B-m)][\partial_t - i(B-m)]$$
$$= \partial_t^4 + 2(B^2+m^2)\,\partial_t^2 + (B^2-m^2)^2$$

is a differential operator with leading term $\Box^2$. ∎

In particular, $D_{2l}\Phi = 0$ has finite propagation speed. By construction, the periodic null pace of D_{2l} for even n is as in Fig. 2.6.

THEOREM 2.12 (Automatic Periodicity). *Suppose n is even. If $n > 2l$, all solutions of $D_{2l}\Phi = 0$ on $\mathbb{R} \times S^{n-1}$ are 2π-periodic in t. If $n \leqslant 2l$, every solution is of the form $\Phi = \Phi' + t\Phi''$, where Φ' and Φ'' are periodic solutions.*

Proof. Again, we reduce the problem to that of getting unique solutions to initial value ODE problems in the variable t, and use the fact that the eigenvalues of B are integers. The only complication comes when the characteristic polynomials of these ODEs have double roots; we never get roots of greater multiplicity. ∎

THEOREM 2.13 (Oddness–evenness). *For even n, all periodic solutions of $D_{2l}\Phi = 0$ have $\Phi(t+\pi, -x) = (-1)^{(n-2l)/2}\Phi(t, x)$. In particular, if $(n-2l)/2$ is even, Φ "lives on" (covers a field on) $\bar{M}^n = (S^1 \times S^{n-1})/\mathbb{Z}_2$.*

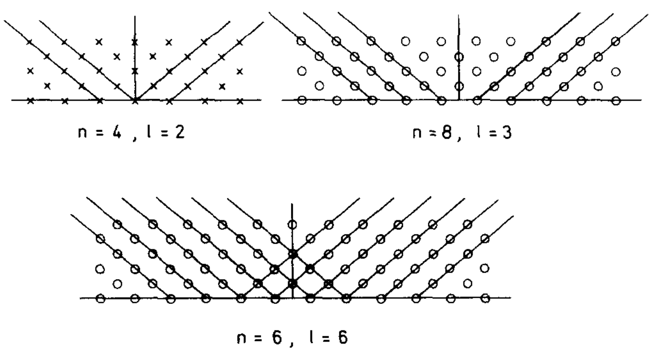


FIGURE 2.6

Proof. By (2.10) or Fig. 2.6, the frequency f of a "simple" solution $e^{ift}\varphi(x)$, $\varphi \in E_j$, has the same parity as $(n-2l)/2+j$, and the jth order spherical harmonic φ has $\varphi(-x)=(-1)^j\varphi(x)$. ∎

THEOREM 2.14 (*Huygens' principle*). *Suppose $n>2l$ is even. If $\Phi \in C^\infty$, $D_{2l}\Phi=0$ and the support of the Cauchy data $\Phi(0,x)$, $(\partial_t\Phi)(0,x),..., (\partial_t^{2l-1}\Phi)(0,x)$ contains no point at a distance t_0 from $x_0 \in S^{n-1}$, $0 \leqslant t_0 \leqslant \pi$, then $\Phi(t_0,x_0)=0$.*

Proof. We just need to look at $D_{2l}\Phi=(\Box^l+\text{lower order})\,\Phi=0$ as a vector-valued wave equation: $(\Box+\text{lower order})\;(\Phi, \Box\Phi,..., \Box^{l-1}\Phi)=0$. Then the result follows from finite propagation speed, the uniqueness of our solution to the Cauchy problem (proof of Theorem 2.12), and oddness–evenness, exactly as for Theorem 2.4. ∎

Remark 2.15. The condition $n>2$ necessary for $D_2\Phi=0$ to have a "perfect" Huygens' principle is replaced here by $n>2l$. This reflects the fact that in Minkowski space M^n, the fundamental solution of $\Box^l\Phi=0$ for $n \leqslant 2l$ is not supported entirely on the light cone (see, e.g., [16]). It can be shown (but we do not do so here) that D_{2l} on $\bar{M}^n$ is related to $\Box^l$ on M^n by the conformal compactification map J,

$$\Box^l(\Omega^{(n-2l)/2}\Phi \circ J)=\Omega^{(n+2l)/2}(D_{2l}\Phi)\circ J,$$

where Ω is the conformal factor in (1.10). As in [27], there is a resultant relation between fundamental solutions.

Running the oddness–evenness argument for $n \leqslant 2l$ shows that if the $t=0$ Cauchy data for $\Phi=\Phi'+t\Phi''$ have small support (say, of radius t_0) near x_0, the $t=\pi$ Cauchy data for $\Phi-\pi\Phi''$ has small support near $-x_0$. Thus the lacuna that opens up starting at time t_0 near x_0 is "filled" with the field $\pi\Phi''$. Similar results hold, as in Remark 2.5, for the earlier and later lacunae.

d. *Representation Theoretic Content of the D_{2l}*

The "forbidden $U_{(n-2l)/2}(\mathfrak{g})$-move" structure of Fig. 2.5 suggests that D_{2l} is conformally covariant, with "inside multiplier" power $(n-2l)/2$. In this section, let $J=(n-2)/2+j$, and denote the $\mathscr{E}_{f,j}$ component of $\Phi \in C^\infty(\bar{\bar{M}}^n)$ by $\Phi_{f,j}$.

THEOREM 2.16 (Covariance of D_{2l}). *Suppose n is even. For* $\Phi \in C^\infty(\bar{\bar{M}}^n)$ and $h \in O(2,n)$,

$$D_{2l}u_{(n-2l)/2}(h)\Phi=u_{(n+2l)/2}(h)\,D_{2l}\Phi.$$

Proof. By the obvious covariance under isometries, Remark 1.4, and symmetry, we are reduced to S-covariance

$$D_{2l} U_{(n-2l)/2}(S)\, \Phi \overset{?}{=} U_{(n+2l)/2}(S)\, D_{2l}\Phi.$$

It is enough to prove this on "simple" fields $\Phi = e^{ift}\varphi(x)$, $\varphi \in E_j$. For such a Φ, $U_a(S)\,\Phi$ is the sum of its 4 components at $(f \pm 1, j \pm' 1)$. In fact, by (2.8), writing $U_\pm$ for $U_{(n \pm 2l)/2}$,

$$\left.\begin{aligned}
(U_\pm(S)\,\Phi)_{f+1,j+1} &= \frac{J+1+f\pm l}{4J}\,\varphi^+ \\
(U_\pm(S)\,\Phi)_{f+1,j-1} &= \frac{J-1-f\mp l}{4J}\,\varphi^- \\
(U_\pm(S)\,\Phi)_{f-1,j+1} &= \frac{J+1-f\pm l}{4J}\,\varphi^+ \\
(U_\pm(S)\,\Phi)_{f-1,j-1} &= \frac{J-1+f\mp l}{4J}\,\varphi^-
\end{aligned}\right\}\quad (n, j) \neq (2, 0), \tag{2.11}$$

$$\left.\begin{aligned}
(U_\pm(S)\, e^{ift}\cdot 1)_{f+1,1} &= \frac{1}{2}(f+1 \pm l)\,\omega \\
(U_\pm(S)\, e^{ift}\cdot 1)_{f-1,1} &= \frac{1}{2}(-f+1 \pm l)\,\omega
\end{aligned}\right\}\quad \text{if } n = 2.$$

D_{2l} acts on such a Φ as multiplication by

$$(-1)^l \prod_{m=0}^{l-1} (f+J-l+1+2m)(f-J+l-1-2m). \tag{2.12}$$

Thus both $D_{2l}U_-(S)\,\Phi$ and $U_+(S)\,D_{2l}\Phi$ have possible nonzero components only at $(f \pm 1, j \pm' 1)$. But, writing D for D_{2l} and assuming $(n, j) \neq (2, 0)$,

$$\begin{aligned}
&(DU_-(S)\,\Phi)_{f+1,j+1} \\
&\quad = \frac{J+1+f-l}{4J}(-1)^l \left[\prod_{m=0}^{l-1} (f+J-l+3+2m)(f-J+l-1-2m)\right]\varphi^+ \\
&\quad = \frac{(-1)^l}{4J}\left[\prod_0^l (f+J-l+1+2m)\right]\left[\prod_0^{l-1} (f-J+l-1-2m)\right]\varphi^+,
\end{aligned}$$

while

$$\begin{aligned}(U_+(S)\,D\Phi)_{f+1,j+1} &= \frac{J+1+f+l}{4J}(-1)^l\left[\prod_{m=0}^{l-1}(f+J-l+1+2m)(f-J+l-1-2m)\right]\varphi^+\\ &= \frac{(-1)^l}{4J}\left[\prod_0^l(f+J-l+1+2m)\right]\left[\prod_0^{l-1}(f-J+l-1-2m)\right]\varphi^+.\end{aligned}$$

Similarly,

$$\begin{aligned}(DU_-(S)\,\Phi)_{f+1,j-1} &= -\frac{(-1)^l}{4J}\left[\prod_0^{l-1}(f+j-l+1+2m)\right]\left[\prod_0^{l}(f-J+l+1-2m)\right]\varphi^-\\ &= (U_+(S)\,D\Phi)_{f+1,j-1}.\end{aligned}$$

In the exceptional case $n=2$ and $j=0$,

$$\begin{aligned}(DU_-(S)\,e^{ift}\cdot 1)_{f+1,1} &= \tfrac{1}{2}(-1)^l\left[\prod_{m=0}^{l}(f-l+1+2m)\right]\left[\prod_{m=0}^{l-1}(f+l-1-2m)\right]\omega\\ &= (U_+(S)\,De^{ift}\cdot 1)_{f+1,1}.\end{aligned}$$

Equality of the $(f-1, j\pm 1)$ components now follows upon application of the time reversal, which commutes with both $U_\pm(S)$ and D. ∎

This gives us representations on $\mathcal{N}(D_{2l})$ and $\mathcal{F}/\mathcal{N}(D_{2l})$, where $\mathcal{F}$ is the space of fields of parity $(-1)^{(n-2l)/2}$. But just as for the wave equation, much more can be said; again we have 5 or 6 invariant subspaces.

LEMMA 2.17. *If n is even, the subspaces*

$$\begin{aligned}W^\pm &= \{J-(l-1)\leqslant \pm f\leqslant J+(l-1)\},\\ F &= W^+\cap W^-,\\ M &= \{|f|\leqslant J+(l-1)\},\\ V^\pm &= \{\pm f\geqslant J-(l-1)\}\end{aligned}$$

of the spaces of fields of parity $(-1)^{(n-2l)/2}$ are $u_{(n-2l)/2}(G)$-invariant.

Proof. As for Lemma 2.8, the problem reduces to $U_{(n-2l)/2}(S)$-invariance, which can be read off from Fig. 2.5 or (2.11). ∎

Note that F is finite dimensional. All the resulting representation spaces admit nondegenerate $U_{(n-2l)/2}(G)$-invariant complex inner products, though not all are unitary.

THEOREM 2.18. *Suppose n is even, and consider the Hermitian inner products*

$$\begin{aligned} \langle \Phi, \Psi \rangle_M &= (\Phi, D_{2l} \Psi)_{L^2} && \text{on} \quad M/(W^+ + W^-), \\ \langle \Phi, \Psi \rangle_{V^\pm} &= (-1)^l (\Phi, D_{2l} \Psi)_{L^2} && \text{on} \quad V^\pm / W^\pm, \\ \langle \Phi, \Psi \rangle_{W^\pm} &= \pm i (\Phi, D'_{2l} \Psi)_{L^2} && \text{on} \quad W^\pm / F, \\ \langle \Phi, \Psi \rangle_F &= (\Phi, D''_{2l} \Psi)_{L^2} && \text{on} \quad F, \end{aligned}$$

where $D'_{2l} = [D_{2l}, t]$ and $D''_{2l} = [D'_{2l}, t]$. All these inner products are nondegenerate and $u_{(n-2l)/2}(G)$-invariant. $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ are positive definite and result, upon completion, in continuous unitary representations. $\langle\ ,\ \rangle_{W^\pm}$ is indefinite unless $l = 1$, and $\langle\ ,\ \rangle_F$ is indefinite unless $n \leqslant 2l$, in which case F is either $\{0\}$ or the one-dimensional space of constants.

Proof. D'_{2l} and D''_{2l} are clearly differential operators with periodic coefficients. Since D_{2l} and t are formally self-adjoint, D'_{2l} is formally skew, so $(1/i) D'_{2l}$ and D''_{2l} are formally self-adjoint. Thus all the inner products are Hermitian. Note that $\mathcal{N}(D'_{2l})$ is exactly F.

Let $u_\pm = u_{(n \pm 2l)/2}, U_\pm = U_{(n \pm 2l)/2}$, and let $D = D_{2l}$. Each inner product is clearly $u_-(K)$-invariant, so it is enough to show either $u_-(G_0)$-invariance or $U_-(\mathfrak{g})$-skewness. For $(\Phi, D\Psi)_{L^2}$, the proof of $u_-(G_0)$-invariance runs exactly as in the case $l = 1$ (Theorem 2.9); the key point is that $(n - 2l)/2 + (n + 2l)/2 = n$. Positive definiteness of $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ results from the fact that the scalar (2.12) has the correct sign at the points $J > |f| + l - 1$ representing $M/(W^+ + W^-)$, and $\pm f > J + l - 1$ representing $V^\pm / W^\pm$. The strength of the resulting uniform structures is between that of $H^{l/2}$ and that of H^l, so we get continuous unitary representations.

For $\langle\ ,\ \rangle_{W^\pm}$, we shall show $U_-(S)$-skewness (and conclude $U_-(\mathfrak{g})$-skewness by symmetry). The argument for this runs exactly as in Theorem 2.9, with D' in place of $2\partial_t$ and the new meaning of $U_\pm$;

$$D[U_-(S), t] + D'U_-(S) = U_+(S) D' + [U_+(S), t] D, \tag{2.13}$$

$$(\Phi, D'U_-(S) \Psi)_{L^2} = (\Phi, U_+(S) D'\Psi)_{L^2}, \qquad \Phi, \Psi \in \mathcal{N}(D),$$

$$U_+(S) = U_-(S)^*. \tag{2.14}$$

As for nondegeneracy, if $\mathscr{E}_{f,j} \subseteq W^+$, then

$$f = J - (l-1) + 2p$$

for some p with $0 \leqslant p \leqslant l-1$, and D' acts on this $\mathscr{E}_{f,j}$ as multiplication by the scalar

$$\frac{(-1)^l}{i}\left[\prod_{m=0}^{l-1}(f+J-l+1+2m)\right]\left[\prod_{\substack{0\leqslant m\leqslant l-1\\ m\neq p}}(f-J+l-1-2m)\right].$$

If $\mathscr{E}_{f,j} \subseteq W^-$,

$$-f = J - (l-1) + 2p$$

for some p with $0 \leqslant p \leqslant l-1$, and D' acts on $\mathscr{E}_{f,j}$ as multiplication by

$$\frac{(-1)^l}{i}\left[\prod_{\substack{0\leqslant m\leqslant l-1\\ m\neq p}}(f+J-l+1+2m)\right]\left[\prod_{m=0}^{l-1}(f-J+l-1-2m)\right].$$

This show that in either case, the sign of $\langle\ ,\ \rangle_{W^\pm}$ on $\mathscr{E}_{f,j} \not\subseteq F$ is $(-1)^p$ (Fig. 2.7). This proves nondegeneracy, and indefiniteness unless $l=1$.

Taking $[\cdot, t]$ of both sides of (2.13) and using the fact that D and D' annihilate F, we get

$$(D[[U_-(S), t], t] + D'[U_-(S), t] + D''U_-(S))\,\Phi = U_+(S)\,D''\Phi, \quad \Phi \in F.$$

But D and D' are formally self-adjoint, so

$$(\Phi, D''U_-(S)\,\Psi)_{L^2} = (\Phi, U_+(S)\,D''\Psi)_{L^2}, \quad \Phi, \Psi \in F;$$

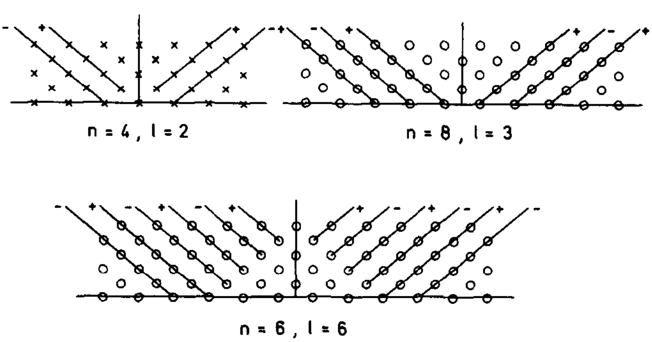
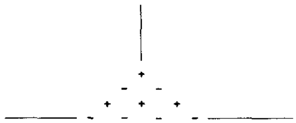


FIGURE 2.7



n = 6, l = 6

FIGURE 2.8

with (2.14), this shows $U_-(S)$-skewness for $\langle\ ,\ \rangle_F$, and thus $u_-(G)$-invariance.

As for nondegeneracy and indefiniteness of $\langle\ ,\ \rangle_F$, for $\mathscr{E}_{f,j} \subseteq F$, we must have

$$f - J + l - 1 - 2p = 0,$$

$$f + J - l + 1 + 2q = 0,$$

for some p, q with $0 \leqslant p, q \leqslant l-1$. On this $\mathscr{E}_{f,j}$, D'' acts as multiplication by

$$(-1)^{l+1}\left[\prod_{\substack{0 \leqslant m \leqslant l-1 \\ m \neq q}} (f + J - l + 1 + 2m)\right]\left[\prod_{\substack{0 \leqslant m \leqslant l-1 \\ m \neq p}} (f - J + l - 1 - 2m)\right],$$

so the sign of $\langle\ ,\ \rangle_F$ on $\mathscr{E}_{f,j}$ is $(-1)^{p+q}$ (Fig. 2.8). ∎

Remark 2.19. The inner products $\langle\ ,\ \rangle_{W^\pm}$ (for any l) on the positive and negative frequency solution spaces $W^\pm/F$ seem to depend on a particular splitting of $\bar{\bar{M}}^n$ into space and time components, through the special role played by t, or equivalently, by ∂_t. (D'_{2l} may also be described as the formal polynomial derivative of D_{2l} in the indeterminate ∂_t.) This also shows up in the polarization approach of Remark 2.10, in which the Hamiltonian quantity corresponding to ∂_t is used to compute the complex structure, and consequently the real part of $\langle\ ,\ \rangle_{W^+}$. But since the intertwining operator which determines the representation spaces is conformally covariant, one would expect these inner products to depend only on the conformal structure of $\bar{\bar{M}}^n$. Zuckerman [54] gives a formula for the well-known invariant inner product on Maxwell fields which makes it easy to see that only the conformal structure is involved (see Remark 4.3). The idea behind his formula is also applicable to all the equations studied here. (See [6, Sect. 3.6] for details in the cases of D_2, D_4, and the $D_{2,k}$, $D_{4,k}$ to be defined in Section 3.)

Remark 2.20. If we want to consider the *full* conformal group $O(2, n)$

of $\bar{\bar{M}}^n$, including the time reversal, we only get invariance of M, $V^+ + V^-$, $W^+ + W^-$, and F. For $l=1$, unitarity of the wave equation subspace is lost: on $(W^+ + W^-)/F$, $\langle\,,\rangle_{W^+} + \langle\,,\rangle_{W^-}$ is not invariant, while $\langle\,,\rangle_{W^+} - \langle\,,\rangle_{W^-}$ is invariant but indefinite. However, for any l, $M/(W^+ + W^-)$, and $(V^+ + V^-)/(W^+ + W^-)$ are unitary in $O(2,n)$-invariant inner products.

Remark 2.21 (odd n). Some elementary remarks are in order concerning the case of *odd* n. Solving the Cauchy problem for the equations $D_2\Phi = 0$, one finds that *all* solutions on $\mathbb{R} \times S^{n-1}$ are 4π-periodic, even for $n < 2l$. Indeed, the characteristic polynomials for the ODEs coming from (2.10) have no double roots: if p and q are integers of the same parity, the jth eigenvalues of $2(B+p)$ and $-2(B+q)$ cannot agree mod 4. Thus all solutions "live on" a double cover $\overset{(4)}{M}{}^n$ of $\bar{\bar{M}}^n$.

Scalar fields on $\overset{(4)}{M}{}^n$ can be grouped by frequency f (an integer or half-integer), and homogeneity $j = 0, 1, 2,\ldots.$ The conformal group of $\overset{(4)}{M}{}^n$ is a double cover of $O(2,n)$, and the orthochronous subgroup of this group acts separately, under any u_a, on 4 types of fields, corresponding to the possible values of $2(f+j)$ (mod 4). The entire conformal group, including the time-reversal, only distinguishes $2(f+j)$ (mod 2); that is, $2f$ (mod 2). The analogues of W^+ and W^- do not even lie within fields of the same type, so they do not reduce the same representation. Still, much can be said about decomposition and invariant inner products, but we shall not pursue this here.

Remark 2.22. Back in the case of even n, by looking at the universal cover $\tilde{M}^n = \mathbb{R} \times S^{n-1}$ of $\bar{\bar{M}}^n$ and its conformal group, the universal cover $\tilde{O}(2,n)$ of $O(2,n)$, we can include the non-periodic solutions of Theorem 2.12 in the representation theory. The ODE argument in Theorem 2.12 actually shows that the general solution of $D_{2l}\Phi = 0$ on $\tilde{M}^n$ is of the form $\Phi + t\Psi$, where Φ is a periodic solution and $\Psi \in F$. But

$$U_a(S)(t\Psi) = tU_a(S)\,\Psi + (\sin\omega)\,\Psi,$$

so that tF is not invariant, even under the orthochronous subgroup $\tilde{O}^{\uparrow}(2,n)$. However, $W^+ + W^- + tF$ is $u_{(n-2l)/2}(G)$-invariant, so that in addition to the representation spaces of Theorem 2.18, $(W^+ + W^- + tF)/W^{\pm}$ should also be considered.

Remark 2.23 (Representations of the Lorentz group). Consider now Riemannian S^{n-1} and its conformal group $O(1,n)$. The Y of Remark 1.5 is the "typical" proper conformal vector field on S^{n-1} in the same sense that

S is such on $\bar{\bar{M}}^n$. By the discussion preceding Lemma 2.8, if $\varphi \in E_j$ and (as before) $J = (n-2)/2 + j$,

$$\begin{aligned} \mathscr{L}(Y)\,\varphi &= \frac{1}{2J}\left\{\left(J - \frac{n-2}{2}\right)\varphi^+ + \left(J + \frac{n-2}{2}\right)\varphi^-\right\}, \\ \omega\varphi &= \frac{1}{2J}\{\varphi^+ - \varphi^-\}. \end{aligned} \tag{2.15}$$

(If $n = 2$ and $j = 0$, use $\mathscr{L}(Y)\,1 = 0$, $\omega \cdot 1 \in E_1$.) We already have one conformally covariant differential operator on functions in S^{n-1}, that given by (2.1)

$$(B^2 - \tfrac{1}{4})\, u_{(n-3)/2}(h)\,\varphi = u_{(n+1)/2}(h)(B^2 - \tfrac{1}{4})\,\varphi, \qquad h \in O(1, n).$$

But the *nonlocal* operator B also satisfies a covariance relation with respect to $\mathfrak{o}(1, n)$,

$$B\left(\mathscr{L}(Y) + \frac{n-2}{2}\,\omega\right)\varphi = \left(\mathscr{L}(Y) + \frac{n}{2}\,\omega\right)B\varphi. \tag{2.16}$$

Indeed, if $\varphi \in E_j$, both sides of (2.16) are $\frac{1}{2}((J+1)\,\varphi^+ + (J-1)\,\varphi^-)$. (If $n = 2$, $j = 0$, both sides are zero.) It can be shown (though we have not set up the machinery here) that as a result, B is $O(1, n)$-covariant. Note that by (2.15), we cannot use evenness and oddness under the antipodal map to decompose the $u_a(O(1, n))$.

The above suggests looking at the sequence of operators B, $(B + \frac{1}{2})(B - \frac{1}{2})$, $(B-1)\,B(B+1)$,..., i.e., at

$$D_p = \prod_{m=0}^{p-1}\left(B - \frac{p-1}{2} + m\right).$$

D_p is $\mathfrak{o}(1, n)$-covariant: if $\varphi \in E_j$,

$$\begin{aligned} &D_p U_{(n-1-p)/2}(Y)\,\varphi \\ &= \frac{1}{2J}\left\{\left[\prod_0^p \left(J - \frac{p-1}{2} + m\right)\right]\varphi^+ + \left[\prod_0^p \left(J - \frac{p+1}{2} + m\right)\right]\varphi^-\right\} \\ &= U_{(n-1+p)/2}(Y)\,D_p\varphi. \end{aligned}$$

(If $n = 2$ and $j = 0$, $D_p U_{(1-p)/2}(Y)\,1 = \prod_0^p (m - ((p-1)/2)) = U_{(1+p)/2}(Y)\,D_p 1$.) The D_{2q} are *differential* operators, so we immediately have $O(1, n)$-covariance for these by Remark 1.4.

The representation theoretic situation is as follows. For odd n,

$$\mathcal{N}(D_{2q}) = \begin{cases} \{0\} & \text{if } q < \dfrac{n-1}{2} \\ E_0 \oplus \cdots \oplus E_{q-(n-1)/2} & \text{if } q \geqslant \dfrac{n-1}{2}, \end{cases}$$

while $\mathcal{N}(D_{2q+1}) = 0$. The D_{2q} are differential operators, so we have representations of $O(1, n)$ on $\mathcal{N}(D_{2q})$ and $\mathscr{C}^\infty/\mathcal{N}(D_{2q})$. The argument of Theorem 2.9 shows that

$$\langle \varphi, \psi \rangle_{2q} \equiv (\varphi, D_{2q}\psi)_{L^2}$$

is a $u_{(n-1-2q)/2}$ $(O(1, n))$-invariant inner product on the infinite dimensional factor $C^\infty/\mathcal{N}(D_{2q})$. $\langle\ ,\ \rangle$ is clearly positive definite, so upon completion, we get continuous unitary representations. The $\langle \varphi, \psi \rangle_{2q+1} = (\varphi, D_{2q+1}\psi)_{L^2}$ are invariant (skew) for $\mathrm{U}_{(n-2-2q)/2}(\mathfrak{o}(1, n))$ on C^∞, and positive definite for $q < (n-2)/2$. For $q > (n-2)/2$, $\langle\ ,\ \rangle_{2q+1}$ is still nondegenerate, but its sign on E_j is

$$\begin{cases} +1, & j > q - \dfrac{n-2}{2} \\ (-1)^{j+1+q-((n-1)/2)}, & j < q - \dfrac{n-2}{2}. \end{cases}$$

For even n, the situation is similar, except that it is the nonlocal operators D_{2q+1} that may have nontrivial null spaces. Note that $O(1, n)$ is the Lorentz group usually associated with $(n+1)$-dimensional Minkowski space, while $\mathrm{O}(2, n)$ is the conformal group associated to n-dimensional Minkowski space. To some extent, the above generalizes the $SL(2, \mathbb{C})$-Riemann sphere picture of the Lorentz group $O(1, 3)$.

3. Differential Form Representations

a. *A General Second Order Conformally Covariant Operator on Forms of Arbitrary Order*

It is natural to ask whether a decomposition like that of Section 2 exists for some of the representations u_a of $O^\uparrow(2, n)$ on differential forms in $\bar{\bar{M}}^n$. A partial answer is given by the theory of the Maxwell equations [8, 13, 32], which gives information about $n/2$- and $(n-2)/2$-forms (Sect. 4.a). But as it turns out, the Maxwell picture is atypical of the general situation; after a gauge is fixed, it resembles the scalar picture more than it does the general differential form picture.

The key to the composition structure of the u_a on forms of *arbitrary* order seems to be a general second order conformally covariant operator $D_{2,k}$ on k-forms introduced by the author in [4].

THEOREM 3.1 [4]. *Let (M, g) be a pseudo-Riemannian manifold of dimension $n \geqslant 3$; let R be the scalar curvature and $(R_{\alpha\beta})$ the Ricci tensor of (M, g). Then the operator*

$$D_{2,k} = (s+1)\,\delta d + (s-1)\,d\delta + (s+1)(s-1)(\tilde{R} - 2V\#),$$

where

$$s = (n-2k)/2,$$

$$\tilde{R} = R/2(n-1), \qquad V_{\alpha\beta} = (R_{\alpha\beta} - \tilde{R}g_{\alpha\beta})/(n-2),$$

and $V\# = (V^{\alpha}{}_{\beta})\#$ is as in Definition 1.1, is conformally covariant of bidegree $(s-1, s+1)$,

$$D_{2,k}u_{s-1}(h)\,\varphi = u_{s+1}(h)\,D_{2,k}\varphi, \qquad h \in \mathscr{C}(M, g). \tag{3.1}$$

That is, $D_{2,k}$ intertwines the representations $u_{s\pm 1}$. As special cases, we get the covariance of

$$D_{2,0} = \frac{n+2}{2} D_2,$$

and the "Maxwell operator on vector potentials"($(n-2)/2$-forms for even n), δd.

Returning to the setting of $\bar{\bar{M}}^n = S^1 \times S^{n-1}$ and $\tilde{M}^n = \mathbb{R} \times S^{n-1}$, the effect of $D_{2,k}$ can be seen concretely as follows. First decompose a k-form Φ as

$$\Phi = dt \wedge \Phi_0 + \Phi_1,$$

where Φ_1 (resp. Φ_0) is a t-dependent k-form (resp. $(k-1)$-form) on S^{n-1}. From now on, we shall reserve the symbols d and δ for the exterior derivative and coderivative in Riemannian S^{n-1}; the analogous operators in $\bar{\bar{M}}^n$ or $\tilde{M}^n$ will be called $d^{(n)}$, $\delta^{(n)}$. Now the only harmonic forms on S^{n-1} are the constant 0-forms and the $(n-1)$-forms which are constant multiples of the normalized volume element, so we may Hodge decompose a p-form φ on S^{n-1} as

$$\varphi = \varphi_\delta + \varphi_d,$$

where

$$\varphi_\delta \in \mathscr{R}(\delta), \qquad \varphi_d \in \mathscr{R}(d), \qquad 1 \leqslant p \leqslant n-2,$$

or $\varphi = \varphi_\delta$ for $p=0$, $\varphi = \varphi_d$ for $p = n-1$ (arbitrarily assigning the constants to the "δ sector" in the bottom order, and the "d sector" in the top order). At fixed t, the components of the above k-form Φ on $\bar{\bar{M}}^n$ or $\tilde{M}^n$ can be so decomposed

$$\Phi_0 = \Phi_{0\delta} + \Phi_{0d}, \qquad \Phi_1 = \Phi_{1\delta} + \Phi_{1d}.$$

We shall call the space of possible 0δ components the 0δ *sector*, and similarly for $0d$, 1δ, $1d$. The $\binom{0\delta}{1d}$ *sector* will refer to the direct sum of the 0δ and $1d$ sectors, and similarly for $\binom{0d}{1\delta}$.

The Ricci tensor of S^{n-1} (and thus of $\bar{\bar{M}}^n$ and $\tilde{M}^n$) is $(n-2)\, g_{S^{n-1}}$, and

$$\begin{aligned} d^{(n)}\Phi &= dt \wedge (\partial_t \Phi_1 - d\Phi_0) + d\Phi_1, \\ \delta^{(n)}\Phi &= -dt \wedge \delta\Phi_0 + \partial_t \Phi_0 + \delta\Phi_1, \end{aligned} \tag{3.2}$$

so

$$\begin{aligned} D_{2,k}\Phi = dt \wedge \{&(s-1)\,\partial_t^2\Phi_0 + (s+1)\,\delta d\Phi_0 + (s-1)\, d\delta\Phi_0 \\ &- 2\delta\partial_t\Phi_1 + (s+1)^2(s-1)\,\Phi_0\} \\ &+ \{(s+1)\,\partial_t^2\Phi_1 + (s+1)\,\delta d\Phi_1 + (s-1)\, d\delta\Phi_1 \\ &- 2d\partial_t\Phi_0 + (s+1)(s-1)^2\Phi_1\}. \end{aligned}$$

Thus $D_{2,k}$ acts separately in the $0d$, 1δ, and $\binom{0\delta}{1d}$ sectors

$$\begin{aligned} (D_{2,k}\Phi)_{0d} &= (s-1)(\partial_t^2 + d\delta + (s+1)^2)\,\Phi_{0d}, \\ (D_{2,k}\Phi)_{1\delta} &= (s+1)(\partial_t^2 + \delta d + (s-1)^2)\,\Phi_{1\delta} \end{aligned}$$

$$\begin{pmatrix} (D_{2,k}\Phi)_{0\delta} \\ (D_{2,k}\Phi)_{1d} \end{pmatrix} = \left(\begin{array}{c|c} (s-1)\,\partial_t^2 + (s+1)\,\delta d + (s+1)^2(s-1) & -2\delta\partial_t \\ \hline -2d\partial_t & (s+1)\,\partial_t^2 + (s-1)\, d\delta + (s+1)(s-1)^2 \end{array}\right) \times \begin{pmatrix} \Phi_{0\delta} \\ \Phi_{1d} \end{pmatrix}.$$

b. *Wave Propagative Properties of* $D_{2,k}$

It is remarkable that the conformally covariant operator $D_{2,k}$ is exactly the correct second-order differential operator on forms to produce results of automatic periodicity, oddness/evenness, and Huygens' principle. For the moment, set aside the Maxwell case $k=(n-2)/2$ ($s=1$) and its "mirror image" under the Hodge star operator, $k=(n+2)/2$ ($s=-1$).

THEOREM 3.2 (Automatic periodicity). *Suppose $n \geqslant 4$ is even, and $s \neq \pm 1$. Then every C^∞ solution of $D_{2,k}\Phi=0$ on $\mathbb{R}\times S^{n-1}$ is 2π-periodic in t.*

Proof. By [11], the eigenvalues of $d\delta$ on closed p-forms in S^{n-1}, $1 \leqslant p \leqslant n-2$, are

$$\mu=\mu_j=\alpha\beta,$$

where (3.3)

$$\alpha=\alpha_j=j-1+p, \qquad \beta=\beta_j=n-1-p+j$$

for $j=1, 2, 3,...,$ and the eigenvalues of δd on coclosed p-forms, $1 \leqslant p \leqslant n-2$, are

$$\lambda=\lambda_j=\sigma\tau,$$

where (3.4)

$$\sigma=\sigma_j=j+p, \qquad \tau=\tau_j=n-2-p+j$$

for $j=1, 2, 3,....$ (The parameters α, β, σ, τ will be important in our treatment of the representation theory of $D_{2,k}$ and its higher order generalizations.) This means that the nonlocal operators

$$\begin{aligned}
A&=\sqrt{d\delta+(s+1)^2} && \text{in the } 0d \text{ sector},\\
B&=\sqrt{\delta d+(s-1)^2} && \text{in the } 1\delta \text{ sector},\\
C&=\begin{cases}\sqrt{\delta d+s^2} \\ \sqrt{d\delta+s^2}\end{cases} && \begin{matrix}\text{in the } 0\delta \text{ sector},\\ \text{in the } 1d \text{ sector}\end{matrix}
\end{aligned}$$

have eigenvalues

$$J=\frac{n-2}{2}+j$$

for $j=1, 2, 3,...,$ plus (taking account of the possible eigenvalue $(n-2)/2$ on constant 0- or $(n-1)$-forms) $j=0$ for B when $k=0$, for C in the 0δ sector when $k=1$, for C in the $1d$ sector when $k=n-1$, and for A when $k=n$.

In terms of A, B, and C, $D_{2,k}$ acts as

$$(s-1)(\partial_t+iA)(\partial_t-iA) \quad \text{in} \quad 0d$$

$$(s+1)(\partial_t+iB)(\partial_t-iB) \quad \text{in} \quad 1\delta$$

$$\mathscr{B}=\left(\begin{array}{c|c} (s-1)\,\partial_t^2+(s+1)(C^2-1) & -2\delta\partial_t \\ \hline -2d\partial_t & (s+1)\,\partial_t^2+(s-1)(C^2-1) \end{array}\right) \quad \text{in} \quad \begin{pmatrix} 0\delta \\ 1d \end{pmatrix}. \tag{3.5}$$

Let $E_{0d,j}$ (resp. $E_{1\delta,j}$, $E_{0\delta,j}$, $E_{1d,j}$) be the $(n-2)/2+j$ eigenspace of A (resp. B, C, C) in the $0d$ (resp. 1δ, 0δ, $1d$) sector. Note that

$$d\colon\ E_{0\delta,j}\to E_{1d,j}, \qquad \delta\colon\ E_{1d,j}\to E_{0\delta,j},$$

and that δ and d commute with C. By the compactness of S^{n-1}, ∂_t commutes, on C^∞ forms, with the Hodge projections and the projections onto the E-spaces. This reduces the Cauchy problem for $D_{2,k}\Phi=0$ to initial value problems for ODEs valued in the (finite-dimensional) E-spaces. Periodicity for the $0d$ and 1δ components is now immediate from (3.5) and the fact that the eigenvalues of A and B are positive integers. For the $\binom{0\delta}{1d}$ component, left multiplication of the operator $\mathscr{B}$ of (3.5) by

$$\tilde{\mathscr{B}}=\left(\begin{array}{c|c} (s+1)\,\partial_t^2+(s-1)(C^2-1) & 2\delta\partial_t \\ \hline 2d\partial_t & (s-1)\,\partial_t^2+(s-1)(C^2-1) \end{array}\right)$$

yields the block diagonal operator

$$\tilde{\mathscr{B}}\mathscr{B}=(s^2-1) \times\left(\begin{array}{c|c} \partial_t^4+2(C^2+1)\,\partial_t^2+(C^2-1)^2 & 0 \\ \hline 0 & \partial_t^4+2(C^2+1)\,\partial_t^2+(C^2-1)^2 \end{array}\right).$$

But

$$\partial_t^4+2(C^2+1)\,\partial_t^2+(C^2-1)^2 = [\partial_t+i(C+1)][\partial_t-i(C+1)][\partial_t+i(C-1)][\partial_t-i(C-1)],$$

and the eigenvalues of $C\pm 1$ are positive integers. ($C-1$ takes on the value 0 only for $n=4$, $k=1$ or 3, in which case $s=\pm 1$.) Thus we also have periodicity for the $\binom{0\delta}{1d}$ component. ∎

Remark 3.3. We have not quite completed the solution of the Cauchy problem for $D_{2,k}\Phi=0$ in the proof of Theorem 3.2. One way of dealing with the $\binom{0\delta}{1d}$ sector is to solve the Cauchy problem for $\tilde{\mathscr{B}}\mathscr{B}\binom{\Phi_{0\delta}}{\Phi_{1d}}=0$, and treat the equation $\mathscr{B}\binom{\Phi_{0\delta}}{\Phi_{1d}}=0$ and its first t-derivative as (conserved) Cauchy data constraints. The general solution of this Cauchy problem is a linear combination of solutions of the 4 forms

$$e^{\pm i(C\pm '1)t}\begin{pmatrix}\varphi_{0\delta}(x)\\ \varphi_{1d}(x)\end{pmatrix}, \tag{3.6}$$

and the Cauchy data constraints on the solution (3.6) reduce to

$$\pm id\varphi_{0\delta}+(C\pm 's)\,\varphi_{1d}=0,$$

or, equivalently,

$$\pm i(C\mp 's)\,\varphi_{0\delta}+\delta\varphi_{1d}=0.$$

Alternatively, one could work directly with the second-order ODE problems produced by $\mathscr{B}\binom{\varphi_{0\delta}}{\varphi_{1d}}=0$. First assume that $k\neq 0, n$ (the scalar case has been dealt with in Sect. 2). With $s\neq\pm 1$, this eliminates $n=4$ from consideration. Next, note that the "constant" components (of $\varphi_{0\delta}$ for $k=1$ or φ_{1d} for $k=n-1$) must be harmonic oscillators of frequency $\pm(n-4)/2$. Thus we may restrict to the space of fields with no "constant" components; on this space, $C\pm 1$ and $C\pm s$ are positive operators. Let ψ be the Cauchy datum for $\partial_t\varphi$, and make the change of variable

$$y_{0\delta}=\frac{1}{s-1}\{-(C+1)\,\varphi_{0\delta}+(C-s)^{-1}\delta\psi_{1d}\},$$

$$y_{1d}=\frac{1}{s+1}\{-(C-1)\,\varphi_{1d}-(C-s)^{-1}d\psi_{0\delta}\},$$

$$z_{0\delta}=\frac{1}{s+1}\{-(C-1)\,\psi_{0\delta}-(C^2-1)(C-s)^{-1}\delta\varphi_{1d}\},$$

$$z_{1d}=\frac{1}{s-1}\{-(C+1)\,\psi_{1d}+(C^2-1)(C-s)^{-1}d\varphi_{0\delta}\}$$

in Cauchy data space. The (φ,ψ) data may be recovered from the (y,z) data by

$$\begin{aligned}2C(C+1)\,\varphi_{0\delta}&=(C+1)(C-s)\,y_{0\delta}+\delta z_{1d},\\ 2C\psi_{0\delta}&=(C-s)\,z_{0\delta}-(C+1)\,\delta y_{1d},\\ 2C(C-1)\,\varphi_{1d}&=(C-1)(C-s)\,y_{1d}-dz_{0\delta},\\ 2C\psi_{1d}&=(C-s)\,z_{1d}+(C-1)\,dy_{0\delta}.\end{aligned}$$

In (y, z) coordinates, the evolution equation becomes

$$\partial_t y_{0\delta} = z_{0\delta}, \qquad \partial_t z_{0\delta} = -(C-1)^2 y_{0\delta},$$
$$\partial_t y_{1d} = z_{1d}, \qquad \partial_t z_{1d} = -(C+1)^2 y_{1d}.$$

This decomposes the system into harmonic oscillators, and separates the "frequency $C \pm 1$" parts.

THEOREM 3.4 (oddness–evenness). *Suppose $n \geqslant 4$ is even, and $s \neq \pm 1$. If Φ is a C^∞ solution of $D_{2,k}\Phi = 0$ on $S^1 \times S^{n-1}$ and $\eta(t, x) = (t + \pi, -x)$, then $\eta^*\Phi = (-1)^{s-1}\Phi$. In particular, if s is odd, Φ "lives on" (covers a field on) $\bar{M}^n = (S^1 \times S^{n-1})/\mathbb{Z}_2$.*

Proof. The μ_j eigenspace of closed p-forms φ on S^{n-1} (recall (3.3)) consists of restrictions (pullbacks under inclusion) to S^{n-1} of closed harmonic p-forms in $\mathbb{R}^n$ whose components (in the standard basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mid i_1 < \cdots < i_p\}$) are $(j-1)$-homogeneous polynomials in the x_i [11]. Such forms have parity $(-1)^{j-1+p}$ under the antipodal map. The λ_j eigenspace of coclosed p-forms on S^{n-1} (recall (3.4)) is dual to the μ_j eigenspace of closed $(n-1-p)$-forms via the Hodge $* = *_{S^{n-1}}$. If ζ: $S^{n-1} \to \mathbb{R}^n$ is the inclusion,

$$*\zeta^*\varphi = \pm\zeta^*\iota(X) *_{\mathbb{R}^n} \varphi, \qquad X = \sum_1^n x_i\, dx_i.$$

This shows that elements of the λ_j eigenspace have parity $(-1)^{j+p}$ under the antipodal map (using the evenness of n).

For $f = 0, \pm 1, \pm 2, \ldots$, let $\mathscr{E}_{0\delta,f,j}$ be the space of fields $e^{ift}\varphi(x)$, where $\varphi \in E_{0\delta,j}$, and similarly for $\mathscr{E}_{0d,f,j}$, $\mathscr{E}_{1\delta,f,j}$, $\mathscr{E}_{1d,f,j}$. By the above,

$$\begin{aligned} \eta^*\Phi &= (-1)^{j+k+f}\Phi, \qquad && \Phi \in \mathscr{E}_{0d,f,j} \text{ or } \mathscr{E}_{1\delta,f,j}, \\ \eta^*\Phi &= (-1)^{j-1+k+f}\Phi, && \Phi \in \mathscr{E}_{1d,f,j} \text{ or } \mathscr{E}_{0\delta,f,j}. \end{aligned} \tag{3.7}$$

But by our solution of the Cauchy problem, L^2 solutions are spanned by the $\mathscr{E}_{0d,f,j} \oplus \mathscr{E}_{1\delta,f,j}$ with $f = \pm((n-2)/2 + j)$, and by subspaces of the $\mathscr{E}_{0\delta,f,j} \oplus \mathscr{E}_{1d,f,j}$ with $f = \pm((n-2)/2 + j \pm' 1)$. (See Remark 2.6 for a discussion of what L^2 means in this setting.) This and $\eta^*dt = dt$ show that the total parity of solutions under η is

$$(-1)^{((n-2)/2)+k} = (-1)^{s-1}. \quad \blacksquare$$

To picture the null space $\mathcal{N}(D_{2,k})$, draw an upper half-plane diagram, this time separating $\binom{0d}{1\delta}$ and $\binom{0\delta}{1d}$ sectors (Fig. 3.1). A point now represents a sum of $\mathscr{E}$-spaces; for example, (f, j) in the $\binom{0\delta}{1d}$ diagram represents

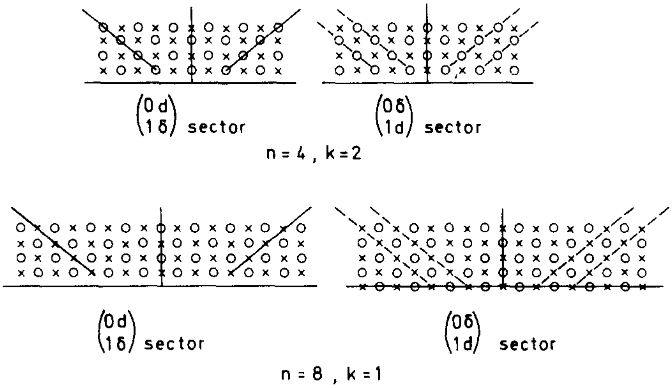


FIGURE 3.1

$\mathscr{E}_{0\delta,f,j} \oplus \mathscr{E}_{1d,f,j}$. Evenness and oddness ($\times$ and $\circ$) are in the sense $\eta^*\Phi = \pm\Phi$. The dotted lines indicate that exactly half of the harmonic oscillators at each point of $f = \pm(J \pm '1)$ are included. The line $j = 0$ is sometimes empty, reflecting the fact that there are no harmonic forms in the orders or sectors in question.

THEOREM 3.5 (Huygens' principle). *Suppose $n \geqslant 4$ is even, and $s \neq \pm 1$. If $\Phi \in C^\infty$, $D_{2,k}\Phi = 0$, and the support of the Cauchy data $\Phi(0, x)$, $(\partial_t \Phi)(0, x)$ contains no point at a distance t_0 from $x_0 \in S^{n-1}$, $0 \leqslant t_0 \leqslant \pi$, then $\Phi(t_0, x_0) = 0$.*

Proof. Since $\{(s+1)^{-1}\delta^{(n)}d^{(n)} + (s-1)^{-1}d^{(n)}\delta^{(n)}\}\, D_{2,k}$ has leading term $\Box^2$, the equation $D_{2,k}\Phi = 0$ has finite propagation speed. (Treat the Cauchy problem for $D_{2,k}\Phi = 0$ as that for $(\Box^2 + \text{lower order})\,\Phi = 0$, with $D_{2,k}\Phi = 0$ and its first t-derivative as conserved fixed-time constraints. By Section 3.a, the constraints can be solved to give $\partial_t^2\Phi$ and $\partial_t^3\Phi$ at time 0 as linear expressions in Φ and $\partial_t\Phi$ at time 0. Thus Cauchy data for $D_{2,k}\Phi = 0$ with small support lead to Cauchy data for the fourth-order constrained system with the same small support. Finite propagation speed for $(\Box^2 + \text{lower order})\,\Phi = 0$ thus implies finite propagation speed for $D_{2,k}\Phi = 0$.) But given finite propagation speed, the Lax–Phillips oddness–evenness method of Theorem 2.4 now works perfectly. ▌

Remark 3.6. The fact that the Hodge projections and the operators A, B, C are nonlocal is not an issue in Theorem 3.5; all we need to apply the Lax–Phillips argument is finite propagation speed, an oddness–evenness principle, and a well-posed Cauchy problem.

c. *Representation Theoretic Content of* $D_{2,k}$

Let S, Y, and ω be as in Section 2. Our analysis of the composition structure of the representations u_a on k-forms will rest, as in the scalar case, on the "moves" in Fig. 3.1 permitted under $U_a(S)$. Using elementary properties of the Lie derivative, one finds that if Φ is a C^∞ k-form on $\bar{\tilde{M}}^n$ or $\tilde{M}^n$,

$$\begin{aligned} U_a(S)\,\Phi = dt \wedge \{&\cos t[\mathscr{L}(Y) + (a+1)\,\omega]\,\Phi_0 \\ &+ \sin t[\omega\partial_t \Phi_0 + \iota(d\omega)\,\Phi_1]\} \\ &+ \cos t[\mathscr{L}(Y) + a\omega]\Phi_1 \\ &+ \sin t[\omega\partial_t \Phi_1 + \varepsilon(d\omega)\,\Phi_0]. \end{aligned} \tag{3.8}$$

From now on, abbreviate $\varepsilon(d\omega)$ and $\iota(d\omega)$ by ε and i. The following four lemmas, analogous to Lemma 2.7 in the scalar case, are needed to convert (3.8) into hard information about Fig. 3.1. This same information will be needed again in the analysis of higher order generalizations $D_{2l,k}$ of the $D_{2,k}$ (Sect. 3.e).

LEMMA 3.7. *Suppose $n \geqslant 2$ (n need not be even here). Let $\lambda = \lambda_j$, $\mu = \mu_j$, etc., be as in* (3.3) and (3.4), *and let $J = (n-2)/2 + j$. If φ is a coclosed p-form in the λ_j eigenspace $E_{p,\delta,j}$ of δd on S^{n-1}, $j \geqslant 1$, then*

$$\varphi^+ \equiv \frac{1}{2J\mu}\,[\delta(\omega d\varphi) + \tau\delta\varepsilon\varphi] \in E_{p,\delta,j+1},$$

$$\varphi^- \equiv \frac{1}{2J\mu}\,[\delta(\omega d\varphi) - \sigma\delta\varepsilon\varphi] \in E_{p,\delta,j-1},$$

$$\varphi^0 \equiv \frac{1}{\mu}\,d\iota\varphi \in E_{p,d,j},$$

where $E_{p,d,j}$ is the μ_j eigenspace of closed p-forms. If $\psi \in E_{p,d,j}$, $j \geqslant 1$, then

$$\psi^+ \equiv \frac{1}{2J\lambda}\,[d(\omega\delta\psi) - \alpha d\iota\psi] \in E_{p,d,j+1},$$

$$\psi^- \equiv \frac{1}{2J\lambda}\,[d(\omega\delta\psi) + \beta d\iota\psi] \in E_{p,d,j-1},$$

$$\psi^0 \equiv \frac{1}{\lambda}\,\delta\varepsilon\psi \in E_{p,\delta,j}.$$

Proof. Recall the identities (1.3), $(d\omega)_\alpha = -Y_\alpha$, and

$$\begin{aligned}
\mathscr{L}(Y) &= \iota(Y)\,d + d\iota(Y) = -\iota d - d\iota,\\
\mathrm{L}(Y)^* &= -\delta\varepsilon - \varepsilon\delta,\\
d\mathscr{L}(Y) &= \mathscr{L}(Y)\,d,\\
\mathscr{L}(Y)^*\delta &= \delta\mathscr{L}(Y)^*.
\end{aligned}$$

Using these, we get

$$\begin{aligned}
\delta d(\delta(\omega d\varphi)) &= \delta d(\omega\lambda\varphi - \iota d\varphi)\\
&= \lambda\delta(\omega d\varphi + \varepsilon\varphi) + \delta\mathscr{L}(Y)\,d\varphi\\
&= \lambda\delta(\omega d\varphi + \varepsilon\varphi) + \delta(-\mathscr{L}(Y)^* - (n-2p-3)\,\omega)\,d\varphi\\
&= \lambda\delta(\omega d\varphi + \varepsilon\varphi) + \delta(\varepsilon\delta - (n-2p-3)\,\omega)\,d\varphi\\
&= [\lambda - (n-2p-3)]\,\delta(\omega d\varphi) + 2\lambda\delta\varepsilon\varphi,
\end{aligned}$$

and

$$\begin{aligned}
\delta d(\delta\varepsilon\varphi) &= -\delta d\mathscr{L}(Y)^*\varphi\\
&= \delta d(\mathscr{L}(Y) + (n-2p-1)\,\omega)\,\varphi\\
&= \delta\mathscr{L}(Y)\,d\varphi + (n-2p-1)\,\delta(\omega d\varphi + \varepsilon\varphi)\\
&= -\delta(\mathscr{L}(Y)^* + (n-2p-3)\,\omega)\,d\varphi + (n-2p-1)\,\delta(\omega d\varphi + \varepsilon\varphi)\\
&= \delta\varepsilon\delta d\varphi + 2\delta(\omega d\varphi) + (n-2p-1)\,\delta\varepsilon\varphi\\
&= (\lambda + n - 2p - 1)\,\delta\varepsilon\varphi + 2\delta(\omega d\varphi).
\end{aligned}$$

But

$$\lambda_{j+1} - \lambda_j = n + 2j - 1,$$

so (after some calculation)

$$\begin{aligned}
\delta d\{\delta(\omega d\varphi) + \tau\delta\varepsilon\varphi\} &= \lambda_{j+1}\{\delta(\omega d\varphi) + \tau\delta\varepsilon\varphi\},\\
\delta d\{\delta(\omega d\varphi) - \sigma\delta\varepsilon\varphi\} &= \lambda_{j-1}\{\delta(\omega d\varphi) - \sigma\delta\varepsilon\varphi\};
\end{aligned}$$

this proves the assertions about $\varphi^{\pm}$. As for φ^0,

$$\begin{aligned}
d\delta(d\iota\varphi) &= d\delta(-\mathscr{L}(Y) - \iota d)\,\varphi\\
&= d\delta(\mathscr{L}(Y)^* + (n-2p-1)\,\omega - \iota d)\,\varphi.
\end{aligned}$$

But $d\varepsilon = -\varepsilon d$ implies $\iota\delta = -\delta\iota$, and since φ is coclosed, $\delta\mathscr{L}(Y)^*\varphi = 0$. Thus

$$\begin{aligned} d\delta(d\iota\varphi) &= d(-(n-2p-1)\,\iota + \iota\delta d\varphi) \\ &= [\lambda - (n-2p-1)]\, d\iota\varphi. \end{aligned}$$

Since

$$\lambda - \mu = n - 2p - 1, \tag{3.9}$$

we have

$$d\delta(d\iota\varphi) = \mu d\iota\varphi,$$

as desired.

Similar arguments (or an application of Hodge $*$-duality) yield

$$\begin{aligned} d\delta(d(\omega\delta\psi)) &= (\mu + n - 2p + 1)\, d(\omega\delta\psi) - 2\mu d\iota\psi, \\ d\delta(d\iota\psi) &= [\mu - (n-2p-1)]\, d\iota\psi - 2d(\omega\delta\psi); \end{aligned}$$

these, along with

$$\mu_{j+1} - \mu_j = n + 2j - 1,$$

imply the assertions about $\psi^{\pm}$. Similarly, using (3.9), we get

$$\delta d(\delta\varepsilon\psi) = \lambda\delta\varepsilon\psi,$$

as desired for the ψ^0 assertion. ∎

Remark 3.8. "Inverting" the formulas of Lemma 3.7, we get

$$\begin{aligned} \delta(\omega d\varphi) &= \mu(\sigma\varphi^+ + \tau\varphi^-), \\ \delta\varepsilon\varphi &= \mu(\varphi^+ - \varphi^-), \\ d\iota\varphi &= \mu\varphi^0, \\ d(\omega\delta\psi) &= \lambda(\beta\psi^+ + \alpha\psi^-), \\ d\iota\psi &= \lambda(-\psi^+ + \psi^-), \\ \delta\varepsilon\psi &= \lambda\psi^0. \end{aligned}$$

LEMMA 3.9. *Under the assumptions of Lemma* 3.7,

$$\begin{aligned}\omega\varphi &= \alpha\varphi^{+} + \beta\varphi^{-} - \varphi^{0},\\ \mathscr{L}(Y)\,\varphi &= \sigma\alpha\varphi^{+} - \tau\beta\varphi^{-} + (n-2p-1)\,\varphi^{0},\\ \omega\psi &= \tau\psi^{+} + \sigma\psi^{-} + \psi^{0},\\ \mathscr{L}(Y)\,\psi &= \lambda(\psi^{+} - \psi^{-}).\end{aligned}$$

Proof. By the identities used in the proof of Lemma 3.7,

$$\begin{aligned}\omega\varphi &= \frac{1}{\lambda}\,\omega\delta\, d\varphi = \frac{1}{\lambda}\,\{\delta(\omega\, d\varphi) + \iota\, d\varphi\}\\ &= \frac{1}{\lambda}\,\{\delta(\omega\, d\varphi) - d\iota\varphi - \mathscr{L}(Y)\,\varphi\}.\end{aligned}$$

But on coclosed p-forms,

$$-\mathscr{L}(Y) = \mathscr{L}(Y)^{*} + (n-2p-1)\,\omega = -\delta\varepsilon + (n-2p-1)\,\omega,$$

so

$$\delta(\omega\, d\varphi) - d\iota\varphi - \delta\varepsilon\varphi = [\lambda - (n-2p-1)]\,\omega\varphi = \mu\omega\varphi.$$

With Remark 3.8, this proves the assertion about $\omega\varphi$. Since

$$\begin{aligned}\mathscr{L}(Y)\,\varphi &= [-\mathscr{L}(Y)^{*} - (n-2p-1)\,\omega]\,\varphi\\ &= \delta\varepsilon\varphi - (n-2p-1)\,\omega\varphi,\end{aligned}$$

we also have the assertion about $\mathscr{L}(Y)\,\varphi$. The other two formulas follow from similar arguments, or by Hodge $*$-duality. ∎

LEMMA 3.10. *Under the assumptions of Lemma* 3.7,

$$\begin{aligned}d\varphi^{+} &= \frac{(\sigma+1)(\tau-1)}{\alpha\sigma}\,(d\varphi)^{+},\\ d\varphi^{-} &= \frac{(\sigma+1)(\tau-1)}{\beta\tau}\,(d\varphi)^{-},\\ \delta\psi^{+} &= \frac{(\alpha-1)(\beta+1)}{\tau\beta}\,(\delta\psi)^{+},\\ \delta\psi^{-} &= \frac{(\alpha-1)(\beta+1)}{\sigma\alpha}\,(\delta\psi)\end{aligned}$$

Proof. Let $\varphi^{(1)}=\delta(\omega d\varphi)$, $\varphi^{(2)}=\delta\varepsilon\varphi$, $\psi^{(1)}=d(\omega\delta\psi)$, $\psi^{(2)}=d\iota\psi$. Then

$$\begin{aligned} d\varphi^{(1)} &= d\delta(\omega\, d\varphi)=d(\omega\delta\, d\varphi-\iota\, d\varphi)\\ &=(d\varphi)^{(1)}-(d\varphi)^{(2)},\\ d\varphi^{(2)} &= d\delta\varepsilon\varphi=-d\mathscr{L}(Y)^*\,\varphi\\ &=d(\mathscr{L}(Y)+(n-2p-1)\,\omega)\,\varphi\\ &=\mathscr{L}(Y)\,d\varphi+\frac{n-2p-1}{\lambda}\,d(\omega\delta\, d\varphi)\\ &=-(d\varphi)^{(2)}+\frac{n-2p-1}{\lambda}\,(d\varphi)^{(1)}. \end{aligned}$$

Remark 3.8 and some computation now give the formulas for $d\varphi^{\pm}$, and similar arguments or duality give the formulas for $\delta\psi^{\pm}$. ▮

LEMMA 3.11. *Under the assumptions of Lemma* 3.7,

$$\begin{aligned} \iota\varphi &= \delta\varphi^0,\\ \varepsilon\varphi &= \frac{\tau-1}{\sigma}\,(d\varphi)^+-\frac{\sigma+1}{\tau}\,(d\varphi)^--(d\varphi)^0,\\ \varepsilon\psi &= d\psi^0,\\ \iota\psi &= -\frac{\alpha-1}{\beta}\,(\delta\psi)^++\frac{\beta+1}{\alpha}\,(\delta\psi)^--(\delta\psi)^0. \end{aligned}$$

Proof. By Lemma 3.9 and the fact that φ, $\varphi^{\pm}$ are coclosed, $\iota\varphi=-\delta(\omega\varphi)=\delta\varphi^0$. Also by Lemma 3.9,

$$\begin{aligned} \varepsilon\varphi &= d(\omega\varphi)-\omega\, d\varphi\\ &=\alpha\, d\varphi^++\beta\, d\varphi^--[(\tau-1)(d\varphi)^++(\sigma+1)(d\varphi)^-+(d\varphi)^0], \end{aligned}$$

so Lemma 3.10 gives the formula for $\varepsilon\varphi$. Similarly (or by duality), we get the formulas for $\varepsilon\psi$, $\iota\psi$. ▮

We can now convert (3.8) into an analogue of (2.8) using Lemmas 3.7, 3.9, and 3.11. If Φ is a k-form in $\bar{\bar{M}}^n$, express Φ as

$$\sum_{f=-\infty}^{\infty}\sum_{j=0}^{\infty} e^{ift}[dt\wedge(\Phi_{0\delta,f,j}+\Phi_{0d,f,j})+\Phi_{1\delta,f,j}+\Phi_{1d,f,j}], \qquad (3.10)$$

TABLE I

$\mathscr{E}_{0\delta,f,j}$ Probe: $\Phi = dt \wedge e^{ift}\varphi_{0\delta}$, $\varphi_{0\delta} \in E_{0\delta,j}$, $j \geqslant 1$. $\Phi' = U_a(S)\,\Phi$

$$\begin{aligned}
\Phi'_{0\delta,f\pm 1,j+1} &= \tfrac{1}{2}(\alpha - 1)(\alpha + a + 1 \pm f)\,\varphi^{+}_{0\delta} \\
\Phi'_{0\delta,f\pm 1,j-1} &= \tfrac{1}{2}(\beta + 1)(-\beta + a + 1 \pm f)\,\varphi^{-}_{0\delta} \\
\Phi'_{0d,f\pm 1,j} &= \tfrac{1}{2}(n - 2k - a \mp f)\,\varphi^{0}_{0\delta} \\
\Phi'_{1\delta,f\pm 1,j} &= \mp(1/2i)(d\varphi_{0\delta})^{0} \\
\Phi'_{1d,f\pm 1,j+1} &= \pm(1/2i)\,\frac{\tau}{\alpha}\,(d\varphi_{0\delta})^{+} \\
\Phi'_{1d,f\pm 1,j-1} &= \mp(1/2i)\,\frac{\sigma}{\beta}\,(d\varphi_{0\delta})^{-}
\end{aligned}$$

where $\Phi_{0\delta,f,j} \in E_{0\delta,j}$, $\Phi_{0d,f,j} \in E_{0d,j}$, $\Phi_{1\delta,f,j} \in E_{1\delta,j}$, and $\Phi_{1d,f,j} \in E_{1d,j}$. Membership of Φ in C^{∞}, L^2, or H^m is determined by the rate of decay of $\|\Phi_{0\delta,f,j}\|_{L^2}$, etc., as $|f| + j \to \infty$, as in Section 2.b. The L^2 norm is defined using any (artificial) Riemannian metric on $\bar{M}^n$ (recall Remark 2.6). Now consider an "$\mathscr{E}_{0\delta,f,j}$ probe," i.e., a k-form Φ whose expansion (3.10) consists of the one term $dt \wedge e^{ift}\varphi_{0\delta}$, where $\varphi_{0\delta} \in E_{0\delta,j}$, $j \geqslant 1$. The components of $U_a(S)\,\Phi$ which are not identically zero are given in Table I. Similarly, the effect of $U_a(S)$ on $\mathscr{E}_{0d,f,j}$, $\mathscr{E}_{1\delta,f,j}$, and $\mathscr{E}_{1d,f,j}$ probes are given in Tables II, III, IV, where α, β, σ, and τ are given by (3.3) and (3.4) with $p = k$.

We also need to take account of the constant 0– and $(n-1)$-forms not already considered in the scalar case. If $k = 1$ and $\Phi = dt \wedge e^{ift} \cdot 1$, then $U_a(S)\,\Phi = \Phi'$, where

$$\begin{aligned}
\Phi'_{0\delta,f\pm 1,1} &= \frac{1}{2}(a + 1 \pm f)\,\omega, \\
\Phi'_{1d,f\pm 1,1} &= \pm\frac{1}{2i}\,d\omega,
\end{aligned} \tag{3.11}$$

TABLE II

$\mathscr{E}_{0d,f,j}$ Probe: $\Phi = dt \wedge e^{ift}\varphi_{0d}$, $\varphi_{0d} \in E_{0d,j}$, $j \geqslant 1$. $\Phi' = U_a(S)\,\Phi$

$$\begin{aligned}
\Phi'_{0\delta,f\pm 1,j} &= \tfrac{1}{2}(a + 1 \pm f)\,\varphi^{0}_{0d} \\
\Phi'_{0d,f\pm 1,j+1} &= \tfrac{1}{2}\beta(\alpha + a + 1 \pm f)\,\varphi^{+}_{0d} \\
\Phi'_{0d,f\pm 1,j-1} &= \tfrac{1}{2}\alpha(-\beta + a + 1 \pm f)\,\varphi^{-}_{0d} \\
\Phi'_{1d,f\pm 1,j} &= \pm(1/2i)\,d\varphi^{0}_{0d}
\end{aligned}$$

TABLE III

$\mathscr{E}_{1\delta,f,j}$ Probe: $\Phi = e^{ift}\varphi_{1\delta}$, $\varphi_{1\delta} \in E_{1\delta,j}$, $j \geq 1$. $\Phi' = U_a(S)\,\Phi$

$\Phi'_{0\delta,f\pm 1,j} = \pm(1/2i)\,\delta\varphi^0_{1\delta}$
$\Phi'_{1\delta,f\pm 1,j+1} = \frac{1}{2}\alpha(\sigma + a \pm f)\,\varphi^+_{1\delta}$
$\Phi'_{1\delta,f\pm 1,j-1} = \frac{1}{2}\beta(-\tau + a \pm f)\,\varphi^-_{1\delta}$
$\Phi'_{1d,f\pm 1,j} = \frac{1}{2}(n - 2k - 1 - a \mp f)\,\varphi^0_{1\delta}$

and all other components are zero. Similarly, if $k = n-1$ and $\Phi = e^{ift} \cdot \mathcal{O}$, where $\mathcal{O}$ is the volume element of S^{n-1}, then $U_a(S)\,\Phi = \Phi'$, where

$$\begin{aligned}\Phi'_{0\delta,f\pm 1,1} &= \pm\frac{1}{2i}\,\iota(d\omega)\,\mathcal{O},\\ \Phi'_{1d,f\pm 1,1} &= \frac{1}{2}\,(n - 1 + a \pm f)\,\omega\mathcal{O}.\end{aligned} \tag{3.12}$$

Superimposing the $\binom{0d}{1\delta}$ and $\binom{0\delta}{1d}$ sectors in Fig. 3.1, Tables I–IV and (3.11), (3.12) show that the general $U_a(S)$ move is as in Fig. 3.2. The "forbidden move" structure under $U_{s-1}(S)$ now provides us with 5 invariant subspaces.

LEMMA 3.12. *Suppose $n \geq 4$ is even. Within the k-forms (C^∞ or H^m) of parity $(-1)^{s-1}$ on $\bar{\bar{M}}^n$, let*

$$Z^{\pm,\pm'} = \{\pm i(C \mp' s)\,\Phi_{0\delta} + \delta\Phi_{1d} = 0\}.$$

TABLE IV

$\mathscr{E}_{1d,f,j}$ Probe: $\Phi = e^{ift}\varphi_{1d}$, $\varphi_{1d} \in E_{1d,j}$, $j \geq 1$. $\Phi' = U_a(S)\,\Phi$

$\Phi'_{0\delta,f\pm 1,j+1} = \mp(1/2i)\dfrac{\alpha - 1}{\beta}\,(\delta\varphi_{1d})^+$
$\Phi'_{0\delta,f\pm 1,j-1} = \pm(1/2i)\dfrac{\beta + 1}{\alpha}\,(\delta\varphi_{1d})^-$
$\Phi'_{0d,f\pm 1,j} = \mp(1/2i)(\delta\varphi_{1d})^0$
$\Phi'_{1\delta,f\pm 1,j} = \frac{1}{2}(a \pm f)\,\varphi^0_{1d}$
$\Phi'_{1d,f\pm 1,j+1} = \frac{1}{2}\tau(\sigma + a \pm f)\,\varphi^+_{1d}$
$\Phi'_{1d,f\pm 1,j-1} = \frac{1}{2}\sigma(-\tau + a \pm f)\,\varphi^-_{1d}$

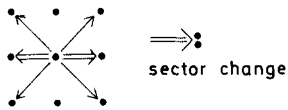


FIGURE 3.2

As usual, put $J=(n-2)/2+j$, *and let*

$$\begin{aligned}
W^+ &= \{f=J\}_{\left(\begin{smallmatrix}0d\\1\delta\end{smallmatrix}\right)} \oplus (\{f=J+1\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)} \cap Z^{+,+}) \\
&\quad \oplus (\{f=J-1\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)} \cap Z^{+,-}), \\
W^- &= \{f=-J\}_{\left(\begin{smallmatrix}0d\\1\delta\end{smallmatrix}\right)} \oplus (\{-f=J+1\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)} \cap Z^{-,+}) \\
&\quad \oplus (\{-f=J-1\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)} \cap Z^{-,-}), \\
V^+ &= \{f \geqslant J+1\} + W^+, \\
V^- &= \{-f \geqslant J+1\} + W^-, \\
M &= \{|f| \leqslant J-1\} + W^+ + W^-,
\end{aligned}$$

where, of course, the subscript $\left(\begin{smallmatrix}0d\\1\delta\end{smallmatrix}\right)$ *(resp.* $\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)$*) indicates that the designated subspace is cut from the* $\left(\begin{smallmatrix}0d\\1\delta\end{smallmatrix}\right)$ *(resp.* $\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)$*) sector. Then* $W^\pm$, $V^\pm$, *and* M *are* $u_{s-1}(G)$*-invariant,* $G=O^\uparrow(2,n)$.

Proof. As in Lemma 2.8, it suffices to show $U_{s-1}(S)$-invariance. The covariance relation (3.1), along with Fig. 3.2, shows that one cannot escape from W^+ or W^- by $U_{s-1}(S)$-moves. To get the invariance of $V^\pm$ and M, we need this plus the following observation: For an $\mathscr{E}_{0\delta,f,j}$ or $\mathscr{E}_{1d,f,j}$ probe with $f=\pm(J\pm 1)$, any move to the adjacent line $f=\pm(J\mp 1)$ in the $\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)$ sector lands in $\mathscr{N}(D_{2,k})$. This says that the pairs of dotted lines in Fig. 3.1, along with the corresponding solid line, are "strong enough" to "catch" any probe originating in $V^\pm$ or M.

Indeed, for an $\mathscr{E}_{0\delta,f,j}$ probe $\Phi=dt\wedge e^{ift}\varphi_0$ with $f=J+1$, Table I gives (with $\Phi'=U_{s-1}(S)\,\Phi$)

$$\Phi'_{0\delta,f-1,j+1} = -\frac{1}{2}(J-s-1)\,\varphi^+_{0\delta},$$

$$\Phi'_{1d,f-1,j+1} = -\frac{1}{2i}\frac{J+s-1}{J-s}(d\varphi_{0\delta})^+;$$

by Lemma 3.10, we have

$$i(C+s)\,\Phi'_{0\delta,f-1,j+1}+\delta\Phi'_{1d,f-1,j+1}=0.$$

For an $\mathscr{E}_{0\delta,f,j}$ probe with $f=J-1$,

$$\Phi'_{0\delta,f+1,j-1} = -\frac{1}{2}(J+s+1)\,\varphi_{0\delta}^{-},$$

$$\Phi'_{1d,f+1,j-1} = -\frac{1}{2i}\frac{J-s+1}{J+s}(d\varphi_{0\delta})^{-};$$

by Lemma 3.10,

$$i(C-s)\,\Phi'_{0\delta,f+1,j-1}+\delta\Phi'_{1d,f+1,j-1}=0.$$

By similar calculations or Hodge duality, we get the analogous statements for an $\mathscr{E}_{1d,J\pm 1,j}$ probe, and by similar calculations or time reversal, we get the desired results on the negative frequency side. ▮

The representation spaces of interest are now $W^{\pm}$, $V^{\pm}/W^{\pm}$, and $M/(W^{+}+W^{-})$, all carrying $u_{s-1}(G)$. As in the scalar case, each admits a nondegenerate invariant Hermitian inner product, but fewer of these representations are unitary. Recall that the natural inner product $(\ ,\ )$ on k-forms (indefinite for $k\neq 0, n$) is

$$(\Phi,\Psi)=\int_{\bar{M}^n} g^k(\Phi,\psi)=2\pi\sum_{f=-\infty}^{\infty}\{(\varphi_{1,f},\psi_{1,f})_{L^2(\Lambda^k(S^{n-1}))} - (\varphi_{0,f},\psi_{0,f})_{L^2(\Lambda^{k-1}(S^{n-1}))}\},$$

where $\Phi=\sum_{f=-\infty}^{\infty}e^{ift}(dt\wedge\varphi_{0,f}+\varphi_{1,f})$.

THEOREM 3.13. *Suppose $n\geq 4$ is even, and $s\neq\pm 1$. The inner products*

$$\langle\Phi,\Psi\rangle_M=(\Phi,D_{2,k}\Psi)\qquad \text{on } M/(W^{+}+W^{-}),$$

$$\langle\Phi,\Psi\rangle_{V^\pm}=-(\Phi,D_{2,k}\Psi)\qquad \text{on } V^{\pm}/W^{\pm},$$

$$\langle\Phi,\Psi\rangle_{W^\pm}=\pm i(\Phi,D'_{2,k}\Psi)\qquad \text{on } W^{\pm},$$

where $D'_{2,k}=[D_{2,k},t]$, are Hermitian, nondegenerate, and $u_{s-1}(G)$-invariant. If we are not in the scalar case ($k=0,n$) already treated, all these inner products are indefinite, except $\langle\ ,\ \rangle_{V^\pm}$ *when $s=0(k=n/2)$. In this case, the completion of $V^{\pm}/W^{\pm}$ in $\langle\ ,\ \rangle_{V^\pm}$ carries $u_{s-1}(G)$ as a continuous unitary representation.*

Proof. The inner products are Hermitian because $D_{2,k}$ and $(1/i)\,D'_{2,k}$ are differential operators formally self-adjoint in $(\ ,\ )$. The argument for invariance of $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ proceeds exactly as in Theorem 2.9 (using the covariance relation (3.1)), once we note that if h is a conformal transformation, $h \cdot g = \Omega^2 g$, then

$$h \cdot g^k = \Omega^{-2k} g^k. \tag{3.13}$$

For nondegeneracy of $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$, note that both $(\ ,\ )$ and $D_{2,k}$ are diagonalized by the decomposition of k-forms into $\mathscr{E}_{0d,f,j}$, $\mathscr{E}_{1\delta,f,j}$, and $\mathscr{E}_{\binom{0\delta}{1d},f,j}$ spaces. Within one of these spaces, if

$$(\Phi, D_{2,k}\Psi) = 0 \qquad \text{all } \Phi,$$

then $D_{2,k}\Psi = 0$.

If $k \neq n/2$, then $s \neq 0$; by assumption $s \neq \pm 1$, so $s+1$ and $s-1$ have the same sign. For an $\mathscr{E}_{0d,f,j}$ probe $\Phi = dt \wedge e^{ift}\varphi_{0d}$, $\varphi_{0d} \in E_{0d,j}$,

$$(\Phi, D_{2,k}\Phi) = -(s-1)(-f^2 + J^2)\, 2\pi\, \|\varphi_{0d}\|^2_{L^2(\Lambda^{k-1}(S^{n-1}))}; \tag{3.14}$$

for an $\mathscr{E}_{1\delta,f,j}$ probe $\Phi = e^{ift}\varphi_{1\delta} \in E_{1\delta,j}$,

$$(\Phi, D_{2,k}\Phi) = (s+1)(-f^2 + J^2)\, 2\pi\, \|\varphi_{1\delta}\|^2_{L^2(\Lambda^k(S^{n-1}))}. \tag{3.15}$$

For $|f| \neq J$, (3.14) and (3.15) have opposite sign; thus $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ are indefinite.

Now let $k = n/2$, and consider the complementary subspaces $Z^\pm = Z^{\pm,\pm'} = \{\pm iC\Phi_{0\delta} + \delta\Phi_{1d} = 0\}$ of the $\binom{0\delta}{1d}$ sector. Since $\delta d = C^2$, Z^+ is self-orthogonal in $(\ ,\ )$, as is Z^-. But $D_{2,k}\colon Z^\pm \to Z^\mp$. Indeed, if $\Phi \in \mathscr{E}_{\binom{0\delta}{1d},f,j} \cap Z^+$ and $\Psi \in \mathscr{E}_{\binom{0\delta}{1d},f,j} \cap Z^-$, say

$$\Phi = e^{ift}(dt \wedge iJ^{-1}\,\delta\varphi_{1d} + \varphi_{1d}),$$
$$\Psi = e^{ift}(-dt \wedge iJ^{-1}\,\delta\psi_{1d} + \psi_{1d}),$$
$$\varphi_{0\delta}, \psi_{0\delta} \in E_{0\delta,j},$$

then

$$D_{2,k}\Phi = [1-(f-J)^2]\, e^{ift}(-dt \wedge iJ^{-1}\,\delta\varphi_{1d} + \varphi_{1d}),$$
$$D_{2,k}\Psi = [1-(f+J)^2]\, e^{ift}(dt \wedge iJ^{-1}\,\delta\psi_{1d} + \psi_{1d}).$$

This shows that $Z^+ \perp Z^-$ in $(\cdot, D_{2,k}\cdot)$, and that for Φ and Ψ as above,

$$(\Phi, D_{2,k}\Phi) = -4\pi[(f-J)^2 - 1]\, \|\varphi_{1d}\|^2_{L^2(\Lambda^k(S^{n-1}))},$$
$$(\Psi, D_{2,k}\Psi) = -4\pi[(f+J)^2 - 1]\, \|\Psi_{1d}\|^2_{L^2(\Lambda^k(S^{n-1}))}.$$

Thus $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ are negative definite in the $\binom{0\delta}{1d}$ sector. But by (3.5), $(\cdot, D_{2,k}\cdot)$ has the sign of $J^2 - f^2$ in the $\binom{0d}{1\delta}$ sector; that is, is positive on $M/(W^+ + W^-)$ and negative on $V^\pm/W^\pm$. This shows that $\langle\ ,\ \rangle_M$ is indefinite, but that $\langle\ ,\ \rangle_{V^\pm}$ is positive definite. On $\mathscr{E}_{\binom{0d}{1\delta},f,j}$ (resp. $\mathscr{E}_{\binom{0\delta}{1d},f,j} \cap Z^{\pm'}$), $\langle\ ,\ \rangle_{V^\pm}$ is $f^2 - J^2$ (resp. $(f \pm 'J)^2 - 1$) times the inner product induced by the artificial Riemannian metric $dt^2 + g_{S^{n-1}}$ on $\bar{\bar{M}}^n$, so the strength of the $\langle\ ,\ \rangle_{V^\pm}$ uniform structure is between that derived from $H^{1/2}$ and that derived from H^1; thus, after completion, we get a continuous unitary representation.

The proof of invariance for $\langle\ ,\ \rangle_{W^\pm}$ proceeds exactly as in Theorem 2.9 or 2.18; to show that $U_{s+1}(S)^* = U_{s-1}(S)$ in $(\ ,\ )$, we need (3.13). For nondegeneracy and indefiniteness, note that $D'_{2,k}$ acts as

$$2(s-1)\,\partial_t \qquad \text{in } 0d$$

$$2(s+1)\,\partial_t \qquad \text{in } 1\delta$$

$$\mathscr{B}' = \left(\begin{array}{c|c} 2(s-1)\,\partial_t & -2\delta \\ \hline -2d & 2(s+1)\,\partial_t \end{array}\right) \qquad \text{in } \begin{pmatrix} 0\delta \\ 1d \end{pmatrix}.$$

Thus on $f = \pm J$, an $\mathscr{E}_{0d,f,j}$ probe $\Phi = dt \wedge e^{ift}\varphi_{0d}$ has

$$\langle \Phi, \Phi \rangle_{W^\pm} = -4\pi(s-1)\,J\,\|\varphi_{0d}\|^2_{L^2(\Lambda^{k-1}(S^{n-1}))},$$

while an $\mathscr{E}_{1\delta,f,j}$ probe $\Phi = e^{ift}\varphi_{1\delta}$ has

$$\langle \Phi, \Phi \rangle_{W^\pm} = 4\pi(s+1)\,J\,\|\varphi_{1\delta}\|^2_{L^2(\Lambda^k(S^{n-1}))}.$$

(This already shows indefiniteness unless $s = 0$.) For $\Phi \in \mathscr{E}_{\binom{0\delta}{1d},J\pm 1,j} \cap Z^{+,\pm}$, say $\Phi = e^{ift}(dt \wedge \varphi_{0\delta} + \varphi_{1d})$ with

$$\varphi_{0\delta} = \frac{i\delta\varphi_{1d}}{J \mp s},$$

the formula

$$d\delta\varphi_{1d} = (C^2 - s^2)\,\varphi_{1d} = (J^2 - s^2)\,\varphi_{1d} \tag{3.16}$$

implies that

$$\mathscr{B}' \begin{pmatrix} \varphi_{0\delta} \\ \varphi_{1d} \end{pmatrix} = 2 \begin{pmatrix} \dfrac{-sJ \pm 1}{J \mp s}\,\delta\varphi_{1d} \\ i(sJ \pm 1)\,\varphi_{1d} \end{pmatrix}.$$

By another application of (3.16) and the fact that δ and d are formal adjoints,

$$\langle \Phi, \Phi \rangle_{W^+} = \mp \frac{4\pi(s^2-1)\,J}{J \mp s}\,\|\varphi_{1d}\|^2_{L^2(\Lambda^k(S^{n-1}))}.$$

Since $s \neq \pm 1$, this completes the proof of the nondegeneracy of $\langle\, ,\, \rangle_{W^+}$, and shows that the sign of $\langle \Phi, \Phi \rangle_{W^+}$ on $f = J + 1$ disagrees with the sign on $f = J - 1$. Thus $\langle\, ,\, \rangle_{W^+}$ is indefinite even in the case $s = 0$. The situation for $\langle\, ,\, \rangle_{W^-}$ is entirely similar. ∎

Remark 3.14 *(Hodge * duality).* We have not decomposed the $n/2$-forms into "self-dual and anti-self-dual" summands (± 1 or $\pm i$ eigenspaces of $*^{(n)}$, the Hodge star operator on $\bar{\bar{M}}^n$ or $\tilde{M}^n$) because the "space reversal" P, being orientation-reversing, reverses duality: $*^{(n)}P^* = -P^* *^{(n)}$ (where P^* is the pullback by P), so

$$*^{(n)}\Phi = \varepsilon\Phi \Rightarrow *^{(n)}P^*\Phi = -\varepsilon P^*\Phi.$$

We would be allowed such a decomposition, however, if we restricted to the identity component $G_0 = SO_0(2, n)$ of $G = O^{\uparrow}(2, n)$. This, however, does not produce any additional unitarity.

More specifically, on an n-dimensional pseudo-Riemannian manifold with q minus signs in its metric signature, we have

$$** = (-1)^{k(n-k)+q} \qquad \text{on } k\text{-forms.}$$

Thus in Lorentzian $\bar{\bar{M}}^n$ or $\tilde{M}^n$ for n even,

$$*^{(n)}*^{(n)} = (-1)^{(n/2)+1}.$$

This means $*^{(n)}$ has ± 1 eigenspaces for $n \equiv 2 \pmod 4$ and $\pm i$ eigenspaces for $n \equiv 0$. Moreover, each $u_a(G_0)$ acts separately on these eigenspaces

$$*^{(n)}u_a(h)\,\Phi = u_a(h)\,*^{(n)}\Phi, \qquad h \in G_0,\ \Phi \text{ an } n/2\text{-form},$$

since $*^{(n)}$ is invariant under orientation-preserving conformal transformations in the middle order. Since

$$*^{(n)}\Phi = dt \wedge (-1)^{n/2} *\Phi_1 - *\Phi_0,$$

where $* = *_{S^{n-1}}$, the condition for ε-duality $*^{(n)}\Phi = \varepsilon\Phi$ is

$$\Phi_0 = -\varepsilon^{-1} * \Phi_1,$$

or

$$\begin{aligned} \Phi_{0d} &= -\varepsilon^{-1} * \Phi_{1\delta}, \\ \Phi_{0\delta} &= -\varepsilon^{-1} * \Phi_{1d}. \end{aligned} \tag{3.17}$$

Let $W^{\pm,\varepsilon}$ be the space of ε-dual forms in $W^{\pm}$. By the proof of Theorem 3.13, $u_{s-1}(G_0)$ will still be nonunitary on $W^{\pm,\varepsilon}$ provided

$$W^{\pm,\varepsilon} \cap \{\pm f = J \pm' 1\} \overset{?}{\neq} \{0\}. \tag{3.18}$$

But $\Phi \in \mathscr{E}_{\binom{0\delta}{1g},f,j}$ with $\pm f = J \pm' 1$ is in the space on the left of (3.18) if and only if (3.17) and

$$\Phi_{0\delta} = \pm \frac{i}{J} \delta \Phi_{1d};$$

that is, if and only if (3.17) and

$$*\delta\Phi_{1d} = \pm iJ\varepsilon^{-1}\Phi_{1d}. \tag{3.19}$$

Now on $E_{n/2,d,j}$, $(*\delta)^2 = (d*)^2 = (-1)^{n/2}\, d\delta = (-1)^{n/2}\, J^2$, so the possible eigenvalues of $*\delta$ on this space are $\pm iJ\varepsilon^{-1}$. Both eigenvalues are realized, since the space reversal P, viewed as a transformation on S^{n-1}, reverses duality

$$\begin{aligned} *\delta P^*\varphi &= (-1)^{\frac{n}{2}}\, d * P^*\varphi = -(-1)^{\frac{n}{2}}\, dP^* * \varphi \\ &= -(-1)^{\frac{n}{2}}\, P^* d * \varphi = -P^* * \delta\varphi. \end{aligned}$$

This shows that both signs in (3.19) are possible for nonzero Φ_{1d}, which in turn proves (3.18).

The proof of Theorem 3.13 also shows that $u_{s-1}(G_0)$ is nonunitary on the ε-dual summand of $M/(W^+ + W^-)$.

Remark 3.15 *(Polarization of symplectic structure).* There is a $u_{s-1}(G)$-invariant symplectic form on the space of real C^∞ solutions to $D_{2,k}\Phi = 0$, viz. (with Ψ the $\partial_t\Phi$ Cauchy datum)

$$\begin{aligned} &\mathscr{A}\left(\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}, \begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix}\right) \\ &\quad = \int_{S^{n-1}} \{(s+1)(\langle \Phi_1, \Psi_1'\rangle - \langle \Phi_1', \Psi_1\rangle) \\ &\quad\quad - (s-1)(\langle \Phi_0, \Psi_0'\rangle - \langle \Phi_0', \Psi_0\rangle) \\ &\quad\quad - 2(\langle \Phi_1, d\Phi_0'\rangle - \langle \Phi_1', d\Phi_0\rangle)\}\, dx. \end{aligned}$$

(Compare formula (4.1) of [44] for flat Minkowski space.) The $\langle\, ,\, \rangle$ are the natural pointwise inner products on forms in S^{n-1}, and the integral can be taken at any fixed t. The corresponding conserved energy is

$$\begin{aligned} \mathscr{E} = \frac{1}{2}\int_{S^{n-1}} &\{(s+1)\,|d\Phi_1|^2 + (s-1)\,|\delta\Phi_1|^2 + (s+1)(s-1)^2\,|\Phi_1|^2 \\ &+ (s+1)\,|\Psi_1|^2 - (s+1)\,|d\Phi_0|^2 - (s-1)\,|\delta\Phi_0|^2 \\ &- (s-1)(s+1)^2\,|\Phi_0|^2 - (s-1)\,|\Psi_0|^2\}\, dx. \end{aligned}$$

Thus the energy Hessian $\mathscr{H}$ is *not* positive definite for $s \neq \pm 1$, $k \neq 0, n$. But we can still define a linear transformation $\mathscr{K}$ on phase space by

$$\mathscr{H}(X, Y) = \mathscr{A}(X, \mathscr{K} Y),$$

and get a complex structure

$$\mathscr{J} = \frac{\mathscr{K}}{\sqrt{-\mathscr{K}^2}}$$

as in Remark 2.10.

These structures are most easily seen in a modified version of the y, z coordinates of Remark 3.3. After a tedious calculation, one finds that

$$\begin{aligned}
&\mathscr{A}\left(\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}, \begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix}\right)\\
&= \int_{S^{n-1}} \Bigg\{(s+1)(\langle \Phi_{1\delta}, \Psi'_{1\delta}\rangle - \langle \Phi'_{1\delta}, \Psi_{1\delta}\rangle)\\
&\quad - (s-1)(\langle \Phi_{0d}, \Psi'_{0d}\rangle - \langle \Phi'_{0d}, \Psi_{0d}\rangle)\\
&\quad + (1-s^2)\left(\left\langle \frac{C-s}{2C(C+1)}\, y_{1d}, z'_{1d}\right\rangle\right.\\
&\quad \left. - \left\langle \frac{C-s}{2C(C+1)}\, y'_{1d}, z_{1d}\right\rangle\right)\\
&\quad + (s^2-1)\left(\left\langle \frac{C-s}{2C(C-1)}\, y_{0\delta}, z'_{0\delta}\right\rangle\right.\\
&\quad \left. - \left\langle \frac{C-s}{2C(C-1)}\, y'_{0\delta}, z_{0\delta}\right\rangle\right)\Bigg\}\, dx.
\end{aligned}$$

First assume $s > 1$ (the $s < -1$ case can be handled by duality), and let

$$a_{1d} = \frac{1}{C-s}\sqrt{\frac{2C(C-1)}{(s^2-1)(C+s)}}\, dy_{0\delta},$$

$$b_{1d} = \frac{1}{C-s}\sqrt{\frac{2C(C-1)}{(s^2-1)(C+s)}}\, dz_{0\delta},$$

$$a_{0\delta} = \frac{1}{C-s}\sqrt{\frac{2C(C+1)}{(s^2-1)(C+s)}}\, \delta y_{1d},$$

$$b_{0\delta} = \frac{1}{C-s}\sqrt{\frac{2C(C+1)}{(s^2-1)(C+s)}}\, \delta z_{1d},$$

$$a_{1\delta}=\frac{1}{\sqrt{s+1}}\,\Phi_{1\delta},$$
$$b_{1\delta}=\frac{1}{\sqrt{s+1}}\,\Psi_{1\delta},$$
$$a_{0d}=\frac{1}{\sqrt{s-1}}\,\Phi_{0d},$$
$$b_{0d}=\frac{1}{\sqrt{s-1}}\,\Psi_{0d}.$$

In a, b coordinates,

$$\mathscr{A}\left(\begin{pmatrix}\Phi\\ \Psi\end{pmatrix},\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix}\right)=\int_{S^{n-1}}\{\langle a_1,b_1'\rangle-\langle a_1',b_1\rangle$$
$$-\langle a_0,b_0'\rangle+\langle a_0',b_0\rangle\}\,dx,$$

and the energy is

$$\mathscr{E}=\tfrac{1}{2}\int_{S^{n-1}}\{|b_1|^2-|b_0|^2+|Ba_{1\delta}|^2-|Aa_{0d}|^2$$
$$+|(C-1)\,a_{1d}|^2-|(C+1)\,a_{0\delta}|^2\}\,dx.$$

Thus, acting on the column vector $(a_{0\delta}, b_{0\delta}, a_{0d}, b_{0d}, a_{1\delta}, b_{1\delta}, a_{1d}, b_{1d})$,

$$\mathscr{J}=\left[\begin{array}{cc|cc|cc|cc} 0 & (C+1)^{-1} & 0 & 0 & 0 & 0 & 0 & 0\\ -(C+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & A^{-1} & 0 & 0 & 0 & 0\\ 0 & 0 & -A & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & -B^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & B & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(C-1)^{-1}\\ 0 & 0 & 0 & 0 & 0 & 0 & C-1 & 0 \end{array}\right]. \tag{3.20}$$

The Hermitian inner product $\mathscr{R}+i\mathscr{A}$, where $\mathscr{R}(X, Y)=\mathscr{A}(X, \mathscr{J}Y)$, is thus positive definite on the 1δ sector and the "frequency $C-1$ part" of the $\binom{0\delta}{1d}$ sector, and negative definite on the $0d$ sector and the "frequency $C+1$

part" of the $\binom{0\delta}{1d}$ sector. This agrees with our findings for W^+ in the proof of Theorem 3.13.

If $s=0$, let

$$a_{1d}=\sqrt{2(C+1)}\, y_{1d}, \qquad b_{1d}=\sqrt{2(C+1)}\, z_{1d},$$
$$a_{0\delta}=\sqrt{2(C-1)}\, y_{0\delta}, \qquad b_{0\delta}=\sqrt{2(C-1)}\, z_{0\delta},$$
$$a_{1\delta}=y_{1\delta}, \qquad b_{1\delta}=z_{1\delta}, \qquad a_{0d}=y_{0d}, \qquad b_{0d}=z_{0d}.$$

Then

$$\begin{aligned}\mathscr{A}\left(\begin{pmatrix}\Phi\\ \Psi\end{pmatrix},\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix}\right)=\int_{S^{n-1}} &\{\langle a_1, b_1'\rangle-\langle a_1', b_1\rangle\\ &+\langle a_{0d}, b_{0d}'\rangle-\langle a_{0d}', b_{0d}\rangle\\ &-\langle a_{0\delta}, b_{0\delta}'\rangle+\langle a_{0\delta}', b_{0\delta}\rangle\}\, dx,\end{aligned}$$

$$\begin{aligned}\mathscr{E}=\tfrac{1}{2}\int_{S^{n-1}} &\{|b_1|^2+|b_{0d}|^2-|b_{0\delta}|^2+|Ba_{1\delta}|^2\\ &+|Aa_{0d}|^2+|(C+1)\,a_{1d}|^2-|(C-1)\,a_{0\delta}|^2\}\, dx,\end{aligned}$$

and in the notation of (3.20),

$$\mathscr{J}=\left(\begin{array}{c|c|c|c}\begin{matrix}0 & (C-1)^{-1}\\ -(C-1) & 0\end{matrix} & 0 & 0 & 0\\ \hline 0 & \begin{matrix}0 & -A^{-1}\\ A & 0\end{matrix} & 0 & 0\\ \hline 0 & 0 & \begin{matrix}0 & -B^{-1}\\ B & 0\end{matrix} & 0\\ \hline 0 & 0 & 0 & \begin{matrix}0 & -(C+1)^{-1}\\ C+1 & 0\end{matrix}\end{array}\right)$$

Thus $\mathscr{R}+i\mathscr{A}$ is negative only in the "frequency $C-1$ part" of the $\binom{0\delta}{1d}$ sector; and this agrees with our findings for W^+ in the proof of Theorem 3.13.

Remark 3.16. The full force of the information in Tables I–IV, (3.11), and (3.12) has not yet been used; for the invariance of $W^\pm$, $V^\pm$, and M (Lemma 3.12), we needed only the information in Fig. 3.2 and part of Tables I, II. However, to prove the covariance of the higher order generalizations $D_{2l,k}$ of $D_{2,k}$ introduced below and to get invariant subspaces analogous to $W^\pm$, $V^\pm$, M, we shall need all this information.

d. *Higher Order Generalizations of the $D_{2,k}$ and Their Wave Propagative Properties*

The "correct" way to generalize $D_{2,k}$ to an operator $D_{2l,k}$ of order $2l$ on k-forms in $\bar{M}^n$ or $\tilde{M}^n$ is as follows. Let

$$\mathscr{B} = \left(\begin{array}{c|c} (s-l)\,\partial_t^2 + (s+l)(C^2 - l^2) & -2l\,\delta\partial_t \\ \hline -2l\,d\partial_t & (s+l)\,\partial_t^2 + (s-l)(C^2 - l^2) \end{array} \right)$$

as an operator in the $\binom{0\delta}{1d}$ sector, and let $D_{2l,k}$ act by

$$\left.\begin{array}{l} (s-l)[\partial_t + iA][\partial_t - iA] \prod_{p=1}^{(l-1)/2} [\partial_t + i(A+2p)][\partial_t - i(A+2p)] \\ \quad \times [\partial_t + i(A-2p)][\partial_t - i(A-2p)] \qquad \text{in the } 0d \text{ sector}, \\ (s+l)[\partial_t + iB][\partial_t - iB] \prod_{p=1}^{(l-1)/2} [\partial_t + i(B+2p)] \\ \quad \times [\partial_t - i(B+2p)][\partial_t + i(B-2p)] \\ \quad \times [\partial_t - i(B-2p)] \text{ in the } 1\delta \text{ sector}, \\ \left\{ \prod_{p=1}^{(l-1)/2} [\partial_t + i(C+(2p-1))][\partial_t - i(C+(2p-1))] \right. \\ \quad \left. \times [\partial_t + i(C-(2p-1))][\partial_t - i(C-(2p-1))] \right\} \mathscr{B} \\ \qquad \text{in the } \binom{0\delta}{1d} \text{ sector}, \end{array}\right\} \quad l \text{ odd}$$

(3.21)

$$\left.\begin{array}{l} (s-l) \prod_{p=1}^{l/2} [\partial_t + i(A+(2p-1))][\partial_t - i(A+(2p-1))] \\ \quad \times [\partial_t + i(A-(2p-1))][\partial_t - i(A-(2p-1))] \qquad \text{in } 0d, \\ (s+l) \prod_{p=1}^{l/2} [\partial_t + i(B+(2p-1))][\partial_t - i(B+(2p-1))] \\ \quad \times [\partial_t + i(B-(2p-1))][\partial_t - i(B-(2p-1))] \text{ in } 1\delta, \\ [\partial_t + iC][\partial_t - iC] \left\{ \prod_{p=1}^{(l-2)/2} [\partial_t + i(C+2p)][\partial_t - i(C+2p)] \right. \\ \quad \left. \times [\partial_t + i(C-2p)][\partial_t - i(C-2p)] \right\} \mathscr{B} \text{ in } \binom{0\delta}{1d}. \end{array}\right\} \quad l \text{ even}$$

(For $k=1$, the "constants" $\mathbb{C}(dt \wedge e^{ift})$ may be assigned arbitrarily to either the 0δ or $0d$ sector; the two definitions of $D_{2l,k}\Phi$ given by (3.21) agree. The same is true of the "constants" $\mathbb{C}e^{ift}\mathcal{O}$, where $\mathcal{O}$ is the volume element of S^{n-1}, for $k=n-1$.) It is shown immediately below that $D_{2l,k}$ is a *differential* operator, in spite of the fact that its ingredients, the Hodge projections and A, B, C, are nonlocal. It is shown in Section e that $D_{2l,k}$ is conformally covariant (on $\bar{\tilde{M}}^n$) of bidegree $(s-l, s+l)$, where, as before, $s=(n-2k)/2$. $D_{4,k}$ is a spacial case of a general fourth order conformally covariant operator introduced by the author in [6]. For $k=0$, $D_{2l,k}$ specializes to the scalar operator D_{2l},

$$D_{2l,0}=\frac{n+2l}{2}D_{2l}.$$

LEMMA 3.17. *$D_{2l,k}$ is a differential operator with leading term*

$$\begin{aligned}&(s+l)(\delta^{(n)}d^{(n)})^l+(s-l)(d^{(n)}\delta^{(n)})^l\\&\quad=((s+l)\,\delta^{(n)}d^{(n)}+(s-l)\,d^{(n)}\delta^{(n)})\,\Box^{l-1}.\end{aligned}$$

Proof. Let $D=D_{2l,k}$. We shall exhibit differential operators $\mathscr{D}_1$ and $\mathscr{D}_2$ such that: (i) $\mathscr{D}_1$ agrees with D in the $\binom{0d}{1\delta}$ sector, and acts as a real polynomial $\mathscr{P}_{0\delta}(\partial_t)$ (resp. $\mathscr{P}_{1d}(\partial_t)$) in ∂_t in the 0δ (resp. $1d$) sector; (ii) $\mathscr{D}_2$ agrees with D in the $\binom{0\delta}{1d}$ sector, and acts as a real polynomial $\mathscr{P}_{0d}(\partial_t)$ (resp. $\mathscr{P}_{1\delta}(\partial_t)$) in ∂_t in the $0d$ (resp. 1δ) sector; and (iii) $\mathscr{P}_{0d}(\partial_t)=\mathscr{P}_{0\delta}(\partial_t)\equiv\mathscr{P}_0(\partial_t)$ and $\mathscr{P}_{1\delta}(\partial_t)=\mathscr{P}_{1d}(\partial_t)\equiv\mathscr{P}_1(\partial_t)$, so that

$$D=\mathscr{D}_1+\mathscr{D}_2-\left(\begin{array}{c|c}\mathscr{P}_0(\partial_t)&0\\\hline 0&\mathscr{P}_1(\partial_t)\end{array}\right),$$

where the blocks are relative to the $\binom{\Phi_0}{\Phi_1}$ decomposition.

Let

$$\mathscr{D}_1=\begin{pmatrix}\mathscr{D}_1^{(0)}&0\\0&\mathscr{D}_1^{(1)}\end{pmatrix},$$

where

$$\mathscr{D}_1^{(0)}=\begin{cases}(s-l)(\partial_t^2+d\delta+(s+1)^2)\displaystyle\prod_{p=1}^{(l-1)/2}[\partial_t^4+2(d\delta+(s+1)^2+(2p)^2)\,\partial_t^2\\\qquad+(d\delta+(s+1)^2-(2p)^2)^2],\qquad l\text{ odd}\\(s-l)\displaystyle\prod_{p=1}^{l/2}[\partial_t^4+2(d\delta+(s+1)^2+(2p-1)^2)\,\partial_t^2\\\qquad+(d\delta+(s+1)^2-(2p-1)^2)^2],\qquad l\text{ even,}\end{cases}$$

$$\mathscr{D}_1^{(1)} = \begin{cases} (s+l)(\partial_t^2 + \delta d + (s-1)^2) \prod_{p=1}^{(l-1)/2} [\partial_t^4 + 2(\delta d + (s-1)^2 + (2p)^2)\,\partial_t^2 \\ \quad + (\delta d + (s-1)^2 - (2p)^2)^2], \qquad l \text{ odd} \\ (s+l) \prod_{p=1}^{l/2} [\partial_t^4 + 2(\delta d + (s-1)^2 + (2p-1)^2)\,\partial_t^2 \\ \quad + (\delta d + (s-1)^2 - (2p-1)^2)^2], \qquad l \text{ even.} \end{cases}$$

Then $\mathscr{D}_1$ is as in (i), where $\mathscr{P}_{0\delta}$ is a real polynomial of degree $2l$ with lead coefficient $s-l$ and roots

$$\begin{aligned} &\pm i(s+1); \quad \pm i(s+1 \pm' 2p), && p = 1,\dots,\frac{l-1}{2};\ l \text{ odd,} \\ &\pm i(s+1 \pm' (2p-1)), && p = 1,\dots, l/2;\ l \text{ even,} \end{aligned} \tag{3.22}$$

and $\mathscr{P}_{1d}$ is a real polynomial of degree $2l$ with lead coefficient $s+l$ and roots

$$\begin{aligned} &\pm i(s-1); \quad \pm i(s-1 \pm' 2p), && p = 1,\dots,\frac{l-1}{2};\ l \text{ odd,} \\ &\pm i(s-1 \pm' (2p-1)), && p = 1,\dots, l/2;\ 1 \text{ even.} \end{aligned} \tag{3.23}$$

Let

$$\mathscr{D}_2 = \begin{cases} \left\{ \prod_{p=1}^{(l-1)/2} \left[\partial_t^4 + 2\left(\left(\begin{array}{c|c} \delta d & 0 \\ \hline 0 & d\delta \end{array} \right) + s^2 + (2p-1)^2 \right) \partial_t^2 \right.\right. \\ \quad \left.\left. + \left(\left(\begin{array}{c|c} \delta d & 0 \\ \hline 0 & d\delta \end{array} \right) + s^2 - (2p-1)^2 \right)^2 \right] \right\} \mathscr{A}, \qquad l \text{ odd,} \\ \left(\partial_t^2 + \left(\begin{array}{c|c} \delta d & 0 \\ \hline 0 & d\delta \end{array} \right) + s^2 \right) \left\{ \prod_{p=1}^{(l-2)/2} \left[\partial_t^4 + 2\left(\left(\begin{array}{c|c} \delta d & 0 \\ \hline 0 & d\delta \end{array} \right) \right.\right.\right. \\ \quad \left. + s^2 + (2p)^2 \right) \partial_t^2 \\ \quad \left.\left. + \left(\left(\begin{array}{c|c} \delta d & 0 \\ \hline 0 & d\delta \end{array} \right) + s^2 - (2p)^2 \right)^2 \right] \right\} \mathscr{A}, \qquad l \text{ even,} \end{cases}$$

where

$$\mathscr{A} = \left(\begin{array}{c|c} (s-l)\,\partial_t^2 + (s+l)(\delta d + s^2 - l^2) & -2l\,\delta\partial_t \\ \hline -2l\,d\partial_t & (s+l)\,\partial_t^2 + (s-l)(d\delta + s^2 - l^2) \end{array}\right).$$

Then $\mathscr{D}_2$ is as in (ii), where $\mathscr{P}_{0d}$ is a real polynomial of degree $2l$ with lead coefficient $s-l$ and roots

$$\begin{array}{llll} \pm i(s+l); & \pm i(s \pm' (2p-1)), & p = 1, ..., \dfrac{l-1}{2}; \, l \text{ odd}, \\ \pm is; \quad \pm i(s+l); & \pm i(s \pm' 2p), & p = 1, ..., \dfrac{l-1}{2}; \, l \text{ even}, \end{array} \tag{3.24}$$

and $\mathscr{P}_{1\delta}$ is a real polynomial of degree $2l$ with lead coefficient $s+l$ and roots

$$\begin{array}{llll} \pm i(s-l); & \pm i(s \pm' (2p-1)), & p = 1, ..., \dfrac{l-1}{2}; \, l \text{ odd}, \\ \pm is; \quad \pm i(s-l); & \pm i(s \pm' 2p), & p = 1, ..., \dfrac{l-2}{2}; \, l \text{ even}. \end{array} \tag{3.25}$$

But the list (3.22) agrees with the list (3.24); and (3.23) agrees with (3.25), so (iii) holds, and D is a differential operator.

The leading term of $\mathscr{D}_1$ is

$$\left(\begin{array}{c|c} (s-l)(\partial_t^2 + d\delta)^l & 0 \\ \hline 0 & (s+l)(\partial_t^2 + \delta d)^l \end{array}\right)$$

(in the sense that $\mathscr{D}_1$ minus this operator is of lower order); the leading term of $\mathscr{D}_2$ is

$$\left[\partial_t^2 + \left(\begin{array}{c|c} \delta d & 0 \\ \hline 0 & d\delta \end{array}\right)\right]^{l-1} \left(\begin{array}{c|c} (s-l)\,\partial_t^2 + (s+l)\,\delta d & -2l\,\delta\partial_t \\ \hline -2l\,d\partial_t & (s+l)\,\partial_t^2 + (s-l)\,d\delta \end{array}\right);$$

and the leading term of diag $(\mathscr{P}_0(\partial_t), \mathscr{P}_1(\partial_t))$ is

$$\left(\begin{array}{c|c} (s-l)\,\partial_t^{2l} & 0 \\ \hline 0 & (s+l)\,\partial_t^{2l} \end{array}\right);$$

(3.2) now implies the statement about the leading term of D. ∎

It is clear from the above that the cases $s = \pm l$ will be qualitatively different from the typical case; as shown in Section 4, these cases can be viewed as higher-order generalizations of the Maxwell equations. Leaving out the cases $s = \pm l$ for now, we can find all solutions of $D_{2l,k}\Phi = 0$ in $\bar{M}^n$ or $\tilde{M}^n$ (n even) in much the same way as for $D_{2,k}\Phi = 0$. It is clear from the definition of $D_{2l,k}$ that periodic solutions occupy the $2l$ lines

$$J-(l-1) \leqslant \pm f \leqslant J+l-1, \qquad J+f \equiv l-1 \pmod 2$$

in the $\binom{0d}{1\delta}$ sector, and include the $2(l-1)$ lines

$$J-(l-2) \leqslant \pm f \leqslant J+l-2, \qquad J+f \equiv l \pmod 2$$

in the $\binom{0\delta}{1d}$ sector. For the rest of the periodic solutions, note that if

$$\tilde{\mathscr{B}} = \left(\begin{array}{c|c} (s+l)\,\partial_t^2 + (s-l)(C^2-l^2) & 2l\,\delta\partial_t \\ \hline 2l\,d\partial_t & (s-l)\,\partial_t^2 + (s+l)(C^2-l^2) \end{array}\right)$$

as an operator in the $\binom{0\delta}{1d}$ sector, then

$$\tilde{\mathscr{B}}\mathscr{B} = (s^2-l^2) \times \left(\begin{array}{c|c} \partial_t^4 + 2(C^2+l^2)\,\partial_t^2 + (C^2-l^2)^2 & 0 \\ \hline 0 & \partial_t^4 + 2(C^2+l^2)\,\partial_t^2 + (C^2-l^2)^2 \end{array}\right).$$

Thus the "extra" solutions lie on the lines

$$f = \pm(J \pm' l),$$

and, arguing as in Section 3.b, satisfy

$$\pm i(J \mp' s)\,\Phi_{0\delta} + \delta\Phi_{1d} = 0.$$

The periodic null space of $D_{2l,k}$ is pictured in Fig. 3.3. The dotted lines again indicate that exactly half of the harmonic oscillators at each point are included. Where two dotted lines intersect, *all* harmonic oscillators are included.

THEOREM 3.18 (Automatic periodicity). *Suppose $n \geqslant 4$ is even, and $s \neq \pm l$. Let $Z^{\pm,\pm'}$ be the space of C^∞ periodic solutions of $D_{2l,k}\Phi = 0$ in the $\binom{0\delta}{1d}$ sector satisfying*

$$\pm i(C \mp' s)\,\Phi_{0\delta} + \delta\Phi_{1d} = 0, \tag{3.26}$$

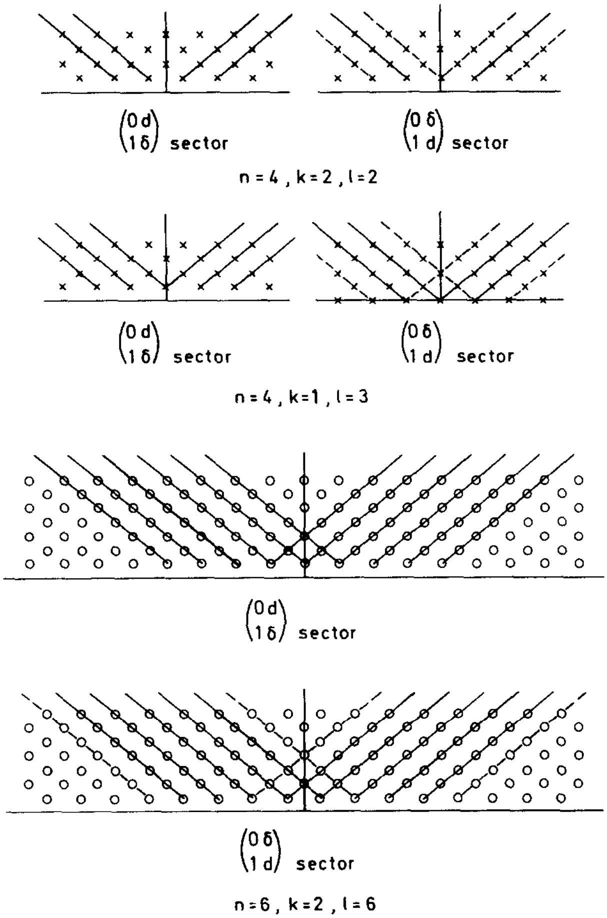


FIGURE 3.3

and let $W^{\pm}$ be the space of C^{∞} periodic solutions given by

$$W^{\pm} = \{J-(l-1) \leqslant \pm f \leqslant J+l-1,\ J+f \equiv l-1 \bmod 2)\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)}$$

$$\oplus \{J-(l-2) \leqslant \pm f \leqslant J+l-2,\ J+f \equiv l \,(\bmod 2)\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)}$$

$$\oplus (\{f = \pm(J+l)\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)} \cap Z^{\pm,+})$$

$$\oplus (\{f = \pm(J-l)\}_{\left(\begin{smallmatrix}0\delta\\1d\end{smallmatrix}\right)} \cap Z^{\pm,-}).$$

Let $F = W^{+} \cap W^{-}$. Then the general solution of $D_{2l,k}\Phi = 0$ on $\tilde{M}^{n}$ is of the form $\Phi + t\Psi$, where Φ and Ψ are 2π-periodic solutions, and in fact $\Psi \in F$. In *particular, if $l < (n-2)/2$, or if $k \neq 0, 1, n-1, n$ and $l < n/2$, all solutions are 2π-periodic.*

Proof. The proof is similar to that of Theorem 3.2; the Cauchy problem is again solved by solving initial value ODE problems. The only complication comes when the characteristic polynomial of one of these ODEs has double roots (higher multiplicity roots do not occur). These correspond exactly to fields in F. ∎

THEOREM 3.19 (Oddness–evenness). *Suppose $n \geqslant 4$ is even, and $s \neq \pm l$. Then every periodic solution of $D_{2l,k}\Phi = 0$ has parity $(-1)^{s-l}$ under the product η of antipodal maps.*

Proof. This follows immediately from the discussion in the proof of Theorem 3.4, together with the above characterization of the periodic solutions. ∎

THEOREM 3.20 (Huygens' principle). *Suppose $n \geqslant 4$ is even, and $s \neq \pm l$. Suppose $l < (n-2)/2$, or $k \neq 0, 1, n-1, n$ and $l < n/2$. Then if Φ is a C^∞ solution of $D_{2l,k}\Phi = 0$ on $\tilde{M}^n$, and the support of the Cauchy data $\Phi(0, x)$, $(\partial_t \Phi)(0, x), \ldots, (\partial_t^{2l-1}\Phi)(0, x)$ contains no point at a distance t_0 from $x_0 \in S^{n-1}$, $0 \leqslant t_0 \leqslant \pi$, we have $\Phi(t_0, x_0) = 0$.*

Proof. Since $\{(s+l)^{-1}\, \delta^{(n)} d^{(n)} + (s-l)^{-1}\, d^{(n)} \delta^{(n)}\}\, D_{2l,k}$ has leading term $\Box^{l+1}$, the equation $D_{2l,k}\Phi = 0$ has finite propagation speed, by the same argument used in the proof of Theorem 3.5. (The crucial point is that the fixed-time constraints $D_{2l,k}\Phi = 0$, $\partial_t D_{2l,k}\Phi = 0$ can be solved to give $\partial_t^{2l}\Phi$ and $\partial_t^{2l+1}\Phi$ at time 0 as linear expressions in $\Phi, \partial_t\Phi, \ldots, \partial_t^{2l-1}\Phi$ at time 0; but this is clear from the proof of Lemma 3.17.) Given finite propagation speed, with the above solution of the Cauchy problem, the Lax–Phillips oddness–evenness argument of Theorem 2.4 implies the result. ∎

Remark 3.21. The guiding principle behind the inequality conditions $l < (n-2)/2$, or $l < n/2$ for $k \neq 0, 1, n-1, n$, is the need to avoid the case $F \neq 0$. Even when $F \neq 0$, however, a modified Huygens' principle (analogous to that of Remark 2.5) holds.

e. *Representation Theoretic Content of the $D_{2l,k}$*

In analogy with the situation for D_2, D_{2l}, and $D_{2,k}$, the operator $D_{2l,k}$ determines a decomposition of $u_{s-l}(G)$ on k-forms into 5 or 6 composition factors, and some infinite families of unitary representations emerge. We first prove the conformal covariance of $D_{2l,k}$ on $\bar{\tilde{M}}^n$, n even, "eigenspace by eigenspace."

THEOREM 3.22 (Covariance of $D_{2l,k}$). *Suppose n is even. For Φ a C^∞ k-form on $\bar{M}^n$ and $h \in O(2, n)$,*

$$D_{2l,k} u_{s-l}(h)\, \Phi = u_{s+l}(h)\, D_{2l,k} \Phi.$$

Proof. By the obvious covariance under isometries and Remark 1.4, we are reduced to $\mathfrak{g}$-covariance; by symmetry, this reduces to S-covariance

$$D_{2l,k} U_{s-l}(S)\, \Phi \overset{?}{=} U_{s+l}(S)\, D_{2l,k} \Phi. \tag{3.27}$$

It is enough to prove this on "simple" fields, i.e., $\mathscr{E}_{0\delta,f,j}$, $\mathscr{E}_{0d,f,j}$, $\mathscr{E}_{1\delta,f,j}$, and $\mathscr{E}_{1d,f,j}$ probes, using the definition of $D_{2l,k}$, Tables I–IV, (3.11), and (3.12). For convenience, we rewrite the information in Tables I–IV for the special representations $U_{s \pm l}$ in terms of f, J, l, and s. Let $D = D_{2l,k}$, and $U_\pm = U_{s \pm l}$. Note that

$$\alpha = J - s, \qquad \beta = J + s,$$
$$\sigma = J - s + 1, \qquad \tau = J + s - 1.$$

(Again, these are the α, β, σ, τ parameters for k-forms.) The "translation" of Lemma 3.10 into J, s terms is given in Table IX.

Consider first an $\mathscr{E}_{1\delta,f,j}$ probe $\Phi = e^{ift}\varphi_{1\delta}$, $\varphi_{1\delta} \in E_{1\delta,j}$. $DU_-(S)\,\Phi$ or $U_+(S)\, D\Phi$ has possibly nonvanishing components only in

$$\mathscr{E}_{1\delta,f \pm 1, j \pm' 1}, \qquad \mathscr{E}_{\binom{0\delta}{1d}, f \pm 1, j}.$$

TABLE V

$\mathscr{E}_{0\delta,f,j}$ Probe: $\Phi = dt \wedge e^{ift}\varphi_{0\delta}$, $\varphi_{0\delta} \in E_{0\delta,j}$, $j \geq 1$. $\Phi' = U_+(S)\, \Phi$

$$\Phi'_{0\delta,f \pm' 1, j+1} = \tfrac{1}{2}(J - s - 1)(J \pm l + 1 \pm' f)\, \varphi^+_{0\delta}$$
$$\Phi'_{0\delta,f \pm' 1, j-1} = \tfrac{1}{2}(J + s + 1)(-J \pm l + 1 \pm' f)\, \varphi^-_{0\delta}$$
$$\Phi'_{0d,f \pm' 1, j} = \tfrac{1}{2}(s \mp l \mp' f)\, \varphi^0_{0\delta}$$
$$\Phi'_{1\delta,f \pm' 1, j} = \mp' (1/2i)(d\varphi_{0\delta})^0$$
$$\Phi'_{1d,f \pm' 1, j+1} = \pm'(1/2i)\frac{J + s - 1}{J - s}(d\varphi_{0\delta})^+$$
$$\Phi'_{1d,f \pm' 1, j-1} = \mp'(1/2i)\frac{J - s + 1}{J + s}(d\varphi_{0\delta})^-$$

TABLE VI

$\mathscr{E}_{0d,f,j}$ Probe: $\Phi = dt \wedge e^{ift}\varphi_{0d}$, $\varphi_{0d} \in E_{0d,j}$, $j \geq 1$. $\Phi' = U_{\pm}(S)\,\Phi$

$\Phi'_{0\delta,f\pm'1,j} = \frac{1}{2}(s \pm l + 1 \pm' f)\,\varphi^0_{0d}$
$\Phi'_{0d,f\pm'1,j+1} = \frac{1}{2}(J+s)(J \pm l + 1 \pm' f)\,\varphi^+_{0d}$
$\Phi'_{0d,f\pm'1,j-1} = \frac{1}{2}(J-s)(-J \pm l + 1 \pm' f)\,\varphi^-_{0d}$
$\Phi'_{1d,f\pm'1,j} = \pm'(1/2i)\,d\varphi^0_{0d}$

For the $\mathscr{E}_{1\delta,f+1,j+1}$ component,

$$(DU_-(S)\,\Phi)_{1\delta,f+1,j+1}$$
$$= (s+l)\left\{\prod_{m=0}^{l-1}\left[-(f+1)^2 + (J-l+2+2m)^2\right]\right\}$$
$$\times \tfrac{1}{2}(J-s)(J-l+1+f)\,\varphi^+_{1\delta},$$
$$(U_+(S)\,D\Phi)_{1\delta,f+1,j+1}$$
$$= (s+l)\left\{\prod_{m=0}^{l-1}\left[-f^2 + (J-l+1+2m)^2\right]\right\}\tfrac{1}{2}(J-s)(J+l+1+f)\,\varphi^+_{1\delta}.$$

Factoring the quadratics and shifting indices, we find that these two expressions are equal. The equality of $\mathscr{E}_{1\delta,f-1,j+1}$ components follows by time reversal (from the equality of $\mathscr{E}_{1\delta,-f+1,j+1}$ components for an $\mathscr{E}_{1\delta,-f,j}$ probe). As for $\mathscr{E}_{1\delta,f+1,j-1}$ components,

$$(DU_-(S)\,\Phi)_{1\delta,f+1,j-1}$$
$$= (s+l)\left\{\prod_{m=0}^{l-1}\left[-(f+1)^2 + (J-l+2m)^2\right]\right\}\tfrac{1}{2}(J+s)(-J-l+1+f)\,\varphi^-_{1\delta}$$
$$= (s+l)\left\{\prod_{m=0}^{l-1}\left[-f^2 + (J-l+1+2m)^2\right]\right\}\tfrac{1}{2}(J+s)(-J+l+1+f)\,\varphi^-_{1\delta}$$
$$= (U_+(S)\,D\Phi)_{1\delta,f+1,j-1},$$

and the equality of $\mathscr{E}_{1\delta,f-1,j-1}$ components follows by time-reversal.

TABLE VII

$\mathscr{E}_{1\delta,f,j}$ Probe: $\Phi = e^{ift}\varphi_{1\delta}$, $\varphi_{1\delta} \in E_{1\delta,j}$, $j \geq 1$. $\Phi' = U_{\pm}(S)\,\Phi$

$\Phi'_{0\delta,f\pm'1,j} = \pm'(1/2i)\,\delta\varphi^0_{1\delta}$
$\Phi'_{1\delta,f\pm'1,j+1} = \frac{1}{2}(J-s)(J \pm l + 1 \pm' f)\,\varphi^+_{1\delta}$
$\Phi'_{1\delta,f\pm'1,j-1} = \frac{1}{2}(J+s)(-J \pm l + 1 \pm' f)\,\varphi^-_{1\delta}$
$\Phi'_{1d,f\pm'1,j} = \frac{1}{2}(s - 1 \mp l \mp' f)\,\varphi^0_{1\delta}$

TABLE VIII

$\mathscr{E}_{1d,f,j}$ Probe: $\Phi = e^{ift}\varphi_{1d}$, $\varphi_{1d} \in E_{1d,j}$, $j \geqslant 1$. $\Phi' = U_{\pm}(S)\,\Phi$

$$\Phi'_{0\delta,f\pm'1,j+1} = \mp'(1/2i)\,\frac{J-s-1}{J+s}\,(\delta\varphi_{1d})^{+}$$

$$\Phi'_{0\delta,f\pm'1,j-1} = \pm'(1/2i)\,\frac{J+s+1}{J-s}\,(\delta\varphi_{1d})$$

$$\Phi'_{0d,f\pm'1,j} = \mp'(1/2i)(\delta\varphi_{1d})^{0}$$

$$\Phi'_{1\delta,f\pm'1,j} = \tfrac{1}{2}(s \pm l \pm' f)\,\varphi^{0}_{1d}$$

$$\Phi'_{1d,f\pm'1,j+1} = \tfrac{1}{2}(J+s-1)(J \pm l + 1 \pm' f)\,\varphi^{+}_{1d}$$

$$\Phi'_{1d,f\pm'1,j-1} = \tfrac{1}{2}(J-s+1)(-J \pm l + 1 \pm' f)\,\varphi^{-}_{1d}$$

The relevant $\mathscr{E}_{0\delta,f+1,j}$ components are

$$(DU_{+}(S)\,\Phi)_{0\delta,f+1,j}$$

$$= \left\{\prod_{m=0}^{l-2}\left[-(f+1)^2 + (J-l+2+2m)^2\right]\right\}\frac{1}{2i}\left\{-(s-l)(f+1)^2 + (s+l)(J^2-l^2) + 2(f+1)\,l(s-1+l-f)\right\}\delta\varphi^{0}_{1\delta},$$

$$(U_{+}(S)\,D\Phi)_{0\delta,f+1,j}$$

$$= (s+l)\left\{\prod_{m=0}^{l-1}\left[-f^2 + (J-l+1+2m)^2\right]\right\}\frac{1}{2i}\,\delta\varphi^{0}_{1\delta}.$$

TABLE IX

$\varphi_{0\delta} \in E_{0\delta,j}$, $\varphi_{0d} \in E_{0d,j}$, $\varphi_{1\delta} \in E_{1\delta,j}$, $\varphi_{1d} \in E_{1d,j}$

$$d\varphi^{+}_{0\delta} = \frac{(J-s+1)(J+s-1)}{(J-s-1)(J-s)}\,(d\varphi_{0\delta})^{+}$$

$$d\varphi^{-}_{0\delta} = \frac{(J-s+1)(J+s-1)}{(J+s+1)(J+s)}\,(d\varphi_{0\delta})^{-}$$

$$\delta\varphi^{+}_{0d} = \frac{(J-s-2)(J+s+2)}{(J+s+1)(J+s)}\,(\delta\varphi_{0d})^{+}$$

$$\delta\varphi^{-}_{0d} = \frac{(J-s-2)(J+s+2)}{(J-s-1)(J-s)}\,(\delta\varphi_{0d})^{-}$$

$$d\varphi^{+}_{1\delta} = \frac{(J-s+2)(J+s-2)}{(J-s)(J-s+1)}\,(d\varphi_{1\delta})^{+}$$

$$d\varphi^{-}_{1\delta} = \frac{(J-s+2)(J+s-2)}{(J+s)(J+s+1)}\,(d\varphi_{1\delta})^{-}$$

$$\delta\varphi^{+}_{1d} = \frac{(J-s-1)(J+s+1)}{(J+s)(J+s-1)}\,(\delta\varphi_{1d})^{+}$$

$$\delta\varphi^{-}_{1d} = \frac{(J-s-1)(J+s+1)}{(J-s)(J-s+1)}\,(\delta\varphi_{1d})^{-}$$

These two expressions will be equal if

$$(s+l)[J-l+1+f][J+l-1-f]\frac{1}{2i}$$
$$\stackrel{?}{=}\frac{1}{2i}\{-(s-l)(f+1)^2+(s+l)(J^2-l^2)+2(f+1)\,l(s-1+l-f)\}. \tag{3.28}$$

But both sides of (3.28) reduce to

$$\frac{1}{2i}\{-(s+l)(f+1)^2+2l(s+l)(f+1)+(s+l)(J^2-l^2)\}.$$

The relevant $\mathscr{E}_{1d,f+1,j}$ components are

$$\begin{aligned}&(DU_-(S)\,\Phi)_{1d,f+1,j}\\ &\quad=\left\{\prod_{m=0}^{l-2}[-(f+1)^2+(J-l+2+2m)^2]\right\}\\ &\quad\quad\times\tfrac{1}{2}\{(s-1+l-f)[-(s+l)(f+1)^2\\ &\quad\quad+(s-l)(J^2-l^2)]-2l(f+1)(J^2-s^2)\}\,\varphi^0_{1\delta},\\ &(U_+(S)\,D\Phi)_{1d,f+1,j}\\ &\quad=\left\{\prod_{m=0}^{l-1}[-f^2+(J-l+1+2m)^2]\right\}\tfrac{1}{2}(s-1-l-f)\,\varphi^0_{1\delta}.\end{aligned}$$

These expressions will be equal if

$$\begin{aligned}&(s+l)[J-l+1+f][J+l-1-f]\,\tfrac{1}{2}(s-1-l-f)\\ &\quad\stackrel{?}{=}\tfrac{1}{2}\{(s-1+l-f)[-(s+l)(f+1)^2+(s-l)(J^2-l^2)]\\ &\quad\quad-2l(J^2-s^2)(f+1)\}.\end{aligned} \tag{3.29}$$

But both sides of (3.29) reduce to

$$\begin{aligned}&\tfrac{1}{2}(s+l)\{(f+1)^3-(s+l)(f+1)^2-(J^2+l^2-2ls)\\ &\quad\times(f+1)+(s-l)(J^2-l^2)\}.\end{aligned}$$

The equality of $\mathscr{E}_{\binom{0\delta}{1d},f-1,j}$ components now follows by time-reversal. This completes the argument for an $\mathscr{E}_{1\delta,f,j}$ probe.

The covariance relation (3.27) for an $\mathscr{E}_{0d,f,j}$ probe now follows by a similar argument, or by an application of Hodge duality.

Next consider an $\mathscr{E}_{1d,f,j}$ probe $\Phi=e^{ift}\varphi_{1d}$, $\varphi_{1d}\in E_{1d,j}$. The only components of $DU_-(S)\,\Phi$ or $U_+(S)\,D\Phi$ that might not vanish are in

$$\mathscr{E}_{\binom{0d}{1\delta},f\pm 1,j},\qquad \mathscr{E}_{\binom{0\delta}{1d},f\pm 1,j\pm' 1}.$$

Starting with the components in the $0d$ sector,

$$(DU_-(S)\,\Phi)_{0d,f+1,j}$$
$$= (s-l)\left\{\prod_{m=0}^{l-1}[-(f+1)^2+(J-l+1+2m)^2]\right\}\left(-\frac{1}{2i}\right)(\delta\varphi_{1d})^0,$$
$$(U_+(S)\,D\Phi)_{0d,f+1,j}$$
$$= \left\{\prod_{m=0}^{l-2}[-f^2+(J-l+2+2m)^2]\right\}\frac{1}{2i}\{(s+l)f^2-(s-l)(J^2-l^2)$$
$$+2lf(s-l-f)\}(\delta\varphi_{1d})^0.$$

These two expressions will be equal if

$$\frac{1}{2i}\{(s+l)f^2-(s-l)(J^2-l^2)+2lf(s-l-f)\}$$
$$\overset{?}{=} -\frac{1}{2i}(s-l)(J-l-f)(J+l+f). \tag{3.30}$$

But (3.28) was proved as an equality of polynomials; replacing l by $-l$ and f by $f+1$, (3.30) becomes exactly problem (3.28). The equality of $\mathscr{E}_{0d,f-1,j}$ components also follows, by time-reversal. As for 1δ components,

$$(DU_-(S)\,\Phi)_{1d,f+1,j}$$
$$= (s+l)\left\{\prod_{m=0}^{l-1}[-(f+1)^2+(J-l+1+2m)^2]\right\}\tfrac{1}{2}(s-l+f)\,\varphi^0_{1d},$$
$$(U_+(S)\,D\Phi)_{1d,f+1,j}$$
$$= \left\{\prod_{m=0}^{l-2}[-f^2+(J-l+2+2m)^2]\right\}\tfrac{1}{2}\{(s+l+f)[-(s+l)f^2$$
$$+(s-l)(J^2-l^2)]+2lf(J^2-s^2)\}\,\varphi^0_{1d}.$$

These expressions will be equal if

$$\tfrac{1}{2}\{(s+l+f)[-(s+l)f^2+(s-l)(J^2-l^2)]+2lf(J^2-s^2)\}$$
$$\overset{?}{=} \tfrac{1}{2}(s+l)(J-l-f)(J+l+f)(s-l+f).$$

But after changing f to $-(f+1)$, this is exactly problem (3.29). We also get equality of $\mathscr{E}_{1d,f-1,j}$ components by time-reversal.

Next consider the effect of the $\mathscr{E}_{1d,f,j}$ probe on the 0δ sector. The relevant $\mathscr{E}_{0\delta,f+1,j+1}$ components are (using Lemma 3.10 in addition to Tables V–VIII)

$$\begin{aligned}
&(DU_-(S)\,\Phi)_{0\delta,f+1,j+1}\\
&\quad=\left\{\prod_{m=0}^{l-2}\left[-(f+1)^2+(J+3-l+2m)^2\right]\right\}\\
&\qquad\times\left\{\left[-(s-l)(f+1)^2+(s+l)((J+1)^2-l^2)\right]\left(-\frac{1}{2i}\right)\frac{J-s-1}{J+s}\right.\\
&\qquad\left.-li(f+1)(J+s-1)(J-l+1+f)\frac{(J-s-1)(J+s+1)}{(J+s-1)(J+s)}\right\}(\delta\varphi_{1d})^+,\\
&(U_+(S)\,D\Phi)_{0\delta,f+1,j+1}\\
&\quad=\left\{\prod_{m=0}^{l-2}\left[-f^2+(J-l+2+2m)^2\right]\right\}\left(-\frac{1}{2i}\right)\\
&\qquad\times\left\{\frac{J-s-1}{J+s}\left[-(s+l)\,f^2+(s-l)(J^2-l^2)\right]-2(J-s-1)\right.\\
&\qquad\left.\times(J+l+1+f)\,lf\right\}(\delta\varphi_{1d})^+.
\end{aligned}$$

These expressions will be equal if

$$\begin{aligned}
&-\frac{J+l+f}{2i}\left\{-(s-l)(f+1)^2+(s+l)((J+1)^2-l^2)\right.\\
&\qquad\left.-2l(f+1)(J-l+1+f)(J+s+1)\right\}\\
&\quad\overset{?}{=}-\frac{J-l+2+f}{2i}\left\{-(s+l)\,f^2+(s-l)(J^2-l^2)\right.\\
&\qquad\left.-2lf(J+s)(J+l+1+f)\right\}.
\end{aligned}\tag{3.31}$$

After a tedious calculation, one find that (3.31) holds as an equality of polynomials in f, J, l, s. Equality of the $\mathscr{E}_{0\delta,f-1,j+1}$ components for an $\mathscr{E}_{1d,f,j}$ probe follows by time-reversal.

For the $\mathscr{E}_{0\delta,f-1,j-1}$ component (using Lemma 3.10),

$$\begin{aligned}
&(DU_-(S)\,\Phi)_{0\delta,f-1,j-1}\\
&\quad=\left\{\prod_{m=0}^{l-2}\left[-(f-1)^2+(J+1-l+2m)^2\right]\right\}\\
&\qquad\times\left\{-\frac{1}{2i}\frac{J+s+1}{J-s}\left[-(s-l)(f+1)^2+(s+l)((J-1)^2-l^2)\right]\right.\\
&\quad-li(J-s+1)(-J-l+1-f)(f-1)\\
&\qquad\left.\times\frac{(J-s-1)(J+s+1)}{(J-s+1)(J-s)}\right\}(\delta\varphi_{1d})^-,
\end{aligned}$$

$$(U_+(S)\,D\Phi)_{0\delta,f-1,j-1}$$
$$= \left\{\prod_{m=0}^{l-2} [f^2+(J-l+2)^2]\right\}\left\{-lif(J+s+1)(-J+l+1-f)\right.$$
$$\left.-\frac{1}{2i}[-(s+l)f^2+(s-l)(J^2-l^2)]\frac{J+s+1}{J-s}\right\}(\delta\varphi_{1d})^-.$$

These expressions will be equal if

$$(J-l+f)\{(s-l)(f-1)^2-(s+l)((J-1)^2-l^2)$$
$$+(-J-l+1-f)\,2l(f-1)(J-s-1)\}$$
$$\stackrel{?}{=} (J+l-2+f)\{2lf(-J+l+1-f)(J-s)$$
$$+(s+l)f^2-(s-l)(J^2-l^2)\}.$$

But changing J to $-J$ and f to $-f$, this is exactly problem (3.31). This also gives (by time-reversal) equality of $\mathscr{E}_{0\delta,f+1,j-1}$ components.

As for $\mathscr{E}_{1d,f+1,j+1}$ components,

$$(DU_-(S)\,\Phi)_{1d,f+1,j+1}$$
$$= \left\{\prod_{m=0}^{l-2} [-(f+1)^2+(J+3-l+2m)^2]\right\}$$
$$\times\left\{l(f+1)\frac{J-s-1}{J+1}\,\frac{(J+s-1)(J+s)}{(J-s-1)(J+s+1)}[(J+1)^2-s^2]\right.$$
$$+[-(s+l)(f+1)^2+(s-l)((J+1)^2-l^2)]\,\tfrac{1}{2}(J+s-1)$$
$$\left.\times(J-l+1+f)\right\}\varphi^+_{1d},$$

$$(U_+(S)\,D\Phi)_{1d,f+1,j+1}$$
$$= \left\{\prod_{m=0}^{l-2} [-f^2+(J-l+2+2m)^2]\right\}$$
$$\times\left\{-\frac{J+s-1}{J-s}lf(J^2-s^2)+\tfrac{1}{2}(J+s-1)(J+l+1+f)\right.$$
$$\left.\times[-(s+l)f^2+(s-l)(J^2-l^2)]\right\}\varphi^+_{1d}.$$

These expressions will be equal if

$$(J+l+f)\{2l(f+1)(J+1-s)+[-(s+l)(f+1)^2+(s-l)$$
$$\times((J+1)^2-l^2)](J-l+1+f)\}$$
$$\stackrel{?}{=} (J-l+2+f)\{-2lf(J+s)+(J+l+1+f)$$
$$\times[-(s+l)f^2+(s-l)(J^2-l^2)]\}, \tag{3.32}$$

and a tedious calculation shows that (3.32) is true as a polynomial identity. By time-reversal, this also gives the equality of $\mathscr{E}_{1d,f-1,j+1}$ components. For the $\mathscr{E}_{1d,f-1,j-1}$ components,

$$(DU_{-}(S)\,\Phi)_{1d,f-1,j-1}$$
$$= \left\{\prod_{m=0}^{l-2} [-(f-1)^2+(J+1-l+2m)^2]\right\}$$
$$\times \left\{\frac{J+s+1}{J-s}\, l(f-1)\, \frac{(J-s+1)(J-s)(J-1+s)}{J+s+1}\right.$$
$$+\tfrac{1}{2}(J-s+1)(-J-l+1-f)[-(s+l)(f-1)^2$$
$$\left.+(s-l)((J-1)^2-l^2)]\right\}\varphi^{-}_{1d},$$

$$(U_{+}(S)\,D\Phi)_{1d,f-1,j-1}$$
$$= \left\{\prod_{m=0}^{l-2} [-f^2+(J-l+2+2m)^2]\right\}$$
$$\times\{-lf(J-s+1)(J-s)+\tfrac{1}{2}[-(s+l)\,f^2+(s-l)(J^2-l^2)]$$
$$\times(J-s+1)(-J+l+1-f)\}\,\varphi^{-}_{1d}.$$

These two expressions will be equal if

$$(J-l+f)\{2l(f-1)(J-1+s)+(-J-l+1-f)[-(s+l)(f-1)^2$$
$$+(s-l)((J-1)^2-l^2)]\}$$
$$\overset{?}{=} (J+l-2+f)\{-2lf(J-s)+[-(s+l)\,f^2$$
$$+(s-l)(J^2-l^2)](-J+l+1-f)\}.$$

But after changing J to $-J$ and f to $-f$, this is exactly problem (3.32). By time-reversal, we also get the equality of $\mathscr{E}_{1d,f+1,j-1}$ components.

The covariance relation (3.27) for a $\Phi_{0\delta,f,j}$ probe now follows by a similar argument, or by an application of Hodge duality.

We still need to treat the special cases of (3.11), (3.12); this just involves more routine calculations of the type above. Finally, since $DU_{-}(S)$ and $U_{+}(S)\,D$ are differential operators, the "eigenspace by eigenspace" proof of covariance, along with the fact that $C^{\infty}=\bigcap H^{m}$, prove (3.27) for C^{∞} forms Φ. ∎

Recalling Fig. 3.2, we now get 5 or 6 $u_{s-l}(G)$-invariant subspaces as was the case for D_{2l}.

LEMMA 3.23. *Suppose $n \geq 4$ is even. Within the k-forms (C^∞ or Sobolev class) of parity $(-1)^{s-l}$ on $\tilde{\bar{M}}^n$, let $W^\pm$ and F be as in Theorem* 3.18. *Then the* 6 *subspaces F, $W^\pm$,*

$$V^\pm = \{\pm f \geq J + l\} + W^\pm,$$

$$M = \{|f| \leq J - l\} + W^+ + W^-$$

are $u_{s-l}(G)$-invariant, $G = O^\uparrow(2, n)$.

Proof. As for Lemma 2.8, it suffices to establish $U_{s-l}(S)$-invariance. Recalling Fig. 3.2, this reduces to the following 6 calculations (corresponding to arrows 1–6 in Fig. 3.4) and their analogues on the other "edge" of W^+. (Time-reversal then takes care of W^-. The observation in the proof of Lemma 3.12 is also needed if $l = 1$.)

(1) Let $\Phi = e^{ift}\varphi_{1\delta}$ be an $\mathscr{E}_{1d,f,j}$ probe, $f = J - l + 1$. By Table VII, $(U_{s-l}(S)\,\Phi)_{1\delta,f-1,j+1} = 0$. Hodge duality gives a corresponding statement about $\mathscr{E}_{0d,f,j}$ probes.

(2) For Φ as above, by Table VII,

$$(U_{s-l}(S)\,\Phi)_{0\delta,f-1,j} = -\frac{1}{2i}\,\delta\varphi^0_{1d},$$

$$(U_{s-l}(S)\,\Phi)_{1d,f-1,j} = \frac{1}{2}(s - 1 + l + f)\,\varphi^0_{1\delta} = \frac{1}{2}(J + s)\,\varphi^0_{1\delta}.$$

Thus by (3.26), the $\binom{0\delta}{1d}$ component of $U_{s-l}(\Phi)$ is in W^+, and similarly for an $\mathscr{E}_{0\delta,f,j}$ probe.

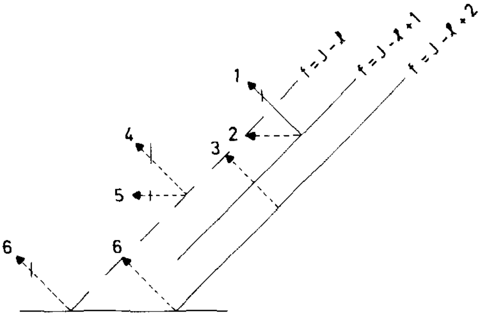


FIGURE 3.4

(3) Let $\Phi = e^{ift}\varphi_{1d}$ be an $\mathscr{E}_{1d,f,j}$ probe, $f = J - l + 2$. By Tables VIII and IX,

$$(U_{s-l}(S)\,\Phi)_{0\delta,f-1,j+1} = \frac{J-s-1}{i(J+s)}\,(\delta\varphi_{1d})^+,$$

$$\delta(U_{s-l}(S)\,\Phi)_{1d,f-1,j+1} = -\frac{(J-s-1)(J+s+1)}{J+s}\,(\delta\varphi_{1d})^+.$$

By (3.26), the $\mathscr{E}_{\binom{0\delta}{1d},f-1,j+1}$ component of $U_{s-l}(S)\,\Phi$ thus lies in W^+, and similarly for an $\mathscr{E}_{0\delta,f,j}$ probe.

(4) Let Φ be an $\mathscr{E}_{\binom{0\delta}{1d},f,j}$ probe on

$$\{f = J - l\} \cap Z^{+,-}, \qquad \Phi = e^{ift}\left\{dt \wedge \frac{i\delta\varphi_{1d}}{J+s} + \varphi_{1d}\right\}.$$

By Tables V and VIII,

$$(U_{s-l}(S)\,\Phi)_{0\delta,f-1,j+1} = (U_{s-l}(S)\,\Phi)_{1d,f-1,j+1} = 0.$$

(5) For Φ as immediately above, by Tables V and VIII,

$$(U_{s-l}(S)\,\Phi)_{0d,f-1,j} = \frac{1}{2i}\,(\delta\varphi_{1d})^0 + \frac{1}{2}\,(J+s)\left(\frac{i\delta\varphi_{1d}}{J+s}\right)^0 = 0,$$

$$(U_{s-l}(S)\,\Phi)_{1\delta,f-1,j} = \frac{1}{2}\,(s-J)\,\varphi^0_{1d} + \frac{1}{2i}\left(d\,\frac{i\delta\varphi_{1d}}{J+s}\right)^0 = 0.$$

(6) The exceptional cases corresponding to (3.11) and (3.12) can also be handled routinely. ∎

The representation spaces of interest are F, $W^\pm/F$, $V^\pm/W^\pm$, and $M/(W^+ + W^-)$.

THEOREM 3.24. *Suppose $n \geq 4$ is even, and $s \neq \pm l$. Then the inner products*

$$\begin{aligned}
\langle \Phi, \Psi\rangle_M &= (\Phi, D_{2l,k}\Psi) && \text{on } M/(W^+ + W^-),\\
\langle \Phi, \Psi\rangle_{V^\pm} &= (-1)^l\,(\Phi, D_{2l,k}\Psi) && \text{on } V^\pm/W^\pm,\\
\langle \Phi, \Psi\rangle_{W^\pm} &= \pm i(\Phi, D'_{2l,k}\Psi) && \text{on } W^\pm,\\
\langle \Phi, \Psi\rangle_F &= (\Phi, D''_{2l,k}\Psi) && \text{on } F,
\end{aligned}$$

where $D'_{2l,k} = [D_{2l,k}, t]$ *and* $D''_{2l,k} = [D'_{2l,k}, t]$, *are Hermitian, nondegenerate, and* $u_{s-l}(G)$*-invariant,* $\langle\ ,\ \rangle_{V^\pm}$ *is positive definite* provided $l > |s|$, *and in this case* $V^\pm/W^\pm$ *completes to a Hilbert space carrying* $u_{s-l}(G)$ *as a continuous unitary representation. With the possible exception of* $\langle\ ,\ \rangle_F$, *all the other inner products are indefinite except in the following limiting cases already treated:* $\langle\ ,\ \rangle_M$ *and* $\langle\ ,\ \rangle_{V^\pm}$ *for* $k = 0, n$, *and* $\langle\ ,\ \rangle_{W^\pm}$ *for* $k = 0, n$ *and* $l = 1$. *(Recall that* $(\ ,\ )$ *itself is indefinite unless* $k = 0, n$.*)*

Remark 3.25. The theorem says that we get a unitary quotient if k is close enough to the middle order $n/2$, or if the order of the fundamental intertwining differential operator is high enough. For $l = 1$, this unitarity condition reduces to the condition $s = 0$ of Theorem 3.13.

Proof of Theorem 3.24. Assume throughout that $s \geqslant 0$, i.e., that $k \leqslant n/2$. (The cases $s < 0$ can be taken care of by applying Hodge duality.) Assume that $k \neq 0, n$, for in these cases, the conclusion of the theorem is contained in Theorem 2.18. For $k = 1$, assign the "constants" $\mathbb{C}\, dt$ to the $0d$ sector, so that $C - s > 0$ in the $\binom{0\delta}{1d}$ sector.

All inner products are Hermitian because $D_{2l,k}$, $(1/i)\, D'_{2l,k}$, and $D''_{2l,k}$ are differential operators formally self-adjoint in $(\ ,\ )$. The argument for invariance of $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ proceeds along the lines of the corresponding argument for Theorem 2.9; the key points are the covariance relation (3.27), the fact that $(s-l)+(s+l) = n-2k$, and the fact that if h is a conformal transformation, $h \cdot g = \Omega^2 g$, then $h \cdot g^k = \Omega^{-2k} g^k$. The proof of nondegeneracy for $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ is exactly as in Theorem 3.13.

Let $D = D_{2l,k}$, and consider now the definiteness question for $(\Phi, D\Psi)$. For an $\mathscr{E}_{0d,f,j}$ probe $\Phi = dt \wedge e^{ift}\varphi_{0d}$,

$$(\Phi, D\Phi) = -2\pi(s-l)\left\{\prod_{m=0}^{l-1} [-f^2 + (J-l+1+2m)^2]\right\} \|\varphi_{0d}\|^2,$$

while for an $\mathscr{E}_{1\delta,f,j}$ probe $\Phi = e^{ift}\varphi_{1\delta}$,

$$(\Phi, D\Phi) = 2\pi(s+l)\left\{\prod_{m=0}^{l-1} [-f^2 + (J-l+1+2m)^2]\right\} \|\varphi_{1\delta}\|^2,$$

where the norms are in the L^2 spaces of forms in S^{n-1}. This already shows the indefiniteness of $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ when $l < s$. When $l > s$, $\langle\ ,\ \rangle_M$ is positive definite on the $\binom{0d}{1\delta}$ sector, while the sign of $\langle\ ,\ \rangle_{V^\pm}$ on the $\binom{0d}{1\delta}$ sector is $(-1)^l$.

For the analysis of the sign of $(\Phi, D\Phi)$ in the $\binom{0\delta}{1d}$ sector, assume that $l > s$, since this is the only case in which unitarity is still possible for $\langle\ ,\ \rangle_{V^\pm}$ or $\langle\ ,\ \rangle_M$. Suppose that $|f| \geqslant J+l$, or $|f| \leqslant J-l$ but $(f, J) \neq (0, l)$. Then we claim there are two $(\cdot, D\cdot)$-orthogonal subspaces $\mathscr{E}^\pm$ of $\mathscr{E} = \mathscr{E}_{\binom{0\delta}{1d}, f, j}$,

$$\mathscr{E}^\pm = \{\Phi_{0\delta} = \pm ib\delta\Phi_{1d}\}$$

with $0 < b = b(f, J, s, l)$ (so that $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$), such that $(\cdot, D\cdot)$ is definite or zero on $\mathscr{E}^\pm$. Since for $\Phi \in \mathscr{E}_{\binom{0\delta}{1d}, f, j}$,

$$D\Phi = \left\{ \prod_{m=0}^{l-2} [-f^2 + (J-l+2+2m)^2] \right\} \mathscr{B}\Phi,$$

we need only show that $\mathscr{E}^+$ and $\mathscr{E}^-$ are $(\cdot, \mathscr{B}\cdot)$-orthogonal, and that $(\cdot, \mathscr{B}\cdot)$ is definite or zero on $\mathscr{E}^\pm$.

The orthogonality condition will be used to determine b. Let

$$\Phi = e^{ift}(dt \wedge ib\,\delta\varphi_{1d} + \varphi_{1d}) \in \mathscr{E}^+,$$
$$\Psi = e^{ift}(-dt \wedge ib\psi_{1d} + \psi_{1d}) \in \mathscr{E}^-.$$

Then

$$(\Phi, \mathscr{B}\Psi) = 2\pi\{(J^2-s^2)[(l+s)(J^2-l^2) + (l-s)f^2]\, b^2 - [(l+s)f^2 + (l-s)(J^2-l^2)]\}\langle \varphi_{1d}, \psi_{1d}\rangle_{L^2(\Lambda^k(S^{n-1}))}.$$

The orthogonality condition thus forces

$$b^2 = \frac{(l+s)f^2 + (l-s)(J^2-l^2)}{(J^2-s^2)[(l+s)(J^2-l^2)+(l-s)f^2]}. \tag{3.33}$$

Both the numerator p and denominator q on the right in (3.33) are positive for $|f| \geqslant J+l$ or $|f| \leqslant J-l$, $(f, J) \neq (0, l)$. Indeed,

$$p\,|_{|f|=J+l} = 2l(J+l)(J+s) > 0, \qquad \frac{\partial p}{\partial(f^2)} = l+s > 0;$$

$$q\,|_{|f|=J+l} = 2l(J+l)(J-s)(J^2-s^2) > 0, \qquad \frac{\partial q}{\partial(f^2)} = (l-s)(J^2-s^2) > 0;$$

$$p\,|_{\substack{|f|=J-l\\(f,J)\neq(0,l)}} = 2l(J-l)(J-s)>0, \qquad \frac{\partial p}{\partial(J^2)}=l-s>0;$$

$$q\,|_{\substack{|f|=J-l\\(f,J)\neq(0,l)}} = 2l(J-l)(J+s)(J^2-s^2)>0, \qquad \frac{\partial q}{\partial(J^2)}=(l+s)(J^2-s^2)>0.$$

Thus we can define b to be the positive square root of the expression on the right in (3.33), and get $(\cdot, D\cdot)$-orthogonality of $\mathscr{E}^+$ and $\mathscr{E}^-$.

For Φ and Ψ as above,

$$(\Phi, \mathscr{B}\Phi)=2\pi\{-qb^2+4lf(J^2-s^2)\,b-p\}\,\|\varphi_{1d}\|^2,$$
$$(\Psi, \mathscr{B}\Psi)=2\pi\{-qb^2-4lf(J^2-s^2)\,b-p\}\,\|\psi_{1d}\|^2.$$

$(\cdot, \mathscr{B}\cdot)$ will certainly be negative definite on both $\mathscr{E}^+$ and $\mathscr{E}^-$ if the quadratic polynomial

$$Q(c)=-qc^2+4lf(J^2-s^2)\,c-p$$

takes on only negative values. But $Q(0)=-p<0$, and the discriminant of Q is

$$\gamma=16l^2f^2(J^2-s^2)^2-4pq.$$

In the special case $f=0$, $J>l$,

$$\gamma=-4pq<0,$$

so $(\cdot, \mathscr{B}\cdot)$ is negative definite, as is $(\cdot, D\cdot)$. With the fact that $\langle\ ,\ \rangle_M$ is positive definite in the $\binom{0d}{1\delta}$ sector, this shows that $\langle\ ,\ \rangle_M$ is indefinite. But for $\langle\ ,\ \rangle_{V^\pm}$,

$$\gamma\,|_{|f|=J+l}=0, \quad \frac{\partial\gamma}{\partial(f^2)}=-8(J^2-s^2)(l^2-s^2)(f^2-J^2-l^2)$$
$$<0 \qquad \text{for} \quad |f|\geqslant J+l.$$

This shows that $(\cdot, \mathscr{B}\cdot)$ is negative definite, so that $(\cdot, D\cdot)$ is $(-1)^l$-definite, on $\mathscr{E}_{\binom{0\delta}{1d},f,j}$ with $|f|>J+l$. To finish the analysis of the sign of $\langle\ ,\ \rangle_{V^\pm}$, we need to look at the cases $f=\pm(J+l)$, in which $b=(J-s)^{-1}$, and

$$Q(\pm' b)=\begin{cases} 0, & \pm=\pm' \\ -4l(J+l)(J+s), & \pm\neq\pm'. \end{cases}$$

That is, on $\mathscr{E}_{\binom{0\delta}{1d},J+l,j} \cap Z^{+,+} \subseteq W^+$ and $\mathscr{E}_{\binom{0\delta}{1d},-(J+l),j} \cap Z^{-,+} \subseteq W^-$, $(\cdot, \mathscr{B}\cdot) = 0$ as expected, but on $\mathscr{E}_{\binom{0\delta}{1d},J+l,j} \cap Z^{+,-}$ and $\mathscr{E}_{\binom{0\delta}{1d},-(J+l),j} \cap Z^{-,-}$, $(\cdot, \mathscr{B}\cdot)$ is negative definite, so that $(\cdot, D\cdot)$ is $(-1)^l$-definite. This completes the proof that $\langle\ ,\ \rangle_{V^\pm}$ is positive definite. The strength of the $\langle\ ,\ \rangle_{V^\pm}$ uniform structure is between that derived from $H^{l/2}$ and that derived from H^l, so after completion, we get a continuous unitary representation. All statements about $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ have now been proved.

The proof of invariance for $\langle\ ,\ \rangle_{W^\pm}$ proceeds exactly as in Theorem 2.9 or 2.18; to show that $U_{s+l}(S)$ and $U_{s-l}(S)$ are (,)-formal adjoints, (3.13) is needed.

$\langle\ ,\ \rangle_{W^\pm}$ is nondegenerate because (,) is nondegenerate on each $\mathscr{E}_{f,j}$, and $\mathscr{N}(D')$ is exactly F. Indefiniteness for the case $l=1$ has already been taken care of in Theorem 3.13. If $l>1$, an $\mathscr{E}_{1\delta,f,j}$ probe $\Phi = e^{ift}\varphi_{1\delta}$ with $\mathscr{E}_{1\delta,f,j} \not\subseteq F$,

$$f^2 = (J-l+1+2p)^2, \qquad 0 \leqslant p \leqslant l-1,$$

has

$$(\Phi, D'\Phi) = (s+l)\left\{\prod_{\substack{0\leqslant m\leqslant l-1\\ m\neq p}} [-f^2 + (J-l+1+2m)^2]\right\} \frac{4\pi f}{i}\, \|\varphi_{1\delta}\|^2. \quad (3.34)$$

Thus on adjacent solid lines of $W^\pm/F$ in the 1δ sector, which will exist if $l>1$, $\langle\ ,\ \rangle_{W^\pm}$ takes on opposing signs. Therefore $\langle\ ,\ \rangle_{W^\pm}$ is always indefinite for $l>1$. ∎

Remark 3.26 ($n=2$). Versions of Theorems 3.18, 3.19, 3.20, 3.22, and 3.24, and Lemma 3.23 hold for $n=2$, $k=1$ (so that $s=0$); but some care must be taken in the proofs concerning the special case $J=0$. We omit the details here.

Remark 3.27 (Odd n). The situation for odd n in the case of forms is similar to the scalar field situation described in Remark 2.21. *All* solutions of $D_{2l,k}\Phi = 0$ in $\tilde{M}^n$, $s \neq \pm l$, are 4π-periodic in t, even if $l > |s|$. There are 4 "parities," corresponding to ± 1 and $\pm i$ eigenspaces of the operator coming from (under any u_a) either element of the double cover of $O(2, n)$ which covers diag $(-1, ..., -1)$. The parity of positive frequency solutions disagrees with that of negative frequency solutions, so the analogues of W^+ and W^- reduce different representations.

Remark 3.28. Back in the case of even n, the non-periodic solution space tF of Theorem 3.23 can be brought into the representation theoretic picture by passing to universal covers (of $\bar{M}^n$ and of G). In addition to the representation spaces of the theorem, $(W^+ + W^- + tF)/W^\pm$ is also $u_{s-l}(\tilde{G})$-invariant.

Remark 3.29 (Hodge * duality). For $k = n/2$ $(s = 0)$, each $u_a(G_0)$ acts on the ε-dual forms, $\varepsilon^2 = (-1)^{(n/2)+1}$ (recall Remark 3.14). But even restricting to G_0, we get no unitary representations beyond those guaranteed by Theorem 3.24. The case $l = 1$ has been taken care of in Remark 3.14, and the proof of Theorem 3.24 shows that the ε-dual summand of $M/(W^+ + W^-)$ is still nonunitary. To show that the ε-dual part of $W^\pm/F$ is nonunitary, note that the analogue of (3.34) for an $\mathscr{E}_{0d,f,j}$ probe $dt \wedge e^{ift}\varphi_{0d}$ is

$$(\Phi, D'\Phi) = -(s-l)\left\{\prod_{\substack{0 \leqslant m \leqslant l-1\\ m \neq p}} [-f^2 + (J - l + 1 + 2m)^2]\right\}\frac{4\pi f}{i}\|\varphi_{0d}\|^2.$$

Thus for $s = 0$, the signs of $\langle\ ,\ \rangle_{W^\pm}$ on adjacent solid lines in the $\binom{0d}{1\delta}$ sector differ, even after restriction to ε-dual forms.

Remark 3.30 (Representations of the Lorentz group). Some interesting observations can be made concerning representations of the Lorentz group $O(1, n)$ on k-forms in S^{n-1}, using the machinery built up above. Let $1 \leqslant k \leqslant n-2$, $s = (n-2k)/2$, let B be the operator $\sqrt{\delta d + (s-1)^2}$ on $\mathscr{R}(\delta)$, and let C be the operator $\sqrt{d\delta + s^2}$ on $\mathscr{R}(d)$. (The eigenvalues of B on C are the $(n-2)/2 + j$, $j = 1, 2,\ldots$.) For $l = 1, 2,\ldots$, define an operator $L_{l,k}$ on C^∞ k-forms in S^{n-1} by

$$L_{l,k} = \begin{cases} \left(\dfrac{n-1}{2} - k + \dfrac{l}{2}\right)\displaystyle\prod_{m=0}^{l-1}\left(B - \dfrac{l-1}{2} + m\right) & \text{on } \mathscr{R}(\delta)\\[2ex] \left(\dfrac{n-1}{2} - k - \dfrac{l}{2}\right)\displaystyle\prod_{m=0}^{l-1}\left(C - \dfrac{l-1}{2} + m\right) & \text{on } \mathscr{R}(d). \end{cases}$$

For even l, $L_{l,k}$ is a differential operator with leading term $((n-1)/2 - k + l/2)\ \ (\delta d)^{l/2} + ((n-1)/2 - k - l/2)\ \ (d\delta)^{l/2} = [((n-1)/2 - k + l/2)\,\delta d + ((n-1)/2 - k - l/2)\,d\delta]\,\Delta^{l/2-1}$:

$$\begin{aligned} L_{l,k} = {} & \left(\frac{n-1}{2} - k + \frac{l}{2}\right)\prod_{m=1}^{l/2}\left(\delta d + (s-1)^2 - \left(m - \frac{1}{2}\right)^2\right)\\ & + \left(\frac{n-1}{2} - k - \frac{l}{2}\right)\prod_{m=1}^{l/2}\left(d\delta + s^2 - \left(m - \frac{1}{2}\right)^2\right)\\ & - \prod_{m=-l/2}^{l/2}\left(\frac{n}{2} - k + m - \frac{1}{2}\right), \qquad l \text{ even}. \end{aligned}$$

By Lemma 3.9, $L_{l,k}$ is Y-covariant

$$L_{l,k}\,U_{((n-1)/2)-k-(l/2)}(Y)=U_{((n-1)/2)-k+(l/2)}(Y)\,L_{l,k}.$$

By symmetry and the obvious $O(n)$-invariance, $L_{l,k}$ is $\mathfrak{o}(1,n)$-covariant, and for even l is $O(1,n)$-covariant by Remark 1.4

$$L_{l,k}u_{((n-1)/2)-k-(l/2)}(h)=u_{((n-1)/2)-k+(l/2)}(h)\,L_{l,k},\qquad h\in O(1,n),\ l\text{ even.} \tag{3.35}$$

The infinite-dimensional representation space is $C^\infty(\Lambda^k)/\mathcal{N}(L_{l,k})$, which admits the invariant inner product $\langle\varphi,\psi\rangle_{l,k}=(\varphi,L_{l,k}\psi)_{L^2}$. (That is, covariance shows that $\langle\ ,\ \rangle_{l,k}$ is $U_{((n-1)/2)-k-(l/2)}(\mathfrak{o}(1,n))$-skew, and for even l is $u_{((n-1)/2)-k-(l/2)}(O(1,n))$-invariant.)

The null space of $L_{l,k}$ is easily computed:

$$\mathcal{N}(L_{l,k})=\begin{cases}\{0\}, & l\equiv n \pmod 2\\ E_1\oplus\cdots\oplus E_{(l+1-n)/2}, & l-1\equiv n \pmod 2),\end{cases}$$

where $E_j=E_{\delta,j,k}\oplus E_{d,j,k}$. $\langle\ ,\ \rangle_{l,k}$ cannot be definite if $n-1-2k-l$ and $n-1-2k+l$ have opposite signs, i.e., if $l>|n-1-2k|$. The case $l=|n-1-2k|$ is treated in Remark 4.12. If $l<|n-1-2k|$, then $l<n-1$, and $\mathcal{N}(L_{l,k})=\{0\}$ regardless of the parity of $l+n$; $\langle\ ,\ \rangle_{l,k}$ is definite (positive for $2k<n-1-l$, negative for $2k>n-1+l$), and continuous unitary representations result for even l.

4. Maxwell's Equations and the Case $s=\pm l$

a. *Maxwell's Equations*

Maxwell's equations in an n-dimensional pseudo-Riemannian manifold, n even, are the conditions $\delta F=dF=0$ on an $n/2$-form F. In $\tilde{M}^n$, we have

THEOREM 4.1. *Suppose $n\geqslant 4$ is even. Then C^∞ $n/2$-form solutions of*

$$\delta^{(n)}F=d^{(n)}F=0 \tag{4.1}$$

in $\tilde{M}^n$ are 2π-periodic in t and even under the antipodal map in $\bar{\bar{M}}^n$. Equations (4.1) *satisfy Huygens' principle in the following sense: if the support of the data $(F_0,F_1,\partial_tF_0,\partial_tF_1)(0,x)$ contains no point at a distance t_0 from $x_0\in S^{n-1}$, $0\leqslant t_0\leqslant\pi$, then $F(t_0,x_0)=0$.*

Proof. If (4.1) holds, then

$$\begin{aligned} &\square F = 0, \\ &F_{1\delta} = F_{0d} = 0 \end{aligned} \tag{4.2}$$

by (3.2). By (4.2),

$$\partial_t^2 F_{0\delta} + \delta\, dF_{0\delta} = \partial_t^2 F_{1d} + d\delta F_{1d} = 0.$$

By (3.3) and (3.4), the eigenvalues of δd on coclosed $(n-2)/2$-forms or of $d\delta$ on $n/2$-forms are $(n-2)/2 + j$, $j = 1, 2, 3, \ldots$. This proves periodicity, and evenness follows immediately from (3.7). The Lax–Phillips oddness–evenness argument of Theorem 2.2 and finite propagation speed (from (4.2)) give Huygens' principle. ∎

The connection of Maxwell's equations to the $D_{2,k}$ is as follows. Since the cohomology $H^{n/2}(\bar{\bar{M}}^n)$ vanishes, $d^{(n)}F = 0$ implies that $F = d^{(n)}A$ for some $(n-2)/2$-form A on $\bar{M}^n$. As a condition on A, Maxwell's equations read $\delta^{(n)} d^{(n)} A = 0$, and $\delta^{(n)} d^{(n)}$ is exactly $D_{2,(n-2)/2}$. Conformal covariance is expressed by

$$\delta^{(n)} d^{(n)} u_0(h) = u_2(h)\, \delta^{(n)} d^{(n)}, \qquad h \in O(2, n). \tag{4.3}$$

The *vector potential* A is not uniquely defined; it is subject to the *gauge ambiguity* $A \mapsto A + d^{(n)}a$ for a an $(n-4)/2$-form. This is at the root of the representation theoretic differences between $D_{2,(n-2)/2}$ (or its "mirror image" $D_{2,(n+2)/2}$) and the other $D_{2,k}$. For the higher order $D_{2l,k}$, this generalizes to a difference between the cases $s = \pm l$ and $s \neq \pm l$. The important representations in the Maxwell case can be described either in terms of Maxwell fields F, or in terms of gauge equivalence classes of Maxwell potentials A; we choose the latter course here. Note that by (3.2), the *pure gauges* $\mathscr{R}(d^{(n)}) = \mathscr{N}(d^{(n)})$ (within the $(n-2)/2$-forms) occupy the entire $0d$ sector, and the subspace $\{\partial_t A_{1d} - dA_{0\delta} = 0\}$ of the $\binom{0\delta}{1d}$ sector. The property of being a pure gauge is $u_0(0(2, n)))$-invariant:

$$d^{(n)} u_0(h) = u_0(h)\, d^{(n)}. \tag{4.4}$$

By (3.2),

$$\delta^{(n)} d^{(n)} A = dt \wedge (\delta d A_0 - \delta \partial_t A_1) - d\partial_t A_0 + (\partial_t^2 + \delta d)\, A_1.$$

Thus $\mathscr{N}(\delta^{(n)} d^{(n)})$ is generated by the pure gauges, together with solutions of $(\partial_t^2 + \delta d)\, \Phi_{1\delta} = 0$ (i.e., the lines $f = \pm J$) in the 1δ sector. In particular, the *Coulomb gauge* can be fixed: if $\delta^{(n)}d^{(n)}A=0$, some gauge equivalent $A+d^{(n)}f$ has only a 1δ component.

Lemma 3.12 still provides us with the 5 $u_0(G)$-invariant subspaces $W^{\pm}$, $V^{\pm}$, M, and by (4.4), $\mathscr{G}=\mathscr{N}(d^{(n)})$ is also invariant. To get representation spaces with nondegenerate invariant inner products, we must "deflate" by the pure gauges $\mathscr{G}$:

THEOREM 4.2. *Suppose $n\geqslant 4$ is even, and let $W^{\pm}$, $V^{\pm}$, M be the subspaces of the $(n-2)/2$-forms on $\bar{\tilde{M}}^n$ (C^∞ or H^m) given in Lemma 3.12. The inner products*

$$\langle \Phi, \Psi\rangle_M=(\Phi, \delta^{(n)}d^{(n)}\Psi) \qquad \text{on } (M+\mathscr{G})/(W^+ + W^- +\mathscr{G})$$

$$\langle \Phi, \Psi\rangle_{V^\pm}= -(\Phi, \delta^{(n)}d^{(n)}\Psi) \qquad \text{on } (V^\pm +\mathscr{G})/(W^\pm +\mathscr{G})$$

$$\langle \Phi, \Psi\rangle_{W^\pm}= \pm i(\Phi, [\delta^{(n)}d^{(n)}, t]\, \Psi) \qquad \text{on } (W^\pm +\mathscr{G})/\mathscr{G}$$

are well-defined, Hermitian, nondegenerate, and $u_0(G)$-invariant. $\langle\ ,\ \rangle_M$ is indefinite but $\langle\ ,\ \rangle_{V^\pm}$ and $\langle\ ,\ \rangle_{W^\pm}$ are positive definite and result, after completion, in continuous unitary representations.

Proof. The inner products are well defined (gauge invariant) because $\delta^{(n)}$ and $d^{(n)}$ are $(\ ,\ )$-formal adjoints, and

$$[\delta^{(n)}d^{(n)}, t]=\delta^{(n)}\varepsilon(dt)-\iota(dt)\, d^{(n)}.$$

They are Hermitian because $\delta^{(n)}d^{(n)}$ and $(1/i)[\delta^{(n)}d^{(n)}, t]$ are formally self-adjoint. Invariance of $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ follows from the argument of Theorem 2.7, using the covariance relation (4.3) and Eq. (3.13). The proof of invariance for $\langle\ ,\ \rangle_{W^\pm}$ is just as in Theorem 2.9.

For an $\mathscr{E}_{1\delta,f,j}$ probe $A=e^{ift}A_{1\delta}$,

$$(A, \delta^{(n)}d^{(n)}A)=2\pi(-f^2+J^2)\, \|A_{1\delta}\|^2,$$

while for an $\mathscr{E}_{\left(\substack{0\delta\\ 1d}\right),f,j}$ probe $A=e^{ift}(dt\wedge A_{0\delta}+A_{1d})$,

$$(A, \delta^{(n)}d^{(n)}A)= -2\pi\, \|\partial_t A_{1d}-dA_{0\delta}\|^2.$$

Since $\partial_t A_{1d}-dA_{0\delta}=0$ is the pure gauge condition, this proves nondegeneracy of $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$, indefiniteness of $\langle\ ,\ \rangle_M$, and positive definiteness of $\langle\ ,\ \rangle_{V^\pm}$.

For an $\mathscr{E}_{1\delta,f,j}$ probe $A=e^{ift}A_{1\delta}$,

$$(A, [\delta^{(n)}d^{(n)}, t]\, A)= -4\pi i f\, \|A_{1\delta}\|^2;$$

by gauge invariance and the possibility of fixing the Coulomb gauge, this shows that $\langle\ ,\ \rangle_{W^\pm}$ is positive definite.

The strength of the $\langle\ ,\ \rangle_{V^\pm}$ uniform structure is between those of the uniform structures derived from $H^{1/2}$ and H^1; the $\langle\ ,\ \rangle_{W^\pm}$ uniform structure is that derived from $H^{1/2}$. Thus the representations with positive definite invariant inner products complete to continuous unitary representations. ∎

Remark 4.3 (*Polarization of symplectic structure*). On real fields, the inner product $\langle\ ,\ \rangle_{W^+}$ is a gauge-invariant *symplectic* form, which may be polarized (on $W^+/\mathscr{G}$) as in Remarks 2.10 and 3.15 to give the complex inner product $\langle\ ,\ \rangle_{W^+}$. This inner product also agrees with that given by Zuckerman [54] for $n=4$, viz.

$$\langle \Phi, \Psi \rangle = \frac{1}{2i} \int_{S^{n-1}} (\Phi \wedge *^{(n)} d^{(n)} \bar{\Psi} - \bar{\Psi} \wedge *^{(n)} d^{(n)} \Phi),$$

the integral being taken at any fixed t: $\langle\ ,\ \rangle_{W^+} = 4\pi \langle\ ,\ \rangle$. After pulling back to Minkowski space, it also agrees with that found by Gross [13] in the case $n=4$.

Remark 4.4 (Hodge $*$ duality). Restricting to G_0, the *solutions* of $\delta^{(n)} d^{(n)} A = 0$ can be split into those leading to $\pm\varepsilon$-dual fields, $\varepsilon^2 = (-1)^{(n/2)+1}$:

$$*^{(n)} d^{(n)} A = \pm \varepsilon d^{(n)} A.$$

This splits the (already unitary) representation on W^+ into two summands carrying $u_0(G_0)$.

Remark 4.5. As is easily verified by applying $*^{(n)}$, the "mirror image" operator $D_{2,(n+2)/2} = d^{(n)} \delta^{(n)}$ carries the same representation theoretic information as $D_{2,(n-2)/2}$.

b. *The Case* $s=l$

The "correct" way to view solutions of $D_{2l,k}\Phi = 0$ with $s=l$ is as vector potentials for a Maxwell-type system on $(k+1)$-forms. The case $s=-l$ is a "mirror image" under $*^{(n)}$.

LEMMA 4.6. *Suppose $n \geq 4$. If $s=l$ (so that n is necessarily even),*

$$D_{2l,k} = \delta^{(n)} \mathscr{D} d^{(n)},$$

where $\mathscr{D}$ is a differential operator of order $2(l-1)$, with leading term $(d^{(n)}\delta^{(n)})^{l-1}$, on $(k+1)$-forms, acting as

$$\begin{cases} 0 & \text{in the } 1\delta \text{ sector} \\ 2l \prod_{m=0}^{l-2} [\partial_t^2 + (C-l+2+2m)^2] & \text{in the } 0d \text{ sector} \\ 2l \left\{ \prod_{m=0}^{l-3} [\partial_t^2 + (B-l+3+2m)^2] \right\} & \\ \quad \times \begin{pmatrix} \partial_t^2 + 4(l-1)^2 & \delta d_t \\ d\partial_t & d\delta \end{pmatrix} & \text{in the } \begin{pmatrix} 0\delta \\ 1d \end{pmatrix} \text{ sector,} \end{cases} \tag{4.5}$$

where $C^2 = d\delta + s^2 = d\delta + l^2$ on closed k-forms, and

$$B^2 = \begin{cases} \delta d + (l-1)^2 & \text{on coclosed } k\text{-forms} \\ d\delta + (l-1)^2 & \text{on closed } (k+1)\text{-forms.} \end{cases}$$

Proof. It is straightforward to compute that if $\mathscr{D}$ is defined by (4.5), then $\delta^{(n)}\mathscr{D}d^{(n)} = D_{2l,k}$ (using the definition of $D_{2l,k}$ and (3.3), (3.4)). That $\mathscr{D}$ is a *differential* operator with leading term $(d^{(n)}\delta^{(n)})^{l-1}$ is established by an argument like that of Lemma 3.17. ∎

LEMMA 4.7. *For $n \geqslant 4$, $s = l$, and $\mathscr{D}$ as above, the system*

$$d^{(n)}F = 0, \qquad \delta^{(n)}\mathscr{D}F = 0 \tag{4.6}$$

on $(k+1)$-forms is conformally invariant in the sense that

$$d^{(n)}u_0(h)\, F = u_0(h)\, d^{(n)}F, \tag{4.7}$$

$$d^{(n)}F = 0 \Rightarrow \delta^{(n)}\mathscr{D}u_0(h)\, F = u_{2l}(h)\, \delta^{(n)}\mathscr{D}F$$

for all $h \in O(2, n)$.

Proof. (4.7) is a general differential geometric identity. If $d^{(n)}F = 0$, the vanishing of the cohomology $H^{k+1}(\bar{M}^n)$ implies that $F = d^{(n)}\Phi$ for some k-form Φ. By the covariance of $D_{2l,k}$,

$$\begin{aligned} \delta^{(n)}\mathscr{D}u_0(h)\, F &= \delta^{(n)}\mathscr{D}u_0(h)\, d^{(n)}\Phi \\ &= \delta^{(n)}\mathscr{D}d^{(n)}u_0(h)\, \Phi = u_{2l}(h)\, \delta^{(n)}\mathscr{D}d^{(n)}\Phi \\ &= u_{2l}(h)\, \delta^{(n)}\mathscr{D}F. \quad ∎ \end{aligned}$$

THEOREM 4.8. *Suppose $n \geqslant 4$, $s = l$, and $k \neq 0$. Then solutions of the system* (4.6) *on $\tilde{M}^n$ are automatically 2π-periodic in t, and even under the antipodal map in $\bar{M}^n$. The system* (4.6) *satisfies Huygens' principle in the following sense: if the support of the data $(F, \partial_t F, ..., \partial_t^{2l-1} F)(0, x)$ contains no point at a distance t_0 from $x_0 \in S^{n-1}$, $0 \leqslant t_0 \leqslant \pi$, then $F(t_0, x_0) = 0$.*

Proof. $d^{(n)} F = 0$ implies that

$$\begin{aligned} F_{1\delta} &= 0, \\ \partial_t F_{1d} - dF_{0\delta} &= 0. \end{aligned} \tag{4.8}$$

$\delta^{(n)} \mathscr{D} F = 0$ implies that

$$\begin{aligned} &\prod_{m=0}^{l-2} [\partial_t^2 + (C - l + 2 + 2m)^2] \, F_{0d} = 0, \\ &(\partial_t \mid \delta) \left\{ \prod_{m=0}^{l-3} [\partial_t^2 + (B - l + 3 + 2m)^2] \right\} \\ &\quad \times \left(\begin{array}{c|c} \partial_t^2 + 4(l-1)^2 & \delta \partial_t \\ \hline d\partial_t & d\delta \end{array} \right) \begin{pmatrix} F_{0\delta} \\ F_{1d} \end{pmatrix} = 0. \end{aligned} \tag{4.9}$$

The latter equation, together with (4.8), implies that

$$\prod_{m=0}^{l-1} [\partial_t^2 + (B - l + 1 + 2m)^2] \, F_{1d} = 0,$$

since

$$\begin{aligned} &(\partial_t \mid \delta) \left(\begin{array}{c|c} \partial_t^2 + 4(l-1)^2 & \delta \partial_t \\ \hline d\partial_t & d\delta \end{array} \right) \begin{pmatrix} [B^2 - (l-1)^2]^{-1} \, \delta \partial_t F_{1d} \\ F_{1d} \end{pmatrix} \\ &= [B^2 - (l-1)^2]^{-1} \, \delta (\partial_t^4 + 2(B^2 + (l-1)^2) \, \partial_t^2 \\ &\quad + (B^2 - (l-1)^2)^2) \, F_{1d}. \end{aligned} \tag{4.10}$$

Now the characteristic polynomials of the ODE determined by (4.9) and (4.10) have no double roots:

$$J - l + 1 = J - s + 1 \geqslant \frac{n}{2} - \frac{n-2}{2} + 1 = 2.$$

This shows periodicity, and (3.7) immediately gives evenness. Huygens' principle follows from the Lax–Phillips oddness/evenness argument of Theorem 2.2, once we note that the leading term of $d^{(n)}\delta^{(n)}\mathscr{D} + (\delta^{(n)}d^{(n)})^l$ is $\Box^l$, so that solutions of (4.6) have finite propagation speed. ∎

Of course, F above plays the role of the Maxwell field in this setting, and the solution A of $D_{2l,k}A = 0$ is the analogue of the vector potential. The case $k = 0$ has been eliminated because it has already been treated in Sec. 1; it is qualitatively different because there is no gauge ambiguity in the definition of A. Since the cohomology $H^{k+1}(\bar{\bar{M}}^n) = 0$, each solution F of (4.6) admits a vector potential. A pure gauge is again defined to be a $(k-1)$-form in $\mathscr{R}(d^{(n)})$. Lemma 3.23 provides the 5 $u_0(G)$-invariant subspaces $W^\pm$, $V^\pm$, M. Since $u_0(G)$ and $d^{(n)}$ commute, $\mathscr{G} = \mathscr{R}(d^{(n)})$ is also $u_0(G)$-invariant. Theorem 4.2 generalizes to

THEOREM 4.9. *Suppose $n \geqslant 4$ is even, $s = l$, and $k \neq 0$. Let $W^\pm$, $V^\pm$, M be the subspaces of the k-forms on $\bar{\bar{M}}^n$ (C^∞ or H^m) given in Lemma 3.23. The inner products*

$$\langle \Phi, \Psi \rangle_M = (\Phi, D_{2l,k}\Psi) \qquad \text{on } (M+\mathscr{G})/(W^+ + W^- + \mathscr{G}),$$
$$\langle \Phi, \Psi \rangle_{V^\pm} = (-1)^l (\Phi, D_{2l,k}\Psi) \qquad \text{on } (V^\pm + \mathscr{G})/(W^\pm + \mathscr{G}),$$
$$\langle \Phi, \Psi \rangle_{W^\pm} = \pm i(\Phi, D'_{2l,k}\Psi) \qquad \text{on } (W^\pm + \mathscr{G})/\mathscr{G},$$

where $D'_{2l,k} = [D_{2l,k}, t]$, are well defined, Hermitian, nondegenerate, and $u_0(G)$-invariant. $\langle\, ,\, \rangle_M$ is indefinite, and $\langle\, ,\, \rangle_{W^\pm}$ is indefinite for $l > 1$. $\langle\, ,\, \rangle_{V^\pm}$ is positive definite, and results, after completion, in a continuous unitary representation.

Proof. $\langle\, ,\, \rangle_M$ and $\langle\, ,\, \rangle_{V^\pm}$ are clearly well defined (gauge invariant). Since

$$\begin{aligned}[D_{2l,k}, t] &= [\delta^{(n)}\mathscr{D}d^{(n)}, t] \\ &= \delta^{(n)}\mathscr{D}[d^{(n)}, t] + \delta^{(n)}[\mathscr{D}, t]\, d^{(n)} + [\delta^{(n)}, t]\, \mathscr{D}d^{(n)} \\ &= \delta^{(n)}\mathscr{D}\varepsilon(dt) + \delta^{(n)}[\mathscr{D}, t]\, d^{(n)} - \iota(dt)\, \mathscr{D}d^{(n)},\end{aligned}$$

$\langle\, ,\, \rangle_{W^\pm}$ is also well-defined. Since $D_{2l,k}$ and $D'_{2l,k}$ are formally self-adjoint (t is formally self-adjoint on $\tilde{M}^n$ and $[D_{2l,k}, t]$ has periodic coefficients), all the inner products are Hermitian. By the covariance of $D_{2l,k}$, (3.13), and the argument of Theorem 2.7, $\langle\, ,\, \rangle_M$ and $\langle\, ,\, \rangle_{V^\pm}$ are $u_0(G)$-invariant. The proof of invariance for $\langle\, ,\, \rangle_{W^\pm}$ is just a slightly modified version of the argument of Theorem 2.9.

As in the Maxwell case, the pure gauges occupy the entire $0d$ sector, and the subspace $\{\partial_t A_{1d} = dA_{0\delta}\}$ of the $\binom{0\delta}{1d}$ sector. For an $\mathscr{E}_{1\delta,f,j}$ probe $A = e^{ift}A_{1\delta}$,

$$(A, D_{2l,k}A) = 4\pi l \prod_{m=0}^{l-1} [-f^2 + (J - l + 1 + 2m)^2],$$

while for an $\mathscr{E}_{\binom{0\delta}{1d},f,j}$ probe $A = e^{ift}(dt \wedge A_{0\delta} + A_{1d})$,

$$(A, D_{2l,k}A) = -4\pi l \left\{ \prod_{m=0}^{l-2} [-f^2 + (J - l + 2 + 2m)^2] \right\} \|\partial_t A_{1d} - dA_{0\delta}\|^2.$$

Since $\partial_t A_{1d} - dA_{0\delta} = 0$ is the pure gauge condition, $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{V^\pm}$ are nondegenerate, $\langle\ ,\ \rangle_M$ is indefinite, and $\langle\ ,\ \rangle_{V^\pm}$ is positive definite. The strength of the $\langle\ ,\ \rangle_{V^\pm}$ uniform structure is between those derived from $H^{l/2}$ and H^l; thus the representation on $(V^\pm + \mathscr{G})/(W^\pm + \mathscr{G})$ completes to a continuous unitary representation.

For an $\mathscr{E}_{1\delta,f,j}$ probe $A = e^{ift}A_{1\delta}$ with $f = J - l + 1 + 2p$, $0 \leqslant p \leqslant l - 1$,

$$D'_{2l,k}A = 8\pi i f l \prod_{\substack{0 \leqslant m \leqslant l-1 \\ m \neq p}} [-f^2 + (J - l + 1 + 2m)^2].$$

Thus the sign of $\langle\ ,\ \rangle_{W^+}$ in the 1δ sector alternates

$$\operatorname{sgn}\langle A, A\rangle_{W^+} = (-1)^{l-1+p}.$$

This shows that $\langle\ ,\ \rangle_{W^+}$ (and similarly, $\langle\ ,\ \rangle_{W^-}$) is indefinite for $l > 1$. ∎

Remark 4.10 (Gauge-fixing). If $l > 1$, solutions of $D_{2l,k}A = 0$ with $s = l$ cannot be reduced to the Coulomb gauge $A_0 = A_{1d} = 0$; if $A - d^{(n)}f$ is in the Coulomb gauge,

$$\partial_t f_{1\delta} = A_{0\delta},$$
$$\partial_t f_{1d} - df_{0\delta} = A_{0d},$$
$$df_{1\delta} = A_{1d}.$$

This can be arranged only if $dA_{0\delta} = \partial_t A_{1d}$; for $l > 1$, this is not necessarily true of solutions. What *can* be done, however, is to give a unique representative in each gauge equivalence class with components only on the lines

$$\pm f = J - l + 1 + 2m, \qquad 0 \leqslant m \leqslant l - 1$$

of the 1δ sector, and the lines

$$\pm f = J - l + 2 + 2m, \qquad 0 \leqslant m \leqslant l - 2$$

of the $1d$ sector. We shall call this the *canonical gauge.*

Remark 4.11 (A partial gauge). In [34] Paneitz showed, in the case $n = 4$, that even though the space L of Coulomb gauge Maxwell potentials is not conformally $(u_0(0(2, n)))$ invariant, the space

$$\tilde{L} = L + d^{(n)}\mathcal{N}(D_4)$$

is. (Recall that D_4 is the fourth-order operator on scalar fields intertwining u_0 and u_4.) Thus to compensate for the non-Coulomb gauge part of a conformal transform of $A \in L$, only the *special* pure gauges $d^{(n)}a$, $a \in \mathcal{N}(D_4)$, are needed. The analogous statement is true for even $n \geqslant 4$: if $k = (n-2)/2$ and L_n is the Coulomb gauge subspace of $\mathcal{N}(\delta^{(n)}d^{(n)})$, then

$$\tilde{L}_n = L_n + d^{(n)}\mathcal{N}(D_{4,(n-4)/2})$$

is $u_0(0(2, n))$-invariant. For by Table VII, if $A = e^{ift}A_{1\delta}$ is an $\mathscr{E}_{1\delta,f,j}$ probe, $f = \pm J$, and $A' = U_0(S)\,A$, then

$$A'_{0\delta,f\pm' 1,j} = \pm' \frac{1}{2i}\,\delta\varphi^0_{1\delta},$$

$$A'_{1d,f\pm' 1,j} = \frac{1}{2}\,(1 \mp' f)\,\varphi^0_{1\delta},$$

so that

$$\partial_t A'_{1d} = \frac{i}{2}\,(f \pm' 1)(1 \mp' f)\,\varphi^0_{1\delta} = \pm' \frac{1}{2i}\,(J^2 - 1)\,\varphi^0_{1\delta} = dA'_{0\delta}.$$

This implies that

$$e^{i(f\pm' 1)t}\{dt \wedge A'_{0\delta,f\pm' 1,j} + A'_{1d,f\pm' 1,j}\} = d^{(n)}a,$$

where the $(n-4)/2$-form a has nonzero components only on the lines $\pm f = J \pm' 1$ of the 1δ sector; in particular, $D_{4,(n-4)/2}\,a = 0$.

The analogue of this for the general $s = l$ case is as follows. If N is the space of canonical gauge solutions of $D_{2l,k}A = 0$, $s = l$, then

$$\tilde{N} = N + d^{(n)}\mathcal{N}(D_{2(l+1),k-1})$$

is $u_0(0(2, n))$-invariant. Indeed, for an "interior" $\mathscr{E}_{1\delta,f,j}$ probe A, i.e., one with $\pm f = J - l + 1 + 2m$, $1 \leqslant m \leqslant l - 2$, it is clear from Fig. 3.2 that

$U_0(S)\,A \in \tilde{N}$. For an "exterior" $\mathscr{E}_{1\delta,f,j}$ probe A, i.e., one with $\pm f = J \pm' (l-1)$, it follows from Table VII that $U_0(S)\,A \in \tilde{N}$. For example, for the case $f = J - (l-1)$, the key point is that

$$dA'_{0\delta,f-1,j} = -\frac{1}{2i}(J^2 - l^2)\,\varphi^0_{1\delta} = \partial_t A'_{1d,f-1,j}.$$

For A an "interior" $\mathscr{E}_{1d,f,j}$ probe $(\pm f = J - l + 2 + 2m,\ 1 \leqslant m \leqslant l-2)$, it is clear from Fig. 3.2 that $U_0(S)\,A \in \tilde{N}$. For A an "exterior" $\mathscr{E}_{1d,f,j}$ probe $(\pm f = J \pm' (l-2))$, the result follows from Tables VIII, IX. For example, in the case $f = J - (l-2)$, the key point is that

$$\begin{aligned} dA'_{0\delta,f-1,j+1} &= -\frac{1}{2i}(J+l-1)(J-l+1)\,\varphi^+_{1d} \\ &= \partial_t A_{1d,f-1,j+1}. \end{aligned}$$

The positive and negative frequency subspaces of $\tilde{L}_n$ and $\tilde{N}$ above are easily shown to be $u_0(G)$-invariant.

Remark 4.12 (Representations of the Lorentz group). For $l = n-1-2k$, the operators $L_{l,k}$ of Remark 3.29 (on k-forms in S^{n-1}), $k \neq 0$, act as

$$\begin{cases} l \displaystyle\prod_{m=0}^{l-1} \left(B - \frac{l-1}{2} + m\right) & \text{on } \mathscr{R}(\delta) \\ 0 & \text{on } \mathscr{R}(d), \end{cases}$$

i.e., as $\delta[\prod_{m=1}^{l-2} (B - ((l-1)/2) + m)]\,d$. (The case $-l = n-1-2k$ is a "mirror image" under the Hodge $*$.) For $l = 2$, $k = (n-3)/2$, $L_{l,k}$ is the (Riemannian) Maxwell operator δd on vector potentials. In any case, $\mathscr{R}(d)$ can be considered as a space of pure gauges. If l is even (so that n must be odd), the covariance relation (3.35) becomes

$$L_{l,k}u_0(h) = u_l(h)\,L_{l,k}, \qquad h \in \mathrm{O}(1, n).$$

The infinite-dimensional representation space is $C^\infty(\Lambda^k)/\mathscr{N}(L_{l,k})$, which admits the $u_0(\mathrm{O}(1,n))$-invariant inner product $\langle \varphi, \psi \rangle_{l,k} = (\varphi, L_{l,k}\psi)_{L^2}$. Note that $\mathscr{N}(L_{l,k}) = \mathscr{R}(d) \oplus E_{\delta,1} \oplus \cdots \oplus E_{\delta,(l+1-n)/2}$, where $E_{\delta,j} = E_{\delta,j,k}$.

5. Remarks

a. *Complete Decomposition*

The principal weakness of the above results compared to, say, those of [40, 31, 32] in special cases, is that we have no proof of the irreducibility of the 5 or 6 composition factors of the representations decomposed here. The tool needed to show that the decomposition is complete, or to find the complete composition lattice, would seem to be the so-called *Gegenbauer forms*. In the scalar field case, a $K = SO(2) \times O(n)$-type is given by a frequency f together with an $O(n)$-type. Each $SO(n)$-type can be represented by its unique (up to a scalar factor) $SO(n-1)$-invariant member, or *Gegenbauer polynomial.* The action of K, together with the action of $\mathfrak{g}$ on the Gegenbauer polynomials, give complete information on the composition series (see [31] for $n = 4$). The same approach should apply in the case of forms, and in arbitrary dimension.

An alternate approach, suggested by [28], is to realize certain forms on $\bar{M}^n$ as boundary values of holomorphic forms in a complex domain. In the presence of a nondegenerate invariant inner product, some reducibility questions become elementary questions in meromorphic function theory.

b. *Dirac Operators*

All the intertwining differential operators treated in this paper are of *even* order; they are generalizations of the wave and Maxwell operators, and of hybrids (the $D_{2,k}$) of these two. The full physical picture includes odd-order operators based on the wave, Maxwell, and *Dirac* operators, and the bundle of spinors, a "tensor product square root" of the bundle of forms. Some results on the decomposition of representations of the conformal group on higher spin bundles are already in place [33, 5]. Eventually, one would like to have complete decompositions, in the sense of Section 5.a, of all such representations.

c. *Particles*

The explicit correlation of the irreducible unitary representations with physical particles, where applicable, should be made. (It should be possible to tap into known information by restricting to the Lorentz group.) One consequence of working on *compactified* Minkowski space is that all our field operators (intertwining differential operators) have discrete spectra. In the case of the wave equation, the eigenvalues of D_2 correspond to the Klein–Gordon equations

$$D_2 \Phi + m^2 \Phi = 0,$$

which have solutions; that is, D_2 can be thought of *mass-squared operator.*

The corresponding V^+/W^+ is the corresponding positive mass-squared, positive frequency representation of G, and massive particles are obtained by decomposing this representation under a smaller (Poincaré-like) group (see [17]). In this sense, all the $D_{2l,k}$ are prospective mass operators, and their eigenvalues may have content for the prediction of physical particle mass ratios.

d. *General Conformal Covariants*

The idea behind the intertwining differential operators $D_{2l,k}$ of this paper comes from work of the author [4, 6] on "general" conformally covariant differential operators (those covariant under conformal deformation of arbitrary pseudo-Riemannian manifolds). This is also the impetus behind [43]. But, intriguingly, the solution of the general classification problem for conformally covariant operators (including *conformal tensors*, which are just zeroth-order conformally covariant operators) may be entirely bound up in these special cases. The "method of Cartan" [51, Sect. IV] reduces the classification problem in principle to questions about representations of $O(p, q)$ and its parabolic subgroups. (See also [47, Sect. I].) The prospect of solving this classification problem (which is motivated by physics, and, more recently, by the theory of CR manifolds [45, 47]) is part of the motivation for the present work.

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