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A Contribution to the Theory of Finite Supersolvable Groups

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In this paper we begin to develop a theory for supersolvable groups which closely parallels the well known theory of nilpotent groups. The results will hinge upon the following weakened concepts of normality and centrality.

Let Σ_H denote a Sylow system of the solvable subgroup H of the group G. The *weak system normalizer* (system quasinormalizer) of Σ_H in G, denoted $N_G^*(\Sigma_H)$, is the subgroup of G defined by

$$N_G^*(\Sigma_H) = \langle x \colon \langle x \rangle A = A \langle x \rangle \text{ for all } A \text{ in } \Sigma_H \rangle.$$

The *weak normalizer* of H in G, denoted $N_G^*(H)$, is defined by

$$N_G^*(H) = HN_G^*(\Sigma_H).$$

(We show that this subgroup of G is independent of the choice of Σ_H). Whenever $G = N_G^*(H)$ $[G = N_G^*(\Sigma_H)]$, we say that H $[\Sigma_H]$ is *weakly normal* in G. The *weak centralizer* (quasi-centralizer) of H in G, denoted $C_G^*(H)$, is defined by

$$C_G^*(H) = \bigcap \{N_G(K) \colon K \leqslant H\}.$$

Should $G = C_G^*(H)$, we will say that H is *weakly central* in G. The *weak center* of G, denoted $Z^*(G)$, is the product of all weakly central subgroups of G.

In terms of these definitions Theorem 2.5 of [7] becomes

THEOREM. *The group G is supersolvable if and only if G has a weakly normal Sylow system.*

While the results of [6] together with those of [8] give

THEOREM. *The group G is supersolvable if and only if each maximal subgroup of G is weakly normal in G.*

In this form these two theorems are clear analogs of well known characterizations of nilpotent groups and represent the type of results found in this work.

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Using the weak center of G one can define in a natural way an ascending weak central series for a group G. In Section 2 we investigate this series and show that it terminates in G precisely when G is supersolvable. In Section 3 we consider the intersection of the weak centralizers of the chief factors in a solvable group G, this intersection provides something of a supersolvable analog to the subgroup of Fitting.

We also include the following generalization of a theorem by Burnside.

THEOREM. *Let P be a Sylow p-subgroup of the group G with $P \leqslant Z^*(N_G(P))$. If p is odd, assume in addition that G is p-solvable. Then G is p-supersolvable.*

The notation used will follow that found in [4]. For Sylow systems we use the conventions of [7], thus a Sylow system will consist of a complete set of permuting Hall subgroups. Throughout this work we assume our groups are finite.

1. WEAK NORMALITY AND WEAK CENTRALITY

Let H be a solvable subgroup of the group G and let Σ_H be a Sylow system of H. In that $N_G^*(\Sigma_H)$ is generated by subgroups which permute with H, $N_G^*(\Sigma_H)$ itself must permute with H. Hence $N_G^*(H) = HN_G^*(\Sigma_H)$ is a subgroup of G. In addition, should $\mathfrak{S}_H$ be some other Sylow system of H, then $\mathfrak{S}_H$ will be conjugate to Σ_H in H. If $\mathfrak{S}_H = (\Sigma_H)^h$, with $h \in H$, observe that $(N_G^*(\Sigma_H))^h = N_G^*(\mathfrak{S}_H)$. With this observation it is readily seen that $N_G^*(H) = HN_G^*(\Sigma_H) = HN_G^*(\mathfrak{S}_H)$ and that $N_G^*(H)$ is in fact independent of the choice of Σ_H.

Since the weak normalizer of a subgroup is only defined when that subgroup is solvable, it is understood that only solvable subgroups of a group may be weakly normal in the group.

We collect below a few simple statements concerning weak normalizers and weak centralizers. The proofs of these are trivial with the possible exception of (e). The proof of (e) depends on little more than the Dedekind identity, however it must be used judiciously.

LEMMA 1.1. *Let H be a solvable subgroup of the group G and let Σ_H be a Sylow system of H. Then*

(a) $C_G(H) \leqslant C_G^*(H) \leqslant N_G(H) \leqslant N_G^*(H)$,

(b) *whenever* $H \leqslant K \leqslant G$, $C_K^*(H) = C_G^*(H) \cap K$ *and* $N_K^*(H) \leqslant N_G^*(H)$,

(c) *if* $\alpha \in \mathrm{Aut}(G)$, $(C_G^*(H))^\alpha = C_G^*(H^\alpha)$ *and* $(N_G^*(H))^\alpha = N_G^*(H^\alpha)$,

(d) *$\langle x \rangle \leqslant G$, permutes with all the elements of Σ_H if and only if $\langle x \rangle$ permutes with the Sylow p-supgroups in Σ_H, and*

(e) *$\langle x \rangle \leqslant G$, permutes with all the elements of Σ_H if and only if $\langle x \rangle$ permutes with the Sylow p-complements in Σ_H.*

For the purposes of this work a Sylow system consists of a complete set of permuting Hall subgroups, however (d) and (e) of the above lemma allow one to use either the Sylow basis or complement basis of a Sylow system when calculating weak normalizers. On the other hand, the weak system normalizer of a subgroup may not be the intersection of the weak normalizers of the individual subgroups in the system. It should also be observed that for $H \leqslant K \leqslant G$, $N_K^*(H)$ need not councide with the subgroup $K \cap N_G^*(H)$. Indeed it is possible that a subgroup H is weakly normal in G, yet not weakly normal in some subgroup of G which contains H.

We now direct our attention to the weak normalizers of Hall subgroups.

LEMMA 1.2. *Let H be a solvable Hall subgroup of the group G. If Σ_H denotes a Sylow system of H and if $x \in G$ with $\langle x\rangle A = A\langle x\rangle$ for all A in Σ_H, then $\langle x^n\rangle A = A\langle x^n\rangle$ for all integers n and all A in Σ_H. In particular $N_G^*(\Sigma_H)$ is generated by elements of prime power order whose cyclic subgroups permute with the elements of Σ_H.*

Proof. Let p be a prime which divides the order of H and let G_p be the Sylow p-subgroup of H (and hence of G) which lies in Σ_H. By [5] the group $T = \langle x\rangle G_p$ is solvable. For the prime r, let T^r be a Sylow r-complement of T containing G_p when $r \neq p$, and T^r be the Sylow r-complement of $\langle x\rangle$ when $r = p$. If Σ_T is the Sylow system of T generated by the r-complements T^r, then by Lemma 1.1 $\langle x\rangle$ permutes with each element of Σ_T. Applying Lemma 1.5 of [7] to the group T, we may conclude that for any integer n, $\langle x^n\rangle$ permutes with each element of Σ_T. As G_p is the Sylow p-subgroup in Σ_T, $\langle x^n\rangle$ permutes with G_p for any integer n. If we now apply Lemma 1.1 to Σ_H, we may conclude that $\langle x^n\rangle$ permutes with each element of Σ_H for any integer n.

We recall that a group G is said to satisfy C_π, where π is a set of primes, whenever G has conjugate Hall π-subgroups. It is of course well known that a π-solvable (even π-separable) group must satisfy C_π and the Sylow theorems assure us that whenever π consists of a single prime that any group satisfies $C_\pi(C_p)$.

THEOREM 1.3. *Let G be a group in which every subgroup satisfies C_π. Let H be a solvable Hall π-subgroup of G and N a normal subgroup of G. Then $N_{G/N}^*(HN/N) = N_G^*(H)N/N$.*

Proof. We proceed by induction on the order of G. Hence we may assume that N is a minimal normal subgroup of G. Let $\bar{A}$ denote the image of the set A under the natural homomorphism of G onto G/N. Clearly $\overline{N_G^*(H)} \leqslant N_{\bar{G}}^*(\bar{H})$. If G is the minimal counterexample we may assume that for some $\bar{x}$ of $\bar{G}$, $\bar{x} \in N_{\bar{G}}^*(\bar{H})$ and $\bar{x} \notin \overline{N_G^*(H)}$. Furthermore in view of Lemma 1.2, we may assume that

(a) $|\bar{x}| = q^t$ for some prime q and integer t, and either

(b) $\bar{x} \in \bar{H}$, or

(c) $\bar{x}$ permutes with the elements of some Sylow system $\Sigma_{\bar{H}}$ of $\bar{H}$.

The proof is now broken into six parts.

(1) $(q, |H|) = 1$ *and* (c) *holds.*

$\bar{H}$ is a Hall subgroup of $\bar{G}$. Should $q \mid |H|$, then either case (b) or (c) would imply that $\bar{x} \in \bar{H}$. Since $\bar{H} \leqslant \overline{N_G^*(H)}$, this would contradict our choice of $\bar{x}$. Thus $(q, |H|) = 1$ and condition (c) must hold.

(2) *$\bar{G}$ is solvable and $\bar{G} = \langle \bar{x} \rangle \bar{H}$.*

Let $N \leqslant T \leqslant G$, with $\bar{T} = \langle \bar{x} \rangle \bar{H}$, observe that $\bar{x} \in N_{\bar{T}}^*(\bar{H})$. If $T \neq G$, then by induction, $\bar{x} \in \overline{N_T^*(H)} \leqslant \overline{N_G^*(H)}$, contrary to its choice. Hence $\bar{T} = \bar{G} = \langle \bar{x} \rangle \bar{H}$. Furthermore, $\langle \bar{x} \rangle$ is a Sylow q-subgroup of $\bar{G}$, and $\langle \bar{x} \rangle$ together with $\Sigma_{\bar{H}}$ will generate a Sylow system of $\bar{G}$. Hence $\bar{G}$ is solvable.

(3) *Either $N \leqslant H$ or $(|N|, |H|) = 1$.*

Set $H_1 = H \cap N$, $N_1 = N \cap N_G(H_1)$ and $S = N_G(H_1)$. Since N satisfies C_π, $G = NS$ and $S \geqslant H$. Let φ be the natural homomorphism of S onto S/N_1. Since $G = NS$, the map ψ with $(g^\varphi)^\psi = \bar{g}$ is an isomorphism of S/N_1 onto $\bar{G}$. We may also assume, by proper choice of representative, that x lies in S.

If $S \neq G$, then by induction $(N_S^*(H))^\varphi = N_{S\varphi}^*(H^\varphi)$. Since $\bar{x} \in N_{\bar{G}}^*(\bar{H})$, $x^\varphi \in N_{S\varphi}^*(H^\varphi)$. It now follows that there exists a $y \in N_S^*(H)$ with $y^\varphi = x^\varphi$. As $N_S^*(H) \leqslant N_G^*(H)$, $\bar{y} = (y^\varphi)^\psi = (x^\varphi)^\psi = \bar{x}$ lies in $\overline{N_S^*(H)}$. This is contrary to the choice of $\bar{x}$, hence $S = G$.

Because N is a minimal normal subgroup of G, either $H_1 = N$ and $N \leqslant H$, or $H_1 = \{1\}$ so that $(|N|, |H|) = 1$.

(4) $(|N|, |H|) = 1$ *and G is π-solvable.*

If $N \leqslant H$, we may conclude from the solvability of H that N is an elementary abelian subgroup of G. As $\bar{G}$ is solvable, so also must G be solvable. Furthermore, since $\langle \bar{x} \rangle$ is a Sylow q-subgroup of $\bar{G}$ with $(q, |H|) = 1$, G itself must have a cyclic Sylow q-subgroup. If Σ is a Sylow system of G containing H and if $\langle y \rangle$ is the Sylow q-subgroup of G in Σ, then it follows that $y \in N_G^*(H)$. But $G = \langle y \rangle H = N_G^*(H)$, and $\bar{G} = \overline{N_G^*(H)} = N_{\bar{G}}^*(\bar{H})$. This is contrary to the choice of G and we may conclude that $N \nleqslant H$, so that $(|N|, |H|) = 1$. Moreover since $\bar{G}$ is solvable, G must be π-solvable.

(5) *G is solvable and N is a normal q-subgroup.*

If q does not divide $|N|$, then N is a normal Hall subgroup of G and hence has a complement T isomorphic with $\bar{G}$. As G is π-solvable we may assume $T \geqslant H$.

Since T is solvable, T has a cyclic Sylow q-subgroup $\langle y \rangle$ which permutes with some Sylow system of H. Hence $T = \langle y \rangle H = N_T^*(H) \leqslant N_G^*(H)$. But then $\bar{G} = \bar{T} = \overline{N_G^*(H)} = N_{\bar{G}}^*(\bar{H})$, contrary to the choice of G. Hence q must divide $|N|$.

Let Q be a Sylow q-subgroup of N, so that $G = NN_G(Q)$. As G is π-solvable, $N_G(Q)$ contains some conjugate of H. By properly choosing Q we may assume that $N_G(Q) \geqslant H$. Set $S = N_G(Q)$ and $N_1 = N \cap N_G(Q)$. By using the same argument as (3), it can be concluded that $S = G$. Hence $N = Q$ and G is solvable.

(6) *G is not a counterexample.*

For each prime p dividing $|H|$, let G_p be a Sylow p-subgroup of G in H with $\bar{G}_p \in \Sigma_{\bar{H}}$. Let G_q be the Sylow q-subgroup of G with $\bar{G}_q = \langle \bar{x} \rangle$. These Sylow subgroups of G form a Sylow basis for a Sylow system Σ of G with Σ reducing into H. By Theorem 2.2 of [7], $N_{\bar{G}}^*(\bar{\Sigma}) = \overline{N_G^*(\Sigma)}$. Since $\bar{x}$ lies in $N_{\bar{G}}^*(\bar{\Sigma})$, $\bar{x}$ also lies in $\overline{N_G^*(\Sigma)}$ and, as $N_G^*(\Sigma) \leqslant N_G^*(\Sigma \cap H) \leqslant N_G^*(H)$, it follows that $\bar{x} \in \overline{N_G^*(H)}$. Thus G is no counterexample and the theorem holds.

We now give three characterizations of supersolvable groups using these concepts. As stated in the introduction, the results of [6] combine with those of [8] to give

THEOREM 1.4. *The group G is supersolvable if and only if each maximal subgroup of G is weakly normal in G.*

DEFINITION. The subgroup H of the group G is called *weakly subnormal* in G whenever there exists a chain of subgroups,

$$H = H_0 \leqslant H_1 \leqslant H_2 \leqslant \cdots \leqslant H_{n-1} \leqslant H_n = G,$$

with H_{i-1} weakly normal in H_i for $i = 1, 2, 3, \ldots, n$.

An equivalent form of Theorem 1.4 may be stated in terms of this weak subnormality, namely

COROLLARY 1.5. *The group G is supersolvable if and only if every proper subgroup of G is weakly subnormal in G.*

We also have

THEOREM 1.6. *A group is supersolvable if and only if it has a complete set of weakly normal Sylow p-complements.*

Proof. If G is supersolvable, Theorem 2.5 of [7] implies that every Sylow p-complement of G is weakly normal in G. Thus we need only show that if the group G has a complete set of weakly normal Sylow p-complements, G is supersolvable. A group G satisfying this condition is of course solvable. Let Σ denote a Sylow system of the group G generated by weakly normal Sylow p-complements of G.

For the fixed prime p, let G^p denote the Sylow p-complement in Σ and G_p the Sylow p-subgroup of G in Σ. Appealing to Lemma 1.2, we see that $N_G^*(G^p)$ is generated by G^p and p-elements of G whose cyclic subgroups permute with the elements of $\Sigma \cap G^p$. Set $T_p = \langle y : y \in G_p$ and $\langle y \rangle A = A \langle y \rangle$ for all $A \in \Sigma \cap G^p \rangle$. If z is any p-element of G whose cyclic subgroup permutes with the elements of $\Sigma \cap G^p$, then some conjugate of z by an element of G^p lies in G_p. Moreover, this conjugate of z will generate a cyclic subgroup which permutes with G^p, so that by Lemma 1.2 of [7], this conjugate of z lies in T_p. We now conclude that $N_G^*(G^p) = T_p G^p = G$, and $T_p = G_p$. Hence we have that $G_p \leqslant N_G^*(\Sigma)$.

Since $N_G^*(\Sigma)$ contains a Sylow p-subgroup of G for each prime p, $G = N_G^*(\Sigma)$. Theorem 2.5 of [7] now gives the supersolvability of G.

It should be noted that in supersolvable groups the Sylow p-subgroups are weakly normal, but that the converse of this need not be true.

Example. Set $B = \langle a_1, a_2, a_3 \rangle$ with the relations $a_i^7 = [a_i, a_j] = a_1 a_2 a_3 = 1$ $(i, j = 1, 2, 3)$. B is the elementary abelian group of order 7^2. Each element of the symmetric group S_3 induces an automorphism on B by $a_i^s = a_{is}$. Let $G = BS_3$ be the semidirect product of B and S_3. The groups B, $\langle (1, 2, 3) \rangle$ and $\langle (1, 2) \rangle$ form a Sylow basis of weakly normal Sylow subgroups of G, yet G is not supersolvable.

2. The Weak Central Series

We recall that a subgroup H of the group G is called weakly central in G whenever H and all of its subgroups are normal in G. As any group is the join of its own cyclic subgroups, it is apparent that a subgroup H will be weakly central in G precisely when the cyclic subgroups of H are normal in G. Moreover, any cyclic normal subgroup of G must be weakly central in G. Thus the weak center of a group G, being the join of all weakly central subgroups of G, coincides with join of all cyclic normal subgroups of G. It is this latter characterization of the weak center which we find most convenient to use in this work. We first consider those groups which coincide with their weak center (i.e., are generated by cyclic normal subgroups).

Definition. A group will be called *semiabelian* whenever it is expressible as the product of cyclic normal subgroups. Any element in a semiabelian groups which generates a cyclic normal subgroup will be called a *normal generator* of the group.

From the definition of semiabelian group it is clear that every homomorphism image of a semiabelian group is again semiabelian and that the direct sum of two semiabelian groups is also semiabelian. However, as seen in the next two examples, subgroups of semiabelian groups need not be semiabelian nor do the semiabelian groups form a formation.

EXAMPLE. Let p denote an odd prime and set $G = \langle x, y, w \rangle$ with the relations

(a) $x^p = y^p = w^{p^2} = 1$,
(b) $[x, y] = w^p$, and
(c) $[x, w] = [y, w] = 1$.

G is semiabelian with normal generators xw, yw, and w. The subgroup $\langle x, y \rangle$ of G is a non abelian group of order p^3 and of exponent p. Hence this subgroup can not be semiabelian.

EXAMPLE. Let p denote an odd prime and set $H = \langle x, y, z, w \rangle$ with the relations

(a) $x^p = y^p = z^p = w^{p^2} = 1$,
(b) $[x, y] = z$, and
(c) $[x, w] = [y, w] = [z, w] = [x, z] = [y, z] = 1$.

The subgroups $N = \langle zw^{-p} \rangle$ and $K = \langle z \rangle$ are normal subgroups of H with H/K abelian and H/N isomorphic to the above group G. Hence both H/K and H/N are semiabelian, but $H/N \cap K$ is isomorphic to H and is not semiabelian.

THEOREM 2.1. *A semiabelian group is nilpotent of class at most two. Each Sylow subgroup of a semiabelian group is semiabelian.*

Proof. Let G denote a semiabelian group and x any of its normal generators. For each prime p, let x_p be a generator of the Sylow p-subgroup of $\langle x \rangle$. x_p is also a normal generator of G. Set $S_p = \langle x_p : x \text{ is a normal generator of } G \rangle$. For each prime p, S_p is a normal semiabelian p-subgroup of G. As G is generated by the normal subgroups S_p, G is nilpotent with Sylow p-subgroup S_p.

For each normal generator x of G, $G/C_G(x)$ is an abelian group and thus $C_G(x)$ contains G'. Since $Z(G) = \bigcap \{C_G(x) : x \text{ is a normal generator of } G\}$, it follows that $Z(G)$ also contains G' so that G has class of at most two.

To gain a little more insight into the structure of semiabelian groups we include the following proposition which may be proven by a simple induction on n.

PROPOSITION 2.2. *For the semiabelian p-group G set $L_n(G) = \langle x^{p^n} : x \text{ is a normal generator of } G \rangle$. Then $\mho_n(G) = L_n(G) = \Phi(\mho_{n-1}(G))$ for all integers $n \geq 1$. In particular $\mho_n(G)$ is semiabelian.*

We now divert our attention back to more general groups. As the weak center of a group is the join of all cyclic normal subgroups we have

THEOREM 2.3. *The weak center of a group is a characteristic semiabelian subgroup.*

THEOREM 2.4. *Every nontrivial normal subgroup of a supersolvable group meets the weak center of the group nontrivially.*

DEFINITION. (a) A series $N_0 \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_k$ of normal subgroups N_i of the group G will be called a *weakly central* series in G whenever $N_{i+1}/N_i \leqslant Z^*(G/N_i)$ for $i = 0, 1, 2,..., k - 1$.

(b) The series

$$\{1\} = Z_0^*(G) \leqslant Z_1^*(G) \leqslant Z_2^*(G) \leqslant \cdots \leqslant Z_i^*(G) \leqslant Z_{i+1}^*(G) \leqslant \cdots$$

with $Z_1^*(G) = Z^*(G)$ and $Z_{i+1}^*(G)/Z_i^*(G) = Z^*(G/Z_i^*(G))$ is called the *ascending weak central* series of G. We set $Z_\infty^*(G) = \bigcup_i Z_i^*(G)$.

LEMMA 2.5. *If* $\{1\} = N_0 \leqslant N_1 \leqslant N_2 \leqslant \cdots$ *is a weakly central series of* G, *then* $N_i = Z_i^*(G)$.

Proof. For $i = 0$ the result is trivial. We proceed by induction and assume $N_k \leqslant Z_k^*(G)$. Let $A/N_k = Z^*(G/N_k)$ so that $A \geqslant N_{k+1}$.

If $x \in A$ with $\langle x\rangle N_k/N_k \trianglelefteq G/N_k$, then $\langle x\rangle N_k \trianglelefteq G$ and hence $\langle x\rangle Z_k^*(G) = \langle x\rangle N_k Z_k^*(G)$ is a normal subgroup of G. It follows that $x \in Z_{k+1}^*(G)$ and, since A is generated by all such elements x, that $A \leqslant Z_{k+1}^*(G)$. In particular $N_{k+1} \leqslant Z_{k+1}^*(G)$ and the lemma is valid.

In [1], Baer calls a normal subgroup K of the group G *supersolvably immersed* in G if to every homomorphism σ of G with $K^\sigma \neq \{1\}$ there exists a cyclic normal subgroup $A \neq \{1\}$ of G^σ with $A \leqslant K^\sigma$. The product of all supersolvably immersed normal subgroups of a group G is a supersolvably immersed characteristic subgroup of G. In the terminology of formations (see Huppert [3]) this maximal supersolvably immersed subgroup of G is called the *supersolvable hypercenter* of G.

It is clear from the definition of supersolvable immersion that the supersolvable hypercenter of G is joined to the identity subgroup by a weakly central series in G. Thus Lemma 2.5 implies that the supersolvable hypercenter of G lies in $Z_\infty^*(G)$. Conversely, each term of the ascending weak central series must be supersolvably immersed in G. Hence we have

THEOREM 2.6. $Z_\infty^*(G)$ *is the supersolvable hypercenter of the group* G.

COROLLARY 2.7. G *is supersolvable if and only if* $G = Z_\infty^*(G)$.

We now establish the analog of the Burnside theorem as mentioned in the introduction. As might be expected the argument when $p = 2$ differs considerably from the odd case, for this reason the proof will be given in two propositions.

PROPOSITION 2.8. *Let G be a p-solvable group and P a Sylow p-subgroup of G. If $P \leqslant Z^*(N_G(P))$, then G is p-supersolvable.*

Proof. We first observe that by the argument used in Lemma 2.1 one can see that the Sylow subgroups of the weak center of a group are also generated by cyclic normal subgroups. Thus it is easily seen that the hypothesis of the proposition goes to quotient groups and subgroups which contain P. We now proceed by induction on the order of G.

Let H be a minimal normal subgroup of G, by induction we may assume that H is a unique minimal normal p-subgroup of G. Furthermore, we may assume that H does not lie in the Frattini subgroup of G so that H is complemented by a maximal subgroup M of G and H is thus self centralizing. As $P \leqslant Z^*(N_G(P))$, P is semiabelian and hence has class at most two. Since H is a self centralizing subgroup of P we must have that $H \geqslant Z(P)$. It follows that $\exp(Z(P)) = p$ and since the class of P is at most two that $\exp(P)$ is either p or p^2.

If $\exp(P) = p$, then since P is semiabelian P is in fact abelian. In this case $H = P$ and $P \trianglelefteq G$. In particular, $P \leqslant Z^*(G)$ and the p-supersolvability of G follows from Theorem 2.6. Hence we need only consider the case $\exp(P) = p^2$.

Since $\exp(Z(P)) = p$ and G is of class two, $P/Z(P)$ is elementary abelian. As $G = HM$, $P = H(P \cap M)$. Set $S = P \cap M$. Since $H \geqslant Z(P)$, S is elementary abelian. If $x \in P$, then $x = yh$ where $y \in S$ and $h \in H$. Since H is an abelian normal subgroup we have that $x^p = (yh)^p = y^p h^p [h, y]^{p(p-1)/2} = [h, y]^{p(p-1)/2}$.

If p is odd, $x^p = 1$ since $[h, y] \in H$ and $\exp(H) = p$. This is contrary to $\exp(P) = p^2$ and the proposition holds for odd primes. We may therefore assume for the balance of the argument that $p = 2$.

As $P \leqslant Z^*(N_G(P))$, Theorem 2.6 implies that $N_G(P)$ is 2-supersolvable and hence $N_G(P)$ is in fact 2-nilpotent. Since H is self centralizing, we must have that $P = N_G(P)$. Since P is semiabelian, every homomorphic image of P is semiabelian. Thus D_4, the dihedral group of order 8, is not a homomorphic image of P. By a result of Wielandt and Yoshida [2, p. 26, Theorem I.6.8], $P \cap G' = P \cap (N_G(P))' = P'$. Should $G' = \{1\}$, the result is trivial, thus we have that $G' \geqslant H$. It now follows that $P' \geqslant H \geqslant Z(P)$, but since P has class at most two it easily follows that $H = Z(P) = P$. Again we have that $P \leqslant Z^*(G)$ and Theorem 2.6 implies that G is 2-supersolvable.

PROPOSITION 2.9. *Let P denote a Sylow 2-subgroup of the group G. If $P \leqslant Z^*(N_G(P))$, then G is 2-nilpotent.*

Proof. As we observed in the proof of Proposition 2.8 the hypothesis goes to quotient groups and subgroups containing P. By Theorem 2.6, $N_G(P)$ is 2-supersolvable and thus 2-nilpotent. We may therefore conclude that $P \cap (N_G(P))' = P'$. Applying the result of Wielandt and Yoshida as before, we may conclude that $P \cap G' = P'$.

If PG' is a proper subgroup of G, then induction gives the 2-nilpotency of PG' and hence the solvability of G. In this case the result follows from Proposition 2.8. Hence we may assume that $G = PG'$.

P' is a Sylow 2-subgroup of G'. If $P' \trianglelefteq G$, then applying induction to G/P' we may conclude that G is solvable and again the result will follow from Proposition 2.8. Thus $N_G(P')$ is a proper subgroup of G and by induction is 2-nilpotent. From this it follows that $N_{G'}(P')$ is also 2-nilpotent. Since P is semi-abelian, P' is abelian. Hence in G', P' lies in the center of its normalizer and from this it follows that G' is itself 2-nilpotent. Thus G is solvable and Proposition 2.8 may be applied again for the result.

We now have

THEOREM 2.10. *Let P be a Sylow p-subgroup of the group G with $P \leqslant Z^*(N_G(P))$. If p is odd, assume in additin that G is p-solvable. Then G is p-supersolvable.*

3. A Characteristic Supersolvable Subgroup

In this final section we introduce a supersolvable analog to the subgroup of Fitting. The Fitting subgroup is of course the product of all nilpotent normal subgroups of the group, but may also be identified as being the intersection of all centralizers of the chief factors of the group. It is well known that the product of normal supersolvable subgroups of a group need not be supersolvable, so it is the second characterization of the Fitting subgroup which we generalize.

DEFINITION. Let K be a normal subgroup of the group G with $G \geqslant H \geqslant K$. The *weak centralizer* of H/K in G, $C_G^*(H/K)$, is defined by $C_G^*(H/K)/K = C_{G/K}^*(H/K)$. If $G = C_G^*(H/K)$, we say that H/K is *weakly central* in G.

A minimal normal subgroup of G is weakly central in G precisely when it has prime order and we have

THEOREM 3.1. *G is supersolvable if and only if each chief factor of G is weakly central in G.*

DEFINITION. Let $\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G$ be a chief series of the group G. The *supersolvable Fitting* subgroup of G, $F^*(G)$, is defined by $F^*(G) = \bigcap \{C_G^*(H_i/H_{i-1}): i = 1, 2, 3, \ldots, n\}$.

It is routine to check that G-isomorphic chief factors of a group G have the same weak centralizer. Hence $F^*(G)$ is a characteristic subgroup of G and is independent of the chief series used in its definition.

THEOREM 3.2. *Let N be a supersolvable normal subgroup of the group G, then $NF^*(G)$ is a normal supersolvable subgroup of G.*

Proof. Let $\{1\} = H_0 \leqslant H_1 \leqslant H_2 \leqslant \cdots \leqslant H_r = NF^*(G)$ be the portion of some chief series of G which lies below $NF^*(G)$. We refine this to the N-composition series

$$(\#) \quad \{1\} = R_0 \leqslant R_1 \leqslant R_2 \leqslant \cdots \leqslant R_t = NF^*(G)$$

of $NF^*(G)$. For each i; $i = 1, 2, 3,..., t$; there is a j such that $H_j \geqslant R_i \geqslant R_{i-1} \geqslant H_{j-1}$. Since $C_G^*(H_j/H_{j-1})$ normalizes every subgroup between H_j and H_{j-1}, so also does $F^*(G)$. Hence $(\#)$ is an $NF^*(G)$-series and moreover if $R_i \geqslant T \geqslant R_{i-1}$, then T is normalized by $F^*(G)$.

Each factor R_i/R_{i-1} of $(\#)$ is either covered or avoided by N. If N covers the factor, it is N-isomorphic to a chief factor of N and since N is supersolvable it follows that the factor has prime order. If N avoids the factor, then the factor is centralized by N. If N centralize the factor R_i/R_{i-1}, then N normalizess every subgroup between R_i and R_{i-1}. Since $F^*(G)$ also normalizes the subgroups between R_i and R_{i-1} we may conclude that R_i/R_{i-1} has prime order. It now follows that the series $(\#)$ is a chief series of $NF^*(G)$ in which all factors have prime order and the result follows.

COROLLARY 3.3. *$F^*(G)$ is a supersolvable normal subgroup of the group G.*

COROLLARY 3.4. *$F^*(G)$ is contained in every maximal supersolvable normal subgroup of the group G.*

Corollary 3.4 might suggest that $F^*(G)$ is simply the intersection of all maximal supersolvable normal subgroups of G, however if one examines the example given at the end of section 1 it is seen that this need not be the case.

By the nature of its definition it is clear that $F^*(G)$ contains the Fitting subgroup of the group G. This next theorem provides more detailed information on the relation between $F^*(G)$ and $\mathrm{Fit}(G)$.

THEOREM 3.5. *In the group G, $F^*(G)/\mathrm{Fit}(G) \leqslant Z(G/\mathrm{Fit}(G))$.*

Proof. Let H/N be a chief factor of the group G. We first show that modulo N, each element of $F^*(G)$ induces a power automorphism on H. When H/N is abelian this claim follows directly from the fact that $F^*(G)$ normalizes every subgroup of H/N. If H/N is not abelian, we must either have $[H, F^*(G)] \leqslant N$ or $H = [H, F^*(G)]N$. In the latter case, since $[H, F^*(G)] \leqslant F^*(G)$, we have

H/N isomorphic to a section of $F^*(G)$ and this would imply that H/N was solvable. Since any solvable chief factor of a group is abelian it follows that $[H, F^*(G)] \leqslant N$ so that $F^*(G)$ centralizes H/N. This establishes the claim.

Let $h \in H$, $g \in G$ and $f \in F^*(G)$, assume further that f induces the power r on H modulo N. Modulo N we may now see that, $h^{g^{-1}f^{-1}gf} = ((h^{g^{-1}})^{r^{-1}})^{gf} = ((h^{r^{-1}})^{g^{-1}})^{gf} = (h^{r^{-1}})^f = h$ (r^{-1} denotes the inverse of r mod $\exp(H/N)$). From this it follows that $[g, f]$ lies in $C_G(H/N)$ and hence in $\mathrm{Fit}(G)$. Thus $[G, F^*(G)] \leqslant \mathrm{Fit}(G)$ and the theorem is valid.

Since a group G is supersolvable precisely when $G = F^*(G)$ we have as a corollary the following well known property of supersolvable groups.

COROLLARY 3.6. *If G is a supersolvable group, then* $G' \leqslant \mathrm{Fit}(G)$.

We conclude with the following property of $F^*(G)$ which is clearly analogous to a well known property of $\mathrm{Fit}(G)$.

THEOREM 3.7. *In the solvable group G, $C_G^*(F^*(G)) \leqslant F^*(G)$.*

Proof. Set $S = F^*(G)$, $C^* = C_G^*(F^*(G))$ and $T = SC^*$. Assume that $T > S$ and consider a chief series of G passes through both T and S. Let L/S be the chief factor in this series lying just above S, so that $T \geqslant L > S$. Let $x \in L - S$, we may write x in the form $x = fc$ where $f \in S$ and $c \in C^*$. If A/B is any factor in this series, then x centralizes A/B when A/B lies above S and hence $x \in C_G^*(A/B)$. We assume now that A/B lies below S, so that $S \geqslant A$. Let $a \in A$, then modulo B, $a^x = a^{fc} = (a^f)^c = (a^r)^c$ for some integer r. Since $a \in S$ and $c \in C_G^*(S)$, there is an integer t with $(a^r)^c = a^{rt}$. Thus, modulo B, $a^x = a^{rt}$ and hence $x \in C_G^*(A/B)$. From this it follows that $x \in F^*(G)$ contrary to its choice. Hence $T = S$ and the result follows.

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