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The Borsuk–Ulam theorem for maps into a surface

Daciberg Lima Gonçalves [a,*], John Guaschi [b,c]

[a] Departamento de Matemática, IME-USP, Caixa Postal 66281, Ag. Cidade de São Paulo, CEP: 05314-970, São Paulo, SP, Brazil
[b] Laboratoire de Mathématiques Nicolas Oresme, UMR CNRS 6139, Université de Caen BP 5186, 14032 Caen Cedex, France
[c] Instituto de Matemáticas, UNAM, León #2, altos, col. centro, Oaxaca de Juárez, Oaxaca, C.P. 68000, Mexico

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ABSTRACT

Let (X,τ,S) be a triple, where S is a compact, connected surface without boundary, and τ is a free cellular involution on a *CW*-complex X. The triple (X,τ,S) is said to satisfy the *Borsuk–Ulam property* if for every continuous map $f:X\longrightarrow S$, there exists a point $x\in X$ satisfying $f(\tau(x))=f(x)$. In this paper, we formulate this property in terms of a relation in the 2-string braid group $B_2(S)$ of S. If X is a compact, connected surface without boundary, we use this criterion to classify all triples (X,τ,S) for which the Borsuk–Ulam property holds. We also consider various cases where X is not necessarily a surface without boundary, but has the property that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of such a surface. If S is different from the 2-sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$, then we show that the Borsuk–Ulam property does not hold for (X,τ,S) unless either $\pi_1(X/\tau)\cong\pi_1(\mathbb{R}P^2)$, or $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected non-orientable surface of genus 2 or 3 and S is non-orientable. In the latter case, the veracity of the Borsuk–Ulam property depends further on the choice of involution τ; we give a necessary and sufficient condition for it to hold in terms of the surjective homomorphism $\pi_1(X/\tau)\longrightarrow\mathbb{Z}_2$ induced by the double covering $X\longrightarrow X/\tau$. The cases $S=\mathbb{S}^2,\mathbb{R}P^2$ are treated separately.

1. Introduction

St. Ulam conjectured that *if* $f:\mathbb{S}^n\longrightarrow\mathbb{R}^n$ *is a continuous map then there exists a point* $p\in\mathbb{S}^n$ *such that* $f(p)=f(-p)$*, where* $-p$ *is the antipodal point of* p [3, footnote, page 178]. The conjecture was solved in 1933 by K. Borsuk [3, Sätz II]. There was another result in Borsuk's paper, Sätz III, which is indeed equivalent to Sätz II (see [16, Section 2, Theorem 2.1.1]). It turned out that Sätz III had been proved three years before by L. Lusternik and L. Schnirelmann [15] (see also [3, footnote, p. 190]). This was the beginning of the history of what we shall refer to as the *Borsuk–Ulam property* or *Borsuk–Ulam type theorem*. We say that *the triple* (X,τ,S) *has the Borsuk–Ulam property if for every continuous map* $f:X\longrightarrow S$*, there is a point* $x\in X$ *such that* $f(x)=f(\tau(x))$. In the past seventy years, the original statement has been greatly generalised in many directions, and has also been studied in other natural contexts. The contributions are numerous, and we do not intend to present here a detailed description of the development of the subject. One may consult [16] for some applications of the Borsuk–Ulam theorem.

In this Introduction, we concentrate on a particular direction that is more closely related to the type of Borsuk–Ulam problem relevant to the main theme of this paper. In [9], a Borsuk–Ulam type theorem for maps from compact surfaces without boundary with free involutions into $\mathbb{R}^2$ was studied. An important feature which appears in these results of that paper is that the validity of the theorem depends upon the choice of involution. This phenomenon did not and could not

* Corresponding author.
E-mail addresses: dlgoncal@ime.usp.br (D.L. Gonçalves), guaschi@math.unicaen.fr (J. Guaschi).

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show up in the case where the domain is the 2-sphere $\mathbb{S}^2$ since up to conjugation there is only one free involution on $\mathbb{S}^2$. In a similar vein, the Borsuk–Ulam property was also analysed for triples for which the domain is a 3-space form in [14], and also for Seifert manifolds in [1]. The study of these papers leads us to formulate a general problem which consists in finding the maximal value n for which the Borsuk–Ulam property is true for triples $(X, \tau, \mathbb{R}^n)$, where X is a given finite-dimensional CW-complex X equipped with a free involution τ. In this paper, we choose a direction closer to that of [9] which is the investigation of maps from a space whose fundamental group is that of a surface, into a compact, connected surface S without boundary. Within this framework, Proposition 13 will enable us to formulate the veracity of the Borsuk–Ulam property in terms of a commutative diagram of the 2-string braid group $B_2(S)$ of S. We shall then apply algebraic properties of $B_2(S)$ to help us to decide whether the Borsuk–Ulam property holds in our setting in all cases.

Throughout this paper, S will always denote a compact, connected surface without boundary, S_g will be a compact, orientable surface of genus $g \geqslant 0$ without boundary, and N_l will be a compact, non-orientable surface of genus $l \geqslant 1$ without boundary. We consider triples (X, τ, S), where X is a CW-complex and τ is a cellular free involution. The following statements summarise our main results.

Corollary 1.

(a) *If X is a CW-complex equipped with a cellular free involution τ, the triple $(X, \tau, \mathbb{S}^2)$ satisfies the Borsuk–Ulam property if and only if the triple $(X, \tau, \mathbb{R}^3)$ satisfies the Borsuk–Ulam property.*
(b) *If X is a 2-dimensional CW-complex, the triple $(X, \tau, \mathbb{S}^2)$ does not satisfy the Borsuk–Ulam property for any cellular free involution τ.*
(c) *The triple $(\mathbb{S}^3, \tau, \mathbb{S}^2)$ satisfies the Borsuk–Ulam property for the unique cellular free involution (up to conjugacy) τ on $\mathbb{S}^3$.*
(d) *The triple $(\mathbb{R}P^3, \tau, \mathbb{S}^2)$ does not satisfy the Borsuk–Ulam property for the unique cellular free involution (up to conjugacy) τ on $\mathbb{R}P^3$.*

If the target is the projective plane $\mathbb{R}P^2$ we have:

Theorem 2. *Let X be a CW-complex equipped with a cellular free involution τ of dimension less than or equal to three, and suppose that $\pi_1(X)$ is isomorphic to the fundamental group of a compact surface without boundary. Then the Borsuk–Ulam property holds for the triple $(X, \tau, \mathbb{R}P^2)$ if and only if X is simply connected. In particular, if X is a compact surface without boundary, then the Borsuk–Ulam property holds for the triple $(X, \tau, \mathbb{R}P^2)$ if and only if X is the 2-sphere.*

These two results thus treat the cases where $S = \mathbb{S}^2$ or $\mathbb{R}P^2$. From now on, assume that S is different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, that X is a finite-dimensional CW-complex, equipped with a cellular free involution τ, and that $\pi_1(X/\tau)$ is either finite or is isomorphic to the fundamental group of a compact surface without boundary. The condition that $\pi_1(X/\tau)$ is finite is of course equivalent to saying that $\pi_1(X)$ is finite.

Remark 3. If the above space X is a finite-dimensional CW-complex that is a $K(\pi, 1)$, the hypothesis that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact surface without boundary is equivalent to saying that $\pi_1(X)$ is isomorphic to the fundamental group of a compact surface without boundary. To see this, observe that X/τ is also a $K(\pi, 1)$ and a finite-dimensional CW-complex. Therefore the group $\pi_1(X/\tau)$ is torsion free and is the middle group of the short exact sequence $1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/\tau) \longrightarrow \mathbb{Z}_2 \longrightarrow 1$. Since $\pi_1(X)$ is a surface group and of finite index in $\pi_1(X/\tau)$, it follows that $\pi_1(X/\tau)$ is also a surface group. Indeed, from [4, Proposition 10.2, Section VIII], $\pi_1(X/\tau)$ is a duality group, and has the same duality module $\mathbb{Z}$ as $\pi_1(X)$. So $\pi_1(X/\tau)$ is a Poincaré duality group over $\mathbb{Z}$. But every PD^2 group over $\mathbb{Z}$ is the fundamental group of a surface as result of [5,6].

In the case that $\pi_1(X)$ is finite, we obtain the following result.

Proposition 4. *Let X be a CW-complex equipped with a cellular free involution τ, and let S be a compact, connected surface without boundary and different from $\mathbb{R}P^2$ and $\mathbb{S}^2$. If $\pi_1(X)$ is finite then the Borsuk–Ulam property holds for the triple (X, τ, S).*

Now suppose that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact surface without boundary. There are four basic cases according to whether S is orientable or non-orientable, and to whether $\pi_1(X/\tau)$ is isomorphic to the fundamental group of an orientable or a non-orientable surface without boundary. In Section 4, we first consider the case where S is non-orientable. The following theorem pertains to the first subcase where $\pi_1(X/\tau)$ is isomorphic to the fundamental group of an orientable surface without boundary.

Theorem 5. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ, and let S be a compact, connected non-orientable surface without boundary and different from $\mathbb{R}P^2$. If $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected orientable surface without boundary then the Borsuk–Ulam property does not hold for the triple (X, τ, S).*

For the second subcase where $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a non-orientable surface without boundary, we have:

Theorem 6. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ, and let S be a compact, connected non-orientable surface without boundary different from $\mathbb{R}P^2$. Suppose that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected non-orientable surface without boundary. Then the Borsuk–Ulam property holds for the triple (X, τ, S) if and only if $\pi_1(X) = \{1\}$.*

In Section 5, we study the second case, where S is orientable. If $\pi_1(X/\tau)$ is isomorphic to the fundamental group of an orientable surface without boundary, we have:

Theorem 7. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ, and let $g > 0$. If $S = S_g$, and if $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected orientable surface without boundary then the Borsuk–Ulam property does not hold for the triple (X, τ, S).*

The remainder of Section 5 is devoted to the study of the subcase where $\pi_1(X/\tau)$ is isomorphic to the fundamental group of the non-orientable surface N_l without boundary and $S = S_g$, where $g \geqslant 1$. Our analysis divides into four subcases:

(1) $l = 1$.
(2) $l \geqslant 4$.
(3) $l = 2$.
(4) $l = 3$.

For subcase (1) we have:

Proposition 8. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ, and let $g \geqslant 1$. If $\pi_1(X/\tau)$ is isomorphic to the fundamental group of the projective plane $\mathbb{R}P^2$, then the Borsuk–Ulam property holds for (X, τ, S_g).*

For subcase (2) we have:

Proposition 9. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ, let $l \geqslant 4$, and let $g \geqslant 1$. If $\pi_1(X/\tau)$ is isomorphic to the fundamental group of the non-orientable surface N_l then the Borsuk–Ulam property does not hold for (X, τ, S_g).*

To describe the results in the remaining two subcases, we first need to introduce some notation and terminology. Let (X, τ, S) be a triple, where τ is a cellular free involution on X and S is a compact surface without boundary, and let

$$\theta_\tau : \pi_1(X/\tau) \longrightarrow \mathbb{Z}_2$$

be the surjective homomorphism defined by the double covering $X \longrightarrow X/\tau$. For subcases (3) and (4), the veracity of the Borsuk–Ulam property depends on the choice of the free involution τ. As we shall see in Proposition 13, the relevant information concerning τ is encoded in θ_τ. The study of the possible θ_τ may be simplified by considering the following equivalence relation (see also the end of Section 2). Let G be a group, and consider the set of elements of $\mathrm{Hom}(G, \mathbb{Z}_2)$ that are surjective homomorphisms (or equivalently the elements that are not the null homomorphism). Two surjective homomorphisms $\phi_1, \phi_2 \in \mathrm{Hom}(G, \mathbb{Z}_2)$ are said to be *equivalent* if there is an isomorphism $\varphi : G \longrightarrow G$ such that $\phi_1 \circ \varphi = \phi_2$. Taking $G = \pi_1(X/\tau)$, and using the results of [1] given in Appendix A, we shall see that many algebraic questions will depend only on the equivalence classes of this relation. This will help to reduce the number of cases to be analysed.

For subcase (3), where $l = 2$, we have:

Proposition 10. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ, and let $g \geqslant 1$. Consider the presentation $\langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1} \rangle$ of the fundamental group of the Klein bottle K. If $\pi_1(X/\tau)$ is isomorphic to $\pi_1(K)$ then the Borsuk–Ulam property holds for the triple (X, τ, S_g) if and only if $\theta_\tau(\alpha) = \bar{1}$.*

For subcase (4), where $l = 3$, we have:

Theorem 11. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ, and suppose that $\pi_1(X/\tau)$ is isomorphic to $\pi_1(N_3)$. Consider the presentation $\langle v, a_1, a_2 \mid v^2 \cdot [a_1, a_2] \rangle$ of the fundamental group of N_3. Then the Borsuk–Ulam property holds for the triple (X, τ, S_g) if and only if θ_τ is equivalent to the homomorphism $\theta : \pi_1(N_3) \longrightarrow \mathbb{Z}_2$ given by $\theta(v) = \theta(a_1) = \bar{1}$ and $\theta(a_2) = \bar{0}$.*

For subcases (3) and (4), observe that as a result of the relations of the given presentation of $\pi_1(N_2)$ (resp. $\pi_1(N_3)$), any map $\theta : J \longrightarrow \mathbb{Z}_2$ satisfying the conditions of Proposition 10 (resp. Theorem 11) extends to a homomorphism, where J is the set of generators of $\pi_1(N_2)$ (resp. $\pi_1(N_3)$). Therefore there is a double covering which corresponds to the kernel of θ, and consequently the cases in question may be realised by some pair (X,τ) for some cellular free involution τ.

Theorems 5, 6, 7 and 11, and Propositions 8, 9 and 10 may be summarised as follows.

Theorem 12. *Let X be a finite-dimensional CW-complex equipped with a cellular free involution τ. Suppose that $S \neq \mathbb{S}^2, \mathbb{R}P^2$. Then the Borsuk–Ulam property holds for (X,τ,S) if and only if one of the following holds*:

(a) $\pi_1(X/\tau) \cong \pi_1(\mathbb{R}P^2)$.
(b) *S is orientable, and either*
  (i) *$\pi_1(X/\tau) \cong \pi_1(N_2)$, and $\theta_\tau(\alpha) = \bar{1}$ for the presentation of N_2 given in Proposition 10.*
  (ii) *$\pi_1(X/\tau) \cong \pi_1(N_3)$, and θ_τ is equivalent to the homomorphism $\theta : \pi_1(N_3) \longrightarrow \mathbb{Z}_2$ given by $\theta(v) = \theta(a_1) = \bar{1}$ and $\theta(a_2) = \bar{0}$ for the presentation of N_3 given in Theorem 11.*

This paper is organised as follows. In Section 2, we recall some general definitions, and state and prove Proposition 13 which highlights the relation between the short exact sequence $1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/\tau) \longrightarrow \mathbb{Z}_2 \longrightarrow 1$, and the short exact sequence $1 \longrightarrow P_2(S) \longrightarrow B_2(S) \longrightarrow \mathbb{Z}_2 \longrightarrow 1$ of the pure and full 2-string braid groups of S. This proposition will play a vital rôle in much of the paper. Part (b) of Proposition 13 brings to light two special cases where $S = \mathbb{S}^2$ or $S = \mathbb{R}P^2$. The case $S = \mathbb{S}^2$ will be treated in Corollary 1. In Section 3, we deal with the case $S = \mathbb{R}P^2$, and prove Theorem 2. In Section 4, we study the case where S is a compact, non-orientable surface without boundary different from $\mathbb{R}P^2$, and prove Theorem 6. Finally, in Section 5, we analyse the case where S is a compact, orientable surface without boundary different from $\mathbb{S}^2$, and prove Theorems 7 and 11 and Propositions 8–10. The proof of Theorem 11 relies on a long and somewhat delicate argument using the lower central series of $P_2(S)$.

2. Generalities

Let S be a compact surface without boundary, and let G be a finite group that acts freely on a topological space X. If $f : X \longrightarrow S$ is a continuous map, we say that an orbit of the action is *singular with respect to* f if the restriction of f to the orbit is non-injective. In particular, if $G = \mathbb{Z}_2$, a singular orbit is an orbit that is sent to a point by f. We study here the existence of singular orbits in the case where the group G is $\mathbb{Z}_2$. The case where G is an arbitrary finite cyclic group will be considered elsewhere.

The existence of a free action of $\mathbb{Z}_2$ on X is equivalent to that of a fixed-point free involution $\tau : X \longrightarrow X$. Let (X,τ,S) be a triple, where τ is a free involution on X, and let $\theta_\tau : \pi_1(X/\tau) \longrightarrow \mathbb{Z}_2$ be the homomorphism defined by the double covering $X \longrightarrow X/\tau$. Recall that $F_2(S) = \{(x,y) \in S \times S \mid x \neq y\}$ is the 2-point configuration space of S, $D_2(S)$ is the orbit space of $F_2(S)$ by the free $\mathbb{Z}_2$-action $\tau_S : F_2(S) \longrightarrow F_2(S)$, where $\tau_S(x,y) = (y,x)$, and $P_2(S) = \pi_1(F_2(S))$ and $B_2(S) = \pi_1(D_2(S))$ are the pure and full 2-string braid groups respectively of S [8]. Let $\pi : B_2(S) \longrightarrow \mathbb{Z}_2$ denote the surjective homomorphism that to a 2-braid of S associates its permutation, and let $p : X \longrightarrow X/\tau$ denote the quotient map.

The following result will play a key rôle in the rest of the paper.

Proposition 13. *Let X be a CW-complex equipped with a cellular free involution τ, and let S be a compact, connected surface without boundary. Suppose that the Borsuk–Ulam property does not hold for the triple (X,τ,S). Then there exists a homomorphism $\phi : \pi_1(X/\tau) \longrightarrow B_2(S)$ that makes the following diagram commute*:

$$\begin{tikzcd} \pi_1(X/\tau) \arrow[rr, dashed, "\phi"] \arrow[dr, "\theta_\tau"'] & & B_2(S) \arrow[dl, "\pi"] \\ & \mathbb{Z}_2 & \end{tikzcd} \qquad (1)$$

Conversely, if such a factorisation ϕ exists then the Borsuk–Ulam property does not hold in the following cases:

(a) *The space X is a CW-complex of dimension less than or equal to two.*
(b) *S is a compact, connected surface without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$.*
(c) *S is the projective plane and X is a CW-complex of dimension less than or equal to three.*

Remark 14. So if X and S are as in the first line of Proposition 13, and if further $S \neq \mathbb{S}^2, \mathbb{R}P^2$ then the Borsuk–Ulam property does not hold for the triple (X,τ,S) if and only if there exists a homomorphism $\phi : \pi_1(X/\tau) \longrightarrow B_2(S)$ that makes the diagram (1) commute.

Proof of Proposition 13. Suppose first that the Borsuk–Ulam property does not hold for the triple (X, τ, S). Then there exists a map $f : X \longrightarrow S$ such that $f(x) \neq f(\tau(x))$ for all $x \in X$. Define the map $\hat{f} : X \longrightarrow F_2(S)$ by $\hat{f}(x) = (f(x), f(\tau(x)))$. Note that $\hat{f}$ is $\mathbb{Z}_2$-equivariant with respect to the actions on X and $F_2(S)$ given respectively by τ and τ_S, and so induces a map $\tilde{f} : X/\tau \longrightarrow D_2(S)$ of the corresponding quotient spaces defined by $\tilde{f}(y) = \{f(x), f(\tau(x))\}$, where $x \in p^{-1}(\{y\})$. On the level of fundamental groups, we obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X) \arrow[r, "p_\#"] \arrow[d, "\hat{f}_\#"'] & \pi_1(X/\tau) \arrow[r, "\theta_\tau"] \arrow[d, "\tilde{f}_\#"'] & \mathbb{Z}_2 \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & P_2(S) \arrow[r] & B_2(S) \arrow[r, "\pi"] & \mathbb{Z}_2 \arrow[r] & 1,
\end{tikzcd}$$

where $\hat{f}_\#, \tilde{f}_\#$ are the homomorphisms induced by $\hat{f}, \tilde{f}$ respectively, and $\rho : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ is the homomorphism induced on the quotients. We claim that ρ is injective. To see this, let $\gamma \in \ker \rho$, let $x_0 \in X/\tau$ be a basepoint, let $\tilde{x}_0 \in X$ be a lift of x_0, and let c be a loop in X/τ based at x_0 such that $\theta_\tau(\langle c \rangle) = \gamma$. Let $\tilde{c}$ be the lift of c based at $\tilde{x}_0$. Thus $\tilde{c}$ is an arc from $\tilde{x}_0$ to a point of $\{\tilde{x}_0, \tau(\tilde{x}_0)\}$. We have that $\pi \circ \tilde{f}_\#(\langle c \rangle) = \rho \circ \theta_\tau(\langle c \rangle) = \bar{0}$, so $\tilde{f}_\#(\langle c \rangle) \in \ker \pi = P_2(S)$. Further, $\tilde{f}(c) = \{f(\tilde{c}), f(\tau(\tilde{c}))\}$. Now $f(\tilde{c})$ (resp. $f(\tau(\tilde{c}))$) is an arc from $f(\tilde{x}_0)$ (resp. $f(\tau(\tilde{x}_0))$) to an element of $\{f(\tilde{x}_0), f(\tau(\tilde{x}_0))\}$. But $\tilde{f}_\#(\langle c \rangle) \in P_2(S)$, so $f(\tilde{c})$ (resp. $f(\tau(\tilde{c}))$) is a loop based at $f(\tilde{x}_0)$ (resp. $f(\tau(\tilde{x}_0))$). Thus $\tilde{c}$ could not be an arc from $\tilde{x}_0$ to $\tau(\tilde{x}_0)$, for otherwise $\tilde{x}_0 \in X$ would satisfy $f(\tilde{x}_0) = f(\tau(\tilde{x}_0))$, which contradicts the hypothesis. Hence $\tilde{c}$ is a loop based at $\tilde{x}_0$, so $\langle \tilde{c} \rangle \in \pi_1(X, \tilde{x}_0)$, and $\langle c \rangle = p_\#(\langle \tilde{c} \rangle)$. Thus $\gamma = \theta_\tau(\langle c \rangle) = \theta_\tau \circ p_\#(\langle \tilde{c} \rangle) = \bar{0}$, and ρ is injective, as claimed, so is an isomorphism. Taking $\phi = \tilde{f}_\#$ yields the required conclusion.

We now prove the converse for the three cases (a)–(c) of the second part of the proposition. Suppose that there exists a homomorphism $\phi : \pi_1(X/\tau) \longrightarrow B_2(S)$ that makes the diagram (1) commute. We treat the three cases of the statement in turn.

(a) By replacing each group G in the algebraic diagram (1) by the space $K(G, 1)$, we obtain a diagram of spaces that is commutative up to homotopy. The first possible non-vanishing homotopy group of the fibre of the classifying map $D_2(S) \longrightarrow K(B_2(S), 1)$ of the universal covering of $D_2(S)$ is in dimension greater than or equal to two. Since X is of dimension at most two, by classical obstruction theory [20, Chapter V, Section 4, Theorem 4.3, and Chapter VI, Section 6, Theorem 6.13], there exists a map $\tilde{f} : X/\tau \longrightarrow D_2(S)$ that induces ϕ on the level of fundamental groups. The composition of a lifting to the double coverings $X \longrightarrow F_2(S)$ of the map $\tilde{f}$ with the projection onto the first coordinate of $F_2(S)$ gives rise to a map that does not collapse any orbit to a point, and the result follows.

(b) Since S is different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, the space $D_2(S)$ is a $K(\pi, 1)$, so all of its higher homotopy groups vanish. Arguing as in case (a), there is no obstruction to constructing a map $\tilde{f}$ that induces ϕ on the level of fundamental groups, which proves the result in this case.

(c) Suppose that $S = \mathbb{R}P^2$. By [11], it follows that the universal covering of $D_2(\mathbb{R}P^2)$ has the homotopy type of the 3-sphere. Since X has dimension less than or equal to three, using classical obstruction theory, we may construct a map $\tilde{f}$ that satisfies the conditions, and once more the result follows. □

Proposition 4 is an immediate consequence of the first part of Proposition 13 above.

Proof of Proposition 4. The finiteness of $\pi_1(X)$ implies that of $\pi_1(X/\tau)$. Since $B_2(S)$ is torsion free, there is no factorisation ϕ of the algebraic diagram (1) of Proposition 13, and the result follows. □

Remark 15. If S is $\mathbb{S}^2$ (resp. $\mathbb{R}P^2$), the difficulty in proving the converse in the case $\dim(X) > 2$ (resp. $\dim(X) > 3$) occurs as a result of the non-vanishing of the higher homotopy groups of the 2-sphere (resp. the 3-sphere).

If S is a compact, connected surface without boundary, by Proposition 13(b), there are two possibilities for S where we do not have equivalence with the existence of a factorisation of the diagram (1). The case of $\mathbb{R}P^2$ will be treated in Section 3. For now, let us consider the case where the target is the sphere $\mathbb{S}^2$.

Proposition 16. *If X is a CW-complex equipped with a cellular free involution τ, a triple $(X, \tau, \mathbb{S}^2)$ satisfies the Borsuk–Ulam property if and only if the classifying map $g : X/\tau \longrightarrow K(\mathbb{Z}_2, 1)$ of the double covering $X \longrightarrow X/\tau$ does not factor (up to homotopy) through the inclusion $\mathbb{R}P^2 \longrightarrow \mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$.*

Proof. First note that the space $D_2(\mathbb{S}^2)$ has the homotopy type of $\mathbb{R}P^2$ [11,12]. If there is a factorisation of g (up to homotopy) through the inclusion $\mathbb{R}P^2 \longrightarrow \mathbb{R}P^\infty$ then we may construct a map $g_1 : X/\tau \longrightarrow D_2(\mathbb{S}^2)$. Consequently, there exists a $\mathbb{Z}_2$-equivariant lifting $\tilde{g}_1 : X \longrightarrow F_2(\mathbb{S}^2)$. The composition of $\tilde{g}_1$ with the projection onto the first coordinate of $F_2(\mathbb{S}^2)$ is a map for which the Borsuk–Ulam property does not hold. Conversely, if the Borsuk–Ulam property does not hold for the triple $(X, \tau, \mathbb{S}^2)$ then by a routine argument, the map which does not collapse any orbit gives rise to the factorisation. □

We are now able to prove Corollary 1.

Proof of Corollary 1.

(a) By Proposition 1 of [14], $(X, \tau, \mathbb{R}^3)$ satisfies the Borsuk–Ulam property if and only if there is no map $f : X/\tau \longrightarrow \mathbb{R}P^2$ such that the pull-back of the non-trivial class of $H^1(\mathbb{R}P^2; \mathbb{Z}_2)$ is the first characteristic class of the $\mathbb{Z}_2$-bundle $X \longrightarrow X/\tau$. But this is exactly the condition given by Proposition 16 for $(X, \tau, \mathbb{S}^2)$.
(b) Since the homomorphism $B_2(\mathbb{S}^2) \longrightarrow \mathbb{Z}_2$ is an isomorphism, the result follows from Proposition 13.
(c) and (d) This is a consequence of the main result of [14]. The fact that there is only one involution on $\mathbb{R}P^3$ up to conjugacy follows from [17]. □

Remark 17. The 'if' part of Corollary 1(a) can also be proved by a very simple geometrical argument. For the converse, we do not know of a more direct proof. One may find other examples of triples such as those given in Corollary 1(c), i.e. triples $(X, \tau, \mathbb{S}^2)$, where X is a CW-complex of dimension 3, for which the Borsuk–Ulam property holds. See [14] for more details.

To conclude this section, recall from the Introduction that if we are given a group G, two surjective homomorphisms $\phi_1, \phi_2 \in \operatorname{Hom}(G, \mathbb{Z}_2)$ are said to be *equivalent* if there is an isomorphism $\varphi : G \longrightarrow G$ such that $\phi_1 \circ \varphi = \phi_2$. We shall see that many algebraic questions will depend only on the equivalence classes of this relation due to the fact that if ϕ_1, ϕ_2 are equivalent then the existence of the commutative diagram (1) for ϕ_1 is equivalent to the existence of the commutative diagram (1) for ϕ_2. A consequence of this is that the number of cases to be analysed may be reduced. From Appendix A, we have the following results:

(a) If G is isomorphic to the fundamental group of an orientable compact, connected surface without boundary and of genus greater than zero then there is precisely one equivalence class.
(b) Suppose that G is isomorphic to the fundamental group of the non-orientable surface N_l, where $l > 1$.
  (i) If $l \neq 2$, there are three distinct equivalence classes.
  (ii) If $l = 2$, there are two distinct equivalence classes.

The knowledge of these classes will be used in conjunction with Proposition 13, notably in Section 5, to study the validity of the Borsuk–Ulam property.

3. The case $S = \mathbb{R}P^2$

In this section, we study the second exceptional case of Proposition 13(b) where the target S is the projective plane $\mathbb{R}P^2$. Indeed, by the proof of the first part of Proposition 13, a triple $(X, \tau, \mathbb{R}P^2)$ does not satisfy the Borsuk–Ulam property if and only if there exists a map $\tilde{f} : X/\tau \longrightarrow D_2(\mathbb{R}P^2)$ for which the choice $\phi = \tilde{f}_\#$ makes the diagram (1) commute. Recall that $B_2(\mathbb{R}P^2)$ is isomorphic to the generalised quaternion group $\mathcal{Q}_{16}$ of order 16 [19].

Proposition 18. *Given the notation of Proposition* 13, *the non-existence of a factorisation $\phi : \pi_1(X/\tau) \longrightarrow \mathcal{Q}_{16}$ of the homomorphism $\theta_\tau : \pi_1(X/\tau) \longrightarrow \mathbb{Z}_2$ through the homomorphism $\mathcal{Q}_{16} \longrightarrow \mathbb{Z}_2$ implies that the Borsuk–Ulam property holds. Conversely, if a factorisation exists, the Borsuk–Ulam property holds if and only if the map $f_1 : X/\tau \longrightarrow K(\mathcal{Q}_{16}, 1)$ obtained from the algebraic homomorphism ϕ does not factor through the map $\mathbb{S}^3/\mathcal{Q}_{16} \longrightarrow K(\mathcal{Q}_{16}, 1)$ given by the Postnikov system, where $K(\mathcal{Q}_{16}, 1)$ is the first stage of the Postnikov tower of $\mathbb{S}^3/\mathcal{Q}_{16}$. In particular, if X has dimension less than or equal to three, if the algebraic factorisation problem has a solution then the Borsuk–Ulam property does not hold.*

Proof. The proof follows straightforwardly from Proposition 13. □

Now we can prove the main result of this section.

Proof of Theorem 2. Since X is of dimension less than or equal to three, the result is equivalent to the existence of the homomorphism ϕ by Proposition 13(c). Suppose first that X is simply connected. Then the fundamental group of the quotient X/τ is isomorphic to $\mathbb{Z}_2$. Since the only element of $B_2(\mathbb{R}P^2)$ of order 2 is the full twist braid, which belongs to $P_2(\mathbb{R}P^2)$, the factorisation of diagram (1) does not exist, and this proves the 'if' part.

Conversely, suppose that X is non-simply connected. Then the fundamental group of X/τ is either isomorphic to the fundamental group of S_g, where $g > 0$, or is isomorphic to the fundamental group of N_l, where $l > 1$ (recall that S_g (resp. N_l) is a compact, connected orientable (resp. non-orientable) surface without boundary of genus g (resp. l)). Let us first prove the result in the case where $\pi_1(X/\tau) \cong \pi_1(S_g)$. The fundamental group of S_g has the following presentation:

$$\left\langle a_1, a_2, \ldots, a_{2g-1}, a_{2g} \;\middle|\; [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] \right\rangle. \tag{2}$$

Consider the presentation $\langle x, y \mid x^4 = y^2,\ yxy^{-1} = x^{-1}\rangle$ of $\mathcal{Q}_{16}$. Then x is of order 8, and defining

$$\phi(a_i) = \begin{cases} x & \text{if } \theta_\tau(a_i) = \bar{1}, \\ x^2 & \text{if } \theta_\tau(a_i) = \bar{0} \end{cases} \tag{3}$$

gives rise to a factorisation. Now suppose that $\pi_1(X/\tau) \cong \pi_1(N_l)$. If $l \geqslant 3$ is odd, $\pi_1(N_l)$ has the following presentation:

$$\left\langle v, a_1, a_2, \dots, a_{l-2}, a_{l-1} \;\middle|\; v^2 \cdot [a_1, a_2] \cdots [a_{l-2}, a_{l-1}] \right\rangle. \tag{4}$$

If $\theta_\tau(v) = \bar{0}$ then we define ϕ by $\phi(v) = e$ (the trivial element of $B_2(\mathbb{R}P^2)$), and $\phi(a_i)$ by Eq. (3). If $\theta_\tau(v) = \bar{1}$ then we define $\phi(v) = xy$. Now $\phi(v^2) = x^4$ which is of order 2, and so $\phi(v^2)$ is the full twist braid. Defining

$$\begin{cases} \phi(a_1) = x^7 y \text{ and } \phi(a_2) = xy & \text{if } \theta_\tau(a_1) = \theta_\tau(a_2) = \bar{1}, \\ \phi(a_1) = x^2 \text{ and } \phi(a_2) = y & \text{if } \theta_\tau(a_1) = \theta_\tau(a_2) = \bar{0}, \\ \phi(a_1) = xy \text{ and } \phi(a_2) = x^2 & \text{if } \theta_\tau(a_1) = \bar{1} \text{ and } \theta_\tau(a_2) = \bar{0}, \\ \phi(a_1) = x^2 \text{ and } \phi(a_2) = xy & \text{if } \theta_\tau(a_1) = \bar{0} \text{ and } \theta_\tau(a_2) = \bar{1}, \end{cases}$$

and the remaining $\phi(a_i)$ by Eq. (3), we obtain a factorisation of the commutative diagram (1), and the result follows. The case where $l \geqslant 2$ is even is similar, and is left to the reader. □

4. The non-orientable case with $S \neq \mathbb{R}P^2$

In this section, we consider the case where the target S is a compact, connected non-orientable surface without boundary and different from $\mathbb{R}P^2$. Recall that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected surface without boundary. In this section, we prove Theorems 5 and 6, which is the case where this surface is orientable or non-orientable respectively.

Proof of Theorem 5. Let $h \geqslant 1$ be such that $\pi_1(X/\tau) \cong \pi_1(S_h)$, and consider the presentation (2) of $\pi_1(X/\tau)$. Let $x \in B_2(S) \setminus P_2(S)$. Then we define

$$\phi(a_i) = \begin{cases} x & \text{if } \theta_\tau(a_i) = \bar{1}, \\ x^2 & \text{if } \theta_\tau(a_i) = \bar{0}. \end{cases} \tag{5}$$

The fact that the relation of $\pi_1(X/\tau)$ is given by a product of commutators implies that ϕ is a well-defined homomorphism that makes the diagram (1) commute. The result then follows by applying Proposition 13(b). □

We now suppose that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of the non-orientable surface N_l.

Proof of Theorem 6. The 'if' part follows because $\pi_1(X/\tau) \cong \mathbb{Z}_2$ and $B_2(S)$ is torsion free. Indeed, there is no algebraic factorisation of the diagram (1) since the only homomorphism that makes the diagram commute is the trivial homomorphism. For the 'only if' part, let $S = N_m$, where $m \geqslant 2$, and let $\pi_1(X/\tau)$ be isomorphic to the fundamental group of the non-orientable surface N_l, where $l \geqslant 2$. We first suppose that l is even. Then $\pi_1(X/\tau)$ has the following presentation:

$$\left\langle \alpha, \beta, a_1, a_2, \dots, a_{2l-3}, a_{2l-2} \;\middle|\; \alpha\beta\alpha\beta^{-1}[a_1, a_2] \cdots [a_{2l-3}, a_{2l-2}] \right\rangle. \tag{6}$$

From [18], we have the following relations in the braid group $B_2(N_m)$: $\rho_{2,1}\rho_{1,1}\rho_{2,1}^{-1} = \rho_{1,1}B^{-1}$, $B = \sigma^2$, $\sigma\rho_{1,1}\sigma^{-1} = \rho_{2,1}$ and $\sigma\rho_{2,1}\sigma^{-1} = B\rho_{1,1}B^{-1}$ (here σ denotes the generator σ_1). We remark that the given elements of $B_2(N_m)$ are those of [18], but we choose to multiply them from left to right, which differs from the convention used in [18]. Other presentations of braid groups of non-orientable surfaces may be found in [2,13]. Now $\rho_{2,1}\rho_{1,1}\rho_{2,1}^{-1} = \rho_{1,1}B^{-1}$ implies that $\rho_{2,1}\rho_{1,1}\rho_{2,1}^{-1}B\rho_{1,1}^{-1}B^{-1} = B^{-1}$. Using the equation $\sigma\rho_{1,1}^{-1}\rho_{2,1}^{-1}\sigma^{-1} = (\sigma\rho_{1,1}^{-1}\sigma^{-1})(\sigma\rho_{2,1}^{-1}\sigma^{-1}) = \rho_{2,1}^{-1}B\rho_{1,1}^{-1}B^{-1}$, this implies in turn that $\rho_{2,1}\rho_{1,1}\sigma\rho_{1,1}^{-1}\rho_{2,1}^{-1}\sigma^{-1} = B^{-1} = \sigma^{-2}$, and hence $\rho_{2,1}\rho_{1,1}\sigma\rho_{1,1}^{-1}\rho_{2,1}^{-1} = \sigma^{-1}$.

Now we construct the factorisation ϕ. If $\theta_\tau(\alpha) = \bar{0}$ then define $\phi(\alpha) = e$, and $\phi(\beta)$ to be equal to any element of $B_2(S) \setminus P_2(S)$ if $\theta_\tau(\beta) = \bar{1}$, and to be equal to e if $\theta_\tau(\beta) = \bar{0}$. If $\theta_\tau(\alpha) = \bar{1}$ and $\theta_\tau(\beta) = \bar{0}$ then we define $\phi(\alpha) = \sigma$ and $\phi(\beta) = \rho_{2,1}\rho_{1,1}$, while if $\theta_\tau(\alpha) = \theta_\tau(\beta) = \bar{1}$, we define $\phi(\alpha) = \sigma$ and $\phi(\beta) = \rho_{2,1}\rho_{1,1}\sigma$. For the remaining generators a_i, we define ϕ as in Eq. (5). It follows from the construction that ϕ is a well-defined homomorphism that makes the diagram (1) commute.

Finally let the fundamental group $\pi_1(X/\tau)$ be isomorphic to $\pi_1(N_l)$, where $l \geqslant 3$ is odd. Consider the presentation (4) of $\pi_1(N_l)$. If $\theta_\tau(v) = \bar{0}$ then the result follows as in the proof of Theorem 5. So suppose that $\theta_\tau(v) = \bar{1}$. We have the relation $\rho_{2,1}B\rho_{2,1}^{-1} = B\rho_{1,1}^{-1}B^{-1}\rho_{1,1}B^{-1}$.

According to Proposition 32 in Appendix A it suffices to consider two cases. The first is $\theta_\tau(a_i) = \bar{0}$ for all i; the second is $\theta_\tau(a_2) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for the other values of i. In the first case, we define $\phi(v) = \sigma$, $\phi(a_1) = \rho_{1,1}^{-1}$, $\phi(a_2) = \rho_{2,1}$ and for the remaining generators a_i, we define $\phi(a_i)$ as in Eq. (5). The result follows via the relation of the presentation (4). As for the second case, we define $\phi(v)=\sigma$, $\phi(a_1)=\sigma^{-1}$, $\phi(a_2)=\rho_{2,1}\rho_{1,1}$, and for the remaining a_i, we define $\phi(a_i)$ as in Eq. (5). The result then follows. □

5. The orientable case with $S \neq \mathbb{S}^2$

The purpose of this section is to study the Borsuk–Ulam property in the case where the target is a compact, connected orientable surface without boundary of genus greater than zero. This is the most delicate case which we will separate into several subcases. As in the previous section, $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected surface without boundary. We first suppose that this surface is orientable.

Proof of Theorem 7. Similar to that of Theorem 5. □

We now suppose that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of the non-orientable surface N_l. Let $S = S_g$, where $g \geqslant 1$. As we mentioned in the Introduction, we consider the following four subcases.

(1) $l=1$.
(2) $l \geqslant 4$.
(3) $l=2$.
(4) $l=3$.

As we shall see, the first two cases may be solved easily. The third case is a little more difficult. The fourth case is by far the most difficult, and will occupy most of this section. Some of the tools used in this last case will appear in the discussion of the first three cases. Let us now study these cases in turn.

Subcase (1): $l=1$. This is the subcase where $\pi_1(X/\tau)$ is isomorphic to the fundamental group of the projective plane $\mathbb{R}P^2$.

Proof of Proposition 8. Since $B_2(S_g)$ is non-trivial and torsion free, it follows that there is no algebraic factorisation of the diagram (1), and the result follows from Proposition 13. □

Subcase (2): $l \geqslant 4$. We recall a presentation of $P_2(S_g)$ that may be found in [7] and that shall be used at various points during the rest of the paper. Other presentations of $P_2(S_g)$ may be found in [2,10].

Theorem 19. *([7]) Let $g \geqslant 1$. The following is a presentation of $P_2(S_g)$.*

generators: $\rho_{i,j}$, *where* $i=1,2$ *and* $j=1,\dots,2g$.
relations:

(I) $[\rho_{1,1},\rho_{1,2}^{-1}]\cdots[\rho_{1,2g-1},\rho_{1,2g}^{-1}] = B_{1,2} = B_{2,1}^{-1} = [\rho_{2,1},\rho_{2,2}^{-1}]\cdots[\rho_{2,2g-1},\rho_{2,2g}^{-1}]$ *(this defines the elements* $B_{1,2}$ *and* $B_{2,1}^{-1}$*).*
(II) $\rho_{2,l}\rho_{1,j} = \rho_{1,j}\rho_{2,l}$ *where* $1 \leqslant j,l \leqslant 2g$*, and* $j<l$ *(resp.* $j<l-1$*) if* l *is odd (resp.* l *is even).*
(III) $\rho_{2,k}\rho_{1,k}\rho_{2,k}^{-1} = \rho_{1,k}[\rho_{1,k}^{-1},B_{1,2}]$ *and* $\rho_{2,k}^{-1}\rho_{1,k}\rho_{2,k} = \rho_{1,k}[B_{1,2}^{-1},\rho_{1,k}]$ *for all* $1 \leqslant k \leqslant 2g$.
(IV) $\rho_{2,k}\rho_{1,k+1}\rho_{2,k}^{-1} = B_{1,2}\rho_{1,k+1}[\rho_{1,k}^{-1},B_{1,2}]$*, and* $\rho_{2,k}^{-1}\rho_{1,k+1}\rho_{2,k} = B_{1,2}^{-1}[B_{1,2},\rho_{1,k}]\rho_{1,k+1}[B_{1,2}^{-1},\rho_{1,k}]$*, for all* k *odd,* $1 \leqslant k \leqslant 2g$.
(V) $\rho_{2,k+1}\rho_{1,k}\rho_{2,k+1}^{-1} = \rho_{1,k}B_{1,2}^{-1}$*, and* $\rho_{2,k+1}^{-1}\rho_{1,k}\rho_{2,k+1} = \rho_{1,k}B_{1,2}[B_{1,2}^{-1},\rho_{1,k+1}]$*, for all* k *odd,* $1 \leqslant k \leqslant 2g$.
(VI) $\rho_{2,l}\rho_{1,j}\rho_{2,l}^{-1} = [B_{1,2},\rho_{1,l}^{-1}]\rho_{1,j}[\rho_{1,l}^{-1},B_{1,2}]$ *and* $\rho_{2,l}^{-1}\rho_{1,j}\rho_{2,l} = [\rho_{1,l},B_{1,2}^{-1}]\rho_{1,j}[B_{1,2}^{-1},\rho_{1,l}]$ *for all* $1 \leqslant l < j \leqslant 2g$ *and* $(j,l) \neq (2t,2t-1)$ *for all* $t \in \{1,\dots,g\}$.

From the above relations, we obtain

$$\rho_{2,k}B_{1,2}\rho_{2,k}^{-1} = B_{1,2}\rho_{1,k}^{-1}B_{1,2}\rho_{1,k}B_{1,2}^{-1}, \tag{7}$$

and $\rho_{2,k}^{-1}B_{1,2}\rho_{2,k} = \rho_{1,k}B_{1,2}\rho_{1,k}^{-1}$. Let $\sigma=\sigma_1$ be the standard generator of $B_2(S_g)$ that swaps the two basepoints, and set $B = B_{1,2} = \sigma^2$. The crucial relation that we shall require is

$$\rho_{2,2i}\rho_{1,2i-1}\rho_{2,2i}^{-1} = \rho_{1,2i-1}B^{-1}, \quad \text{where } i \in \{1,\dots,g\}.$$

Proof of Proposition 9. First assume that l is odd. Then N_l has the presentation given by Eq. (4). Using Proposition 32, for at least two generators a_{2i-1}, a_{2i} with $1 < i \leqslant g$, we have $\theta_\tau(a_{2i-1}) = \theta_\tau(a_{2i}) = \bar{0}$. If $\theta_\tau(v) = \bar{0}$ then the factorisation is defined as in the corresponding case of the proof of Theorem 5. So assume that $\theta_\tau(v) = \bar{1}$, and define $\phi(v)=\sigma$, $\phi(a_{2i-1}) =$

$\rho_{1,1}^{-1}$, $\phi(a_{2i})=\rho_{2,2}$, and for $j\notin\{2i-1,2i\}$, set $\phi(a_j)=\sigma$ if $\theta_\tau(a_j)=\bar{1}$, and $\phi(a_j)=e$ if $\theta_\tau(a_j)=\bar{0}$. It follows from the relation of the presentation of $\pi_1(N_l)$ given in Eq. (4) and the first relation of (V) of Theorem 19 that ϕ is a well-defined homomorphism that makes the diagram (1) commute. The result follows from Proposition 13.

If $l\geqslant 4$ is even, the proof is similar. Once more, from Proposition 32, we have $\theta_\tau(a_{2i-1})=\theta_\tau(a_{2i})=\bar{0}$ for some $i\in\{1,\dots,g\}$. The fundamental group of the surface N_l has the presentation given by Eq. (6). Define $\phi(\alpha)=\sigma$, and set $\phi(\beta)=e$ if $\theta_\tau(\beta)=\bar{0}$ and $\phi(\beta)=\sigma$ if $\theta_\tau(\beta)=\bar{1}$. We define $\theta_\tau(a_i)$ as in the case l odd, and the result follows in a similar manner. □

Before going any further, we define some notation that shall be used to discuss the remaining two cases. For $i=1,2$, the two projections $p_i:P_2(S_g)\longrightarrow P_1(S_g)$ furnish a homomorphism $p_1\times p_2:P_2(S_g)\longrightarrow P_1(S_g)\times P_1(S_g)$ (which is the homomorphism induced by the inclusion $F_2(S_g)\longrightarrow S_g\times S_g$). Let N denote the kernel of $p_1\times p_2$. We thus have a short exact sequence

$$1\longrightarrow N\longrightarrow P_2(S_g)\xrightarrow{p_1\times p_2} P_1(S_g)\times P_1(S_g)\longrightarrow 1. \tag{8}$$

Let (x_1,x_2) be a basepoint in $F_2(S_g)$, let

$$\mathbb{F}_1=P_1\big(S_g\setminus\{x_2\},x_1\big),\quad \text{and let}\quad \mathbb{F}_2=P_1\big(S_g\setminus\{x_1\},x_2\big). \tag{9}$$

We know that for $i=1,2$, $\mathbb{F}_i=\ker p_j$, where $j\in\{1,2\}$ and $j\neq i$, and that $\mathbb{F}_i$ is a free subgroup of $P_2(S_g)$ of rank $2g$ with basis $\{\rho_{i,1},\dots,\rho_{i,2g}\}$. Now N is also equal to the normal closure of B in $P_2(S_g)$ (see [7], and Proposition 3.2 in particular), and is a free group of infinite rank with basis $\{B_\eta=\eta B\eta^{-1}\mid \eta\in\mathbb{S}_1\}$, where $\mathbb{S}_1$ is a Reidemeister–Schreier system for the projection $\pi_1(S_g\setminus\{x_2\},x_1)\longrightarrow\pi_1(S_g,x_1)$.

Subcase (3): $l=2$. Suppose that $\pi_1(X/\tau)\cong\pi_1(K)$, where K denotes the Klein bottle.

Proof of Proposition 10. If $\theta_\tau(\alpha)=\bar{0}$, it is straightforward to check that we have a factorisation of diagram (1), and so by Proposition 13, the Borsuk–Ulam property does not hold for the triple (X,τ,S_g). Conversely, assume that $\theta_\tau(\alpha)=\bar{1}$, and suppose that the Borsuk–Ulam property does not hold for the triple (X,τ,S_g). We will argue for a contradiction. Since $\theta_\tau(\alpha)=\bar{1}$ we may assume by Proposition 32 that $\theta_\tau(\beta)=\bar{0}$. By Proposition 13, we have a factorisation as in diagram (1). So there are elements which by abuse of notation we also denote $\alpha,\beta\in B_2(S_g)$ satisfying $\beta\alpha\beta^{-1}=\alpha^{-1}$. This relation implies that

$$\beta\alpha^2\beta^{-1}=\alpha^{-2}, \tag{10}$$

of which both sides belong to $P_2(S_g)$. Applying this homomorphism to Eq. (10), we obtain two similar equations, each in $P_1(S_g)$. For each of these two equations, the subgroup of $P_1(S_g)$ generated by $p_i(\alpha^2)$ and $p_i(\beta)$, for $i=1,2$, must necessarily have rank at most one (the subgroup is free Abelian if $g=1$, and is free if $g>1$, so must have rank one as a result of the relation). This implies that $p_i(\alpha^2)$ is trivial. Therefore $\alpha^2\in N$. The Abelianisation N_{Ab} of N is isomorphic to the group ring $\mathbb{Z}[\pi_1(S_g)]$, by means of the natural bijection $\mathbb{S}_1\longrightarrow\pi_1(S_g)$. Let $\lambda:N\longrightarrow N_{\mathrm{Ab}}$ denote the Abelianisation homomorphism, and let $\exp:\mathbb{Z}[\mathbb{S}_1]\longrightarrow\mathbb{Z}$ denote the evaluation homomorphism.

Since $\alpha^2\in N$, both sides of Eq. (10) belong to N. Eq. (7) implies that $\exp\circ\lambda(\beta\alpha^2\beta^{-1})=\exp\circ\lambda(\alpha^2)$, and so $\exp\circ\lambda(\alpha^2)=0$ by Eq. (10). On the other hand, $\alpha\in B_2(S_g)\setminus P_2(S_g)$, so there exists $\gamma\in P_2(S_g)$ satisfying $\alpha=\gamma\sigma$. Hence

$$\alpha^2=\gamma\sigma\gamma\sigma^{-1}.B, \tag{11}$$

and since $\alpha^2,B\in N$, we see that $\gamma\sigma\gamma\sigma^{-1}\in N$. Now $\gamma\in P_2(S_g)$, so we may write $\gamma=w_1w_2$, where for $i=1,2$, $w_i\in\mathbb{F}_i$. Setting $w_i'=\sigma w_i\sigma^{-1}$ for $i=1,2$, we have that $w_i'\in\mathbb{F}_j$, where j satisfies $\{i,j\}=\{1,2\}$. Further, $1=(p_1\times p_2)(w)=(p_1\times p_2)(w_1w_2w_1'w_2')=(w_1w_2',w_2w_1')$ (we abuse notation slightly by writing the elements of the factors of $P_1(S_g)\times P_1(S_g)$ in the same form as the corresponding elements of $P_2(S_g)$). Thus w_1w_2' and $w_1'w_2$, considered as elements of $P_2(S_g)$, belong to N. We have that

$$\sigma w_1w_2'\sigma^{-1}=w_1'.\sigma w_2'\sigma^{-1}=w_1'Bw_2B^{-1}=w_1'w_2.w_2^{-1}Bw_2.B^{-1},$$

and since $\exp\circ\lambda(\sigma w_1w_2'\sigma^{-1})=\exp\circ\lambda(w_1w_2')$, it follows that

$$\exp\circ\lambda\big(w_1w_2'\big)=\exp\circ\lambda\big(w_1'w_2\big). \tag{12}$$

Now

$$\gamma\sigma\gamma\sigma^{-1}=w_1w_2w_1'w_2'=w_1w_2'.w_2'^{-1}w_2\big(w_1'w_2\big)w_2^{-1}w_2',$$

and thus $\exp\circ\lambda(\gamma\sigma\gamma\sigma^{-1})=2\exp\circ\lambda(w_1w_2')$ by Eq. (12). In particular, $\exp\circ\lambda(\alpha^2)$ is odd by Eq. (11), which contradicts the fact that $\exp\circ\lambda(\alpha^2)=0$. We thus conclude that the equation $\beta\alpha\beta^{-1}=\alpha^{-1}$, where $\alpha\in B_2(S_g)\setminus P_2(S_g)$, $\beta\in P_2(S_g)$, has no solution, and hence the Borsuk–Ulam property holds for the triple (X,τ,S_g). □

Subcase (4): $l=3$. Using the results of Proposition 32, it suffices to consider the following three cases:

(a) $\theta_\tau(v) = \theta_\tau(a_2) = \bar{0}$ and $\theta_\tau(a_1) = \bar{1}$.
(b) $\theta_\tau(v) = \bar{1}$ and $\theta_\tau(a_1) = \theta_\tau(a_2) = \bar{0}$.
(c) $\theta_\tau(v) = \theta_\tau(a_1) = \bar{1}$ and $\theta_\tau(a_2) = \bar{0}$.

Most of the rest of this section is devoted to analysing case (c), which is by far the most difficult of the three cases. Using the transformations of Proposition 30, we may show that case (c) is equivalent to $\theta_\tau(v) = \theta_\tau(a_2) = \bar{1}$ and $\theta_\tau(a_1) = \bar{0}$, and so by the discussion at the end of Section 2, it suffices to consider the latter case. So in what follows, let $\theta_\tau : \pi_1(N_3) \longrightarrow \mathbb{Z}_2$ be the homomorphism given by $\theta_\tau(v) = \theta_\tau(a_2) = \bar{1}$ and $\theta_\tau(a_1) = \bar{0}$. We first define some notation. By Proposition 13, we must decide whether there exist $a, c \in B_2(S_g)$ and $w \in P_2(S_g)$ such that

$$a^2[w, c] = 1. \tag{13}$$

Set

$$a = \rho^{-1}\sigma, \quad \text{and} \quad c = \sigma v, \quad \text{where } \rho, v \in P_2(S_g). \tag{14}$$

In order to determine the existence of solutions to Eq. (13), we begin by studying its projection onto $P_1(S_g) \times P_1(S_g)$ via the short exact sequence (8), and its projection onto $(P_1(S_g))_{\mathrm{Ab}} \times (P_1(S_g))_{\mathrm{Ab}}$ under the homomorphism

$$P_1(S_g) \times P_1(S_g) \longrightarrow \big(P_1(S_g)\big)_{\mathrm{Ab}} \times \big(P_1(S_g)\big)_{\mathrm{Ab}}, \tag{15}$$

where $(P_1(S_g))_{\mathrm{Ab}} \cong \mathbb{Z}^{2g}$ is the Abelianisation of $P_1(S_g)$. Since ρ, v and w belong to $P_2(S_g)$, we may write

$$\rho = \rho_1\rho_2, \qquad v = v_1 v_2 \quad \text{and} \quad w = w_1 w_2, \tag{16}$$

where for $i = 1, 2$, $\rho_i, v_i, w_i \in \mathbb{F}_i$, and $\mathbb{F}_i$ is as defined in Eq. (9). Given a word w in $\mathbb{F}_i$ written in terms of the basis $\{\rho_{i,k} \mid 1 \leqslant k \leqslant 2g\}$, let $\tilde{w}$ denote the word in $\mathbb{F}_j$, obtained by replacing each $\rho_{i,k}$ by $\rho_{j,k}$, where $j \in \{1, 2\}$ and $j \neq i$. The automorphism ι_σ of $P_2(S_g)$ given by conjugation by σ has the property that its restriction to $\mathbb{F}_1$ (resp. to $\mathbb{F}_2$) coincides with the map that sends w to $\tilde{w}$ (resp. to $B\tilde{w}B^{-1}$). The restriction of ι_σ to the intersection $\mathbb{F}_1 \cap \mathbb{F}_2$, which is the normal closure of B, is invariant under ι_σ. We have

$$\iota_\sigma(w) = \sigma w \sigma^{-1} = \begin{cases} \tilde{w} & \text{if } w \in \mathbb{F}_1, \\ B\tilde{w}B^{-1} & \text{if } w \in \mathbb{F}_2, \end{cases} \tag{17}$$

where in the first (resp. second) case, w is written in terms of the basis $\{\rho_{1,k} \mid 1 \leqslant k \leqslant 2g\}$ (resp. $\{\rho_{2,k} \mid 1 \leqslant k \leqslant 2g\}$) of $\mathbb{F}_1$ (resp. $\mathbb{F}_2$). We will later consider the automorphism induced by ι_σ on a quotient of $P_2(S_g)$ by a term of the lower central series.

Lemma 20. *With the notation introduced above, θ_τ factors as in diagram* (1) *if and only if for $i = 1, 2$, there exist $\rho_i, v_i, w_i \in \mathbb{F}_i$ such that*

$$B = \sigma\rho_1\rho_2\sigma^{-1}\rho_1\rho_2\sigma v_1 v_2 w_1 w_2 v_2^{-1} v_1^{-1}\sigma^{-1} w_2^{-1} w_1^{-1}, \tag{18}$$

or equivalently, such that

$$B = \tilde{\rho}_1 B\tilde{\rho}_2 B^{-1}\rho_1\rho_2\tilde{v}_1 B\tilde{v}_2 B^{-1}\tilde{w}_1 B\tilde{w}_2\tilde{v}_2^{-1}B^{-1}\tilde{v}_1^{-1}w_2^{-1}w_1^{-1}. \tag{19}$$

Furthermore, if we project Eq. (19) *onto each of the factors of $P_1(S_g) \times P_1(S_g)$ then the following equations hold in $P_1(S_g)$:*

$$\tilde{\rho}_2\rho_1\tilde{v}_2\tilde{w}_2\tilde{v}_2^{-1} = w_1 \quad \textit{and} \quad \tilde{\rho}_1\rho_2\tilde{v}_1\tilde{w}_1\tilde{v}_1^{-1} = w_2, \tag{20}$$

where by abuse of notation, we use the same notation for elements of $P_2(S_g)$ and their projection in $P_1(S_g)$.

Proof. Substituting Eq. (14) into Eq. (13) leads to $(\rho^{-1}\sigma)^2[w, \sigma v] = 1$, which is equivalent in turn to $(\rho^{-1}\sigma)(\rho^{-1}\sigma^{-1})\sigma^2[w, \sigma v] = 1$, and to $\sigma^2 = \sigma\rho\sigma^{-1}\rho[\sigma v, w]$. Substituting Eq. (16) into this last equation yields Eq. (18). Using Eq. (17), we obtain Eq. (19). The second part is also straightforward, using the fact that $\ker p_1 = \mathbb{F}_2$ and $\ker p_2 = \mathbb{F}_1$. $\square$

From Eq. (8), the two equations of (20) in $P_1(S_g)$ are equivalent respectively to the equations

$$\tilde{\rho}_2\rho_1\tilde{v}_2\tilde{w}_2\tilde{v}_2^{-1}z_1 = w_1 \quad \text{and} \quad \tilde{\rho}_1\rho_2\tilde{v}_1\tilde{w}_1\tilde{v}_1^{-1}z_2 = w_2 \quad \text{in } P_2(S_g), \tag{21}$$

where $z_1, z_2 \in N$. An easy calculation proves the following:

Lemma 21. *Eq.* (19) *may be rewritten in the form*:

$$\begin{aligned} B = {} & \big[\tilde{\rho}_1, B\tilde{\rho}_2B^{-1}\rho_1\big]\big(B\tilde{\rho}_2B^{-1}\rho_1\big[\tilde{\rho}_1\rho_2\tilde{v}_1, B\tilde{v}_2B^{-1}\big]\rho_1^{-1}B\tilde{\rho}_2^{-1}B^{-1}\big) \\ & \times \big(B\tilde{\rho}_2B^{-1}\rho_1B\tilde{v}_2B^{-1}\big[\tilde{\rho}_1\rho_2\tilde{v}_1\tilde{w}_1, B\tilde{w}_2\tilde{v}_2^{-1}B^{-1}\big]B\tilde{v}_2^{-1}B^{-1}\rho_1^{-1}B\tilde{\rho}_2^{-1}B^{-1}\big) \\ & \times \big[B\tilde{\rho}_2B^{-1}\rho_1B\tilde{v}_2\tilde{w}_2\tilde{v}_2^{-1}B^{-1}, \tilde{\rho}_1\rho_2\tilde{v}_1\tilde{w}_1\tilde{v}_1^{-1}w_2^{-1}\big] \\ & \times \big(\tilde{\rho}_1\rho_2\tilde{v}_1\tilde{w}_1\tilde{v}_1^{-1}w_2^{-1}\big)\big(B\tilde{\rho}_2B^{-1}\rho_1B\tilde{v}_2\tilde{w}_2\tilde{v}_2^{-1}B^{-1}w_1^{-1}\big). \end{aligned} \tag{22}$$

Remark 22. Observe that the commutators in Eq. (22) have the property that one of the terms belongs to $\mathbb{F}_1$, while the other belongs to $\mathbb{F}_2$. Consequently, each commutator belongs to N by Eq. (8).

Corollary 23. *The elements $\tilde{\rho}_1\rho_2\tilde{v}_1\tilde{w}_1\tilde{v}_1^{-1}w_2^{-1}$ and $B\tilde{\rho}_2B^{-1}\rho_1B\tilde{v}_2\tilde{w}_2\tilde{v}_2^{-1}B^{-1}w_1^{-1}$ of $P_2(S_g)$ belong to N. If we further project onto the Abelianisation* (*cf.* Eq. (15)), *then the projections of $\tilde{\rho}_2\rho_1$ and $\tilde{\rho}_1\rho_2$ belong to the commutator subgroup of the factors $P_1(S_g)\times\{1\}$ and $\{1\}\times P_1(S_g)$ of $P_1(S_g)\times P_1(S_g)$ respectively.*

Proof. From Remark 22, the commutators on the right-hand side of Eq. (22) belong to N, and hence the last line of this equation also belongs to N. Projecting each of the factors of this last line onto $P_1(S_g)\times P_1(S_g)$ and using Eq. (21) yield the first part of the corollary. For the second part, the projection of $\tilde{\rho}_1\rho_2\tilde{v}_1\tilde{w}_1\tilde{v}_1^{-1}w_2^{-1}$ onto the second factor of $(P_1(S_g))_{\mathrm{Ab}}\times(P_1(S_g))_{\mathrm{Ab}}$ via $P_1(S_g)\times P_1(S_g)$ yields $\tilde{\rho}_1\rho_2\tilde{w}_1w_2^{-1}=1$, where once more we do not distinguish notationally between an element of $P_2(S_g)$ and its projection in $(P_1(S_g))_{\mathrm{Ab}}\times(P_1(S_g))_{\mathrm{Ab}}$. Consider $\xi=\tilde{\rho}_2\rho_1\tilde{v}_2\tilde{w}_2\tilde{v}_2^{-1}w_1^{-1}\in P_2(S_g)$. By Eq. (21), $\xi\in N$. Now $\xi\in\mathbb{F}_1$, so $\iota_\sigma(\xi)=\rho_2\tilde{\rho}_1v_2w_2v_2^{-1}\tilde{w}_1^{-1}$ by Eq. (17), and since N is equal to the normal closure of B in $P_2(S_g)$, it is invariant under ι_σ. The projection of $\iota_\sigma(\xi)$ onto the second factor of $(P_1(S_g))_{\mathrm{Ab}}\times(P_1(S_g))_{\mathrm{Ab}}$ via $P_1(S_g)\times P_1(S_g)$ thus yields $\rho_2\tilde{\rho}_1w_2\tilde{w}_1^{-1}=1$. So in this factor of $(P_1(S_g))_{\mathrm{Ab}}$, we have $\tilde{\rho}_1\rho_2\tilde{w}_1=w_2$ and $\rho_2\tilde{\rho}_1w_2=\tilde{w}_1$. Substituting the second of these equations into the first gives $1=\rho_2\tilde{\rho}_1\tilde{\rho}_1\rho_2=(\tilde{\rho}_1\rho_2)^2$ since $(P_1(S_g))_{\mathrm{Ab}}$ is Abelian. The fact that the group $(P_1(S_g))_{\mathrm{Ab}}\cong\mathbb{Z}^{2g}$ is torsion free implies that $\tilde{\rho}_1\rho_2=1$ in $(P_1(S_g))_{\mathrm{Ab}}$. Hence $(1,\tilde{\rho}_1\rho_2)$, considered as an element of $\{1\}\times P_1(S_g)$ belongs to its commutator subgroup. A similar argument proves the result for $\tilde{\rho}_2\rho_1$. □

Let $k\in\{1,2\}$ and $i\in\{1,\dots,2g\}$. Using Theorem 19, it is not hard to see that if x is an element of $P_2(S_g)$ written as a word w in the generators of that theorem then the sum of the exponents of $\rho_{k,i}$ appearing in w, which we denote by $|x|_{\rho_{k,i}}$, is a well-defined integer that does not depend on the choice of w.

Lemma 24. *Let $i\in\{1,\dots,2g\}$.*

(a) *Let $k\in\{1,2\}$. The map $P_2(S_g)\longrightarrow\mathbb{Z}$ given by $x\longmapsto|x|_{\rho_{k,i}}$ is a homomorphism whose kernel contains N.*
(b) *Given a solution of Eq.* (22), *we have*:

$$|\rho_1|_{\rho_{1,i}}+|\rho_2|_{\rho_{2,i}}=-|w_1|_{\rho_{1,i}}+|w_2|_{\rho_{2,i}} \quad \textit{and}$$
$$|\rho_1|_{\rho_{1,i}}+|\rho_2|_{\rho_{2,i}}=|w_1|_{\rho_{1,i}}-|w_2|_{\rho_{2,i}}.$$

Hence

$$|\rho_1|_{\rho_{1,i}}=-|\rho_2|_{\rho_{2,i}} \quad \textit{and} \quad |w_1|_{\rho_{1,i}}=|w_2|_{\rho_{2,i}}. \tag{23}$$

Proof. (a) follows easily using the presentation of $P_2(S_g)$ given in Theorem 19.
(b) This is a consequence of applying part (a) to Eq. (21), and using the fact that $|x|_{\rho_{1,i}}=|\tilde{x}|_{\rho_{2,i}}$ for all $x\in P_2(S_g)$. □

Let $G=P_2(S_g)$, and for $i\in\mathbb{N}$, let $\Gamma_i(G)$ denote the terms of its lower central series. Recall that by definition, $\Gamma_1(G)=G$ and $\Gamma_{i+1}(G)=[\Gamma_i(G),G]$ for all $i\in\mathbb{N}$. By Corollary 23, Eq. (22) may be interpreted as a relation in $\Gamma_2(P_2(S_g))$. We shall study this equation by means of its projection onto $K\otimes\mathbb{Z}_2$, where K is a certain quotient of $\Gamma_2(G)/\Gamma_3(G)$, which we shall define presently. We first recall some properties of $G/\Gamma_3(G)$.

Lemma 25. *We have the following relations in the group $G/\Gamma_3(G)$*:

(a) *$[ab,c]=[a,c][b,c]$, $[a,bc]=[a,b][a,c]$ and $[a^s,b^t]=[a,b]^{st}$ for all $a,b,c\in G/\Gamma_3(G)$ and all $s,t\in\mathbb{Z}$.*
(b) *The automorphism of $G/\Gamma_3(G)$ induced by ι_σ is given by the map which sends the class of a word w in G to the class of the word $\tilde{w}$.*
(c) *Let $1\leqslant i,j\leqslant 2g$. In $G/\Gamma_3(G)$ we have that $[\rho_{2,i+1},\rho_{1,i}]=B^{-1}$ and $[\rho_{2,i},\rho_{1,i+1}]=B$ for i odd, and $[\rho_{2,i},\rho_{1,j}]=1$ otherwise* (*notationally, we do not distinguish between an element of G and its class in $G/\Gamma_3(G)$*).

Proof. Part (a) is a consequence of the well-known formulas $[ab,c]=a[b,c]a^{-1}[a,c]$ and $[a,bc]=[a,b]b[a,c]b^{-1}$, and the fact that $\Gamma_2(G)/\Gamma_3(G)$ is central in $G/\Gamma_3(G)$. The fact that $[a^s,b^t]=[a,b]^{st}$ then follows by an inductive argument. Part (b) is a consequence of the description of the automorphism ι_σ given by Eq. (17), and the fact that the class of w is the same as the class of BwB^{-1} in $G/\Gamma_3(G)$ because $B\in\Gamma_2(G)$. Part (c) follows from the presentation of $P_2(S_g)$ given in Theorem 19, using once more the fact that $B\in\Gamma_2(G)$. □

Proposition 26. *The projection of Eq.* (22) *onto $G/\Gamma_3(G)$ is given by*:

$$B=[\tilde v_1,\tilde w_2]\big[\tilde w_1,\tilde v_2^{-1}\big]\big(\tilde\rho_1\rho_2\tilde v_1\tilde w_1\tilde v_1^{-1}w_2^{-1}\big)\big(\tilde\rho_2\rho_1\tilde v_2\tilde w_2\tilde v_2^{-1}w_1^{-1}\big). \tag{24}$$

Proof. First note by Theorem 19 that $G_{\mathrm{Ab}}=(P_1(S_g))_{\mathrm{Ab}}\times(P_1(S_g))_{\mathrm{Ab}}\cong\mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$, where a basis of the first (resp. second) $(P_1(S_g))_{\mathrm{Ab}}$-factor consists of the images of $\rho_{1,i}$ (resp. $\rho_{2,i}$), for $i=1,\dots,2g$. The element $\tilde\rho_1\rho_2$ of G belongs to $\mathbb{F}_2$, and so $|\tilde\rho_1\rho_2|_{\rho_{1,i}}=0$ for all $1\leqslant i\leqslant 2g$. Further,

$$|\tilde\rho_1\rho_2|_{\rho_{2,i}}=|\tilde\rho_1|_{\rho_{2,i}}+|\rho_2|_{\rho_{2,i}}=|\rho_1|_{\rho_{1,i}}+|\rho_2|_{\rho_{2,i}}=0$$

by Eq. (23). Thus $|\tilde\rho_1\rho_2|_{\rho_{k,i}}=0$ for all $k\in\{1,2\}$ and $i\in\{1,\dots,2g\}$. This implies that $\tilde\rho_1\rho_2\in\Gamma_2(G)$. A similar argument shows that $\tilde\rho_2\rho_1\in\Gamma_2(G)$.

We now take Eq. (22) modulo $\Gamma_3(G)$. Since $\tilde\rho_1\rho_2, B, \tilde\rho_2\rho_1\in\Gamma_2(G)$, and using the fact that $\Gamma_2(G)/\Gamma_3(G)$ is central in $G/\Gamma_3(G)$ as well as Lemma 25(a), we obtain

$$\begin{aligned}B&=[\tilde v_1,\tilde v_2]\big[\tilde v_1\tilde w_1,\tilde w_2\tilde v_2^{-1}\big]\big[\tilde v_2\tilde w_2\tilde v_2^{-1},\tilde v_1\tilde w_1\tilde v_1^{-1}w_2^{-1}\big]\big(\tilde\rho_1\rho_2\tilde v_1\tilde w_1\tilde v_1^{-1}w_2^{-1}\big)\big(\tilde\rho_2\rho_1\tilde v_2\tilde w_2\tilde v_2^{-1}w_1^{-1}\big)\\&=[\tilde v_1,\tilde v_2]\big[\tilde v_1\tilde w_1,\tilde w_2\tilde v_2^{-1}\big]\big[\tilde w_2,\tilde w_1w_2^{-1}\big]\big(\tilde\rho_1\rho_2\tilde v_1\tilde w_1\tilde v_1^{-1}w_2^{-1}\big)\big(\tilde\rho_2\rho_1\tilde v_2\tilde w_2\tilde v_2^{-1}w_1^{-1}\big)\\&=[\tilde v_1,\tilde v_2][\tilde v_1,\tilde w_2]\big[\tilde v_1,\tilde v_2^{-1}\big]\big[\tilde w_1,\tilde v_2^{-1}\big]\big[\tilde w_2,w_2^{-1}\big]\big(\tilde\rho_1\rho_2\tilde v_1\tilde w_1\tilde v_1^{-1}w_2^{-1}\big)\big(\tilde\rho_2\rho_1\tilde v_2\tilde w_2\tilde v_2^{-1}w_1^{-1}\big)\\&=[\tilde v_1,\tilde w_2]\big[\tilde w_1,\tilde v_2^{-1}\big]\big[\tilde w_2,w_2^{-1}\big]\big(\tilde\rho_1\rho_2\tilde v_1\tilde w_1\tilde v_1^{-1}w_2^{-1}\big)\big(\tilde\rho_2\rho_1\tilde v_2\tilde w_2\tilde v_2^{-1}w_1^{-1}\big)\end{aligned} \tag{25}$$

in $G/\Gamma_3(G)$. Using Lemma 25(c), we see that in $G/\Gamma_3(G)$, the only non-trivial contributions in $[\tilde w_2,w_2^{-1}]$ come from terms of the form $[\rho_{1,i},\rho_{2,i+1}]=B$ and $[\rho_{1,i+1},\rho_{2,i}]=B^{-1}$ for i odd. Thus in $G/\Gamma_3(G)$, the B-coefficient of $[\tilde w_2,w_2^{-1}]$ is given by:

$$\begin{aligned}-\sum_{\substack{1\leqslant i\leqslant 2g\\ i\text{ odd}}}|\tilde w_2|_{\rho_{1,i}}|w_2|_{\rho_{2,i+1}}+\sum_{\substack{1\leqslant i\leqslant 2g\\ i\text{ odd}}}|\tilde w_2|_{\rho_{1,i+1}}|w_2|_{\rho_{2,i}}&=-\sum_{\substack{1\leqslant i\leqslant 2g\\ i\text{ odd}}}|w_2|_{\rho_{2,i}}|w_2|_{\rho_{2,i+1}}\\&\quad+\sum_{\substack{1\leqslant i\leqslant 2g\\ i\text{ odd}}}|w_2|_{\rho_{2,i+1}}|w_2|_{\rho_{2,i}}=0.\end{aligned}$$

Hence $[\tilde w_2,w_2^{-1}]=1$ in $G/\Gamma_3(G)$, and Eq. (25) thus reduces to Eq. (24). □

Remark 27. We summarise some properties of the factors of Eq. (24):

(a) The factors $[\tilde v_1,\tilde w_2], [\tilde w_1,\tilde v_2^{-1}]$ belong to N because $\tilde v_1,\tilde w_1\in\mathbb{F}_2$ and $\tilde w_2,\tilde v_2^{-1}\in\mathbb{F}_1$.
(b) The factors $\tilde\rho_1\rho_2\tilde v_1\tilde w_1\tilde v_1^{-1}w_2^{-1}$ and $\tilde\rho_2\rho_1\tilde v_2\tilde w_2\tilde v_2^{-1}w_1^{-1}$ belong to N since their images in $P_1(S_g)\times P_1(S_g)$ belong to the subgroups $P_1(S_g)\times\{1\}$, $\{1\}\times P_1(S_g)$ respectively, and B projects to the trivial element.
(c) The elements $(\tilde\rho_1\rho_2), [\tilde v_1,\tilde w_1], \tilde w_1w_2^{-1}$ belong to $\mathbb{F}_2\cap\Gamma_2(G)$, and $\tilde\rho_2\rho_1, [\tilde v_2,\tilde w_2], \tilde w_2w_1^{-1}$ belong to $\mathbb{F}_1\cap\Gamma_2(G)$.

We now compute the group $\Gamma_2(G)/\Gamma_3(G)$.

Proposition 28.

(a) *The group $\Gamma_2(G)/\Gamma_3(G)$ is free Abelian of rank $2g(2g-1)-1$; a basis is given by the classes of the elements of $\{e_{k,i,j}, B \mid k=1,2,\ 1\leqslant i<j\leqslant 2g$ and $i\neq 2g-1\}$, where $e_{k,i,j}=[\rho_{k,i},\rho_{k,j}]$ for all $k=1,2$ and $1\leqslant i<j\leqslant 2g$.*
(b) *Given $v,w\in P_2(S_g)$, the commutator $[v,w]$, considered as an element of $G/\Gamma_3(G)$, belongs to $\Gamma_2(G)/\Gamma_3(G)$, and*
 (i) *$|[v,w]|_{e_{k,i,j}}=d_{k,i,j}(v,w)$ for $k=1,2$, $1\leqslant i<j\leqslant 2g$ and $(i,j)\neq(2t-1,2t)$ for all $1\leqslant t\leqslant g$,*
 (ii) *$|[v,w]|_{e_{k,2i-1,2i}}=d_{k,2i-1,2i}(v,w)-d_{k,2g-1,2g}(v,w)$ for all $k=1,2$ and $1\leqslant i<g$,*
 (iii) *$|[v,w]|_B=-d_{1,2g-1,2g}(v,w)-d_{2,2g-1,2g}(v,w)+\sum_{1\leqslant i\leqslant g}a_{2i-1,2i}(v,w)$,*
 where $|u|_B$ and $|u|_{e_{k,i,j}}$ denote the exponent sum of the element $u\in\Gamma_2(G)/\Gamma_3(G)$ with respect to the basis elements of part (a), *and where*

$$d_{k,i,j}(v,w)=\begin{vmatrix}|v|_{\rho_{k,i}} & |v|_{\rho_{k,j}}\\ |w|_{\rho_{k,i}} & |w|_{\rho_{k,j}}\end{vmatrix}\quad\text{and}\quad a_{2i-1,2i}(v,w)=\begin{vmatrix}|v|_{\rho_{2,2i-1}} & |v|_{\rho_{2,2i}}\\ |w|_{\rho_{1,2i-1}} & |w|_{\rho_{1,2i}}\end{vmatrix}+\begin{vmatrix}|v|_{\rho_{1,2i-1}} & |v|_{\rho_{1,2i}}\\ |w|_{\rho_{2,2i-1}} & |w|_{\rho_{2,2i}}\end{vmatrix}.$$

Proof. (a) Let G_{12} denote the group defined by a presentation with generating set

$$\{a_{k,1},\dots,a_{k,2g},b_{k,i,j},\beta \mid k=1,2,\ 1\leqslant i<j\leqslant 2g,\ i\neq 2g-1\},$$

and defining relations:

(I) $b_{k,i,j}=[a_{k,i},a_{k,j}]$ for $k=1,2$, $1\leqslant i<j\leqslant 2g$, where $i\neq 2g-1$.
(II) $\beta=[a_{k,1},a_{k,2}^{-1}]\cdots[a_{k,2g-1},a_{k,2g}^{-1}]=[a_{2,2i-1},a_{1,2i}]=[a_{1,2i-1},a_{2,2i}]$ for all $k\in\{1,2\}$ and $1\leqslant i\leqslant g$.
(III) $[a_{1,i},a_{2,j}]=1$ for all $1\leqslant i,j\leqslant 2g$, where $\{i,j\}\neq\{2t-1,2t\}$ for all $1\leqslant t\leqslant g$.
(IV) For $k=1,2$ and $1\leqslant i<j\leqslant 2g$, the elements $b_{k,i,j}$ and β belong to the centre of the group G_{12}.

We will construct a homomorphism from G_{12} to $G/\Gamma_3(G)$ and conversely. To define a homomorphism from G_{12} to $G/\Gamma_3(G)$, consider the map defined on the generators of G_{12} by $\beta\longmapsto B$, $a_{k,l}\longmapsto\rho_{k,l}$, and $b_{k,i,j}\longmapsto e_{k,i,j}$ for all $k\in\{1,2\}$, $1\leqslant l\leqslant 2g$ and $1\leqslant i<j\leqslant 2g$. Using Theorem 19 and Lemma 25, a straightforward calculation shows that the images of the relations of the presentation of G_{12} are satisfied in the group $G/\Gamma_3(G)$, and thus we obtain a homomorphism from G_{12} onto $G/\Gamma_3(G)$. Conversely, consider the map from $\phi: G\longrightarrow G_{12}$ defined on the generators of G by $\rho_{k,j}\longmapsto a_{k,j}$ for all $k\in\{1,2\}$ and $j\in\{1,\dots,2g\}$. Since $[a_{k,2i-1},a_{k,2i}^{-1}]=a_{k,2i}^{-1}(a_{k,2i}a_{k,2i-1}a_{k,2i}^{-1}a_{k,2i-1}^{-1})a_{k,2i}=a_{k,2i}^{-1}b_{k,2i-1,2i}^{-1}a_{k,2i}=b_{k,2i-1,2i}^{-1}$ for all $k\in\{1,2\}$ and $1\leqslant i\leqslant g$, we conclude from relations (I) and (IV) above that β is central in G_{12}. Taking the image of relation (I) of Theorem 19 shows that $\phi(B)=\beta$, and applying ϕ to the remaining relations of G and using these two facts about β, we conclude that ϕ extends to a homomorphism of G onto G_{12}. Since β and the $b_{k,i,j}$ belong to the centre of G_{12}, we see that $\Gamma_2(G_{12})$ is the Abelian group generated by the $b_{k,i,j}$, and that $\Gamma_3(G_{12})$ is trivial. It follows that ϕ factors through $G/\Gamma_3(G)$. Since $\phi([a_{k,i},a_{k,j}])=b_{k,i,j}$ for all $k\in\{1,2\}$ and $1\leqslant i<j\leqslant 2g$, we thus obtain two homomorphisms between G_{12} to $G/\Gamma_3(G)$, where one is the inverse of the other. In particular, G_{12} and $G/\Gamma_3(G)$ are isomorphic, and hence $\Gamma_2(G_{12})$ is isomorphic to $\Gamma_2(G)/\Gamma_3(G)$. By considering the Abelianisation of G_{12}, one may check using the relations (I)–(IV) above that $\Gamma_2(G_{12})$ is a free Abelian subgroup of G_{12} with basis $\{\beta,b_{k,i,j}\mid k=1,2,\ 1\leqslant i<j\leqslant 2g,\ i\neq 2g-1\}$, and this proves part (a).

(b) Let $v,w\in G$, and consider their classes modulo $\Gamma_3(G)$, which we also denote by v,w respectively. Then in $G/\Gamma_3(G)$, we have

$$v=\left(\prod_{k=1}^{2}\left(\prod_{i=1}^{2g}\rho_{k,i}^{|v|_{\rho_{k,i}}}\right)\right)\cdot v' \quad\text{and}\quad w=\left(\prod_{k=1}^{2}\left(\prod_{i=1}^{2g}\rho_{k,i}^{|w|_{\rho_{k,i}}}\right)\right)\cdot w', \tag{26}$$

where $v',w'\in\Gamma_2(G)/\Gamma_3(G)$. We now calculate the coefficients of $[v,w]$ in the given basis of $\Gamma_2(G)/\Gamma_3(G)$, noting that v',w' may be ignored since they are central in $G/\Gamma_3(G)$. From Lemma 25 and part (a), if $1\leqslant i<j\leqslant 2g$ and $k,l\in\{1,2\}$, we have that

$$\left[\rho_{k,i}^{s},\rho_{l,j}^{t}\right]=\begin{cases} e_{k,i,j}^{st} & \text{if } k=l,\\ B^{st} & \text{if } k\neq l \text{ and } (i,j)=(2t-1,2t) \text{ for some } t\in\{1,\dots,g\},\\ 1 & \text{if } k\neq l \text{ and } (i,j)\neq(2t-1,2t) \text{ for all } t\in\{1,\dots,g\},\end{cases} \tag{27}$$

and from relation (I) of Theorem 19 and Lemma 25, we have

$$[\rho_{k,2g-1},\rho_{k,2g}]=e_{k,1,2}^{-1}\cdots e_{k,2g-3,2g-2}^{-1}B^{-1}. \tag{28}$$

Thus if $(i,j)\neq(2t-1,2t)$ for all $t\in\{1,\dots,g\}$, we obtain

$$\big|[v,w]\big|_{e_{k,i,j}}=|v|_{\rho_{k,i}}|w|_{\rho_{k,j}}-|v|_{\rho_{k,j}}|w|_{\rho_{k,i}}=d_{k,i,j}(v,w)$$

obtained from the coefficients of $\rho_{k,i}$ and $\rho_{k,j}$ in Eq. (26) which gives (i), while if $i\in\{1,\dots,g-1\}$, we obtain an extra term in the expression for the coefficient of $e_{k,2i-1,2i}$ from the coefficients of $\rho_{k,2g-1}$ and $\rho_{k,2g}$ via Eq. (28), and so

$$\big|[v,w]\big|_{e_{k,2i-1,2i}}=d_{k,2i-1,2i}(v,w)-d_{k,2g-1,2g}(v,w),$$

which gives (ii). Finally, the B-coefficient of $[v,w]$ is obtained from three different types of expression: the first emanates from the coefficients of $\rho_{1,2i-1}$ and $\rho_{2,2i}$ for each $1\leqslant i\leqslant g$, which gives rise to a coefficient

$$\begin{vmatrix} |v|_{\rho_{1,2i-1}} & |v|_{\rho_{2,2i}}\\ |w|_{\rho_{1,2i-1}} & |w|_{\rho_{2,2i}}\end{vmatrix},$$

the second comes from the coefficients of $\rho_{2,2i-1}$ and $\rho_{1,2i}$ for each $1\leqslant i\leqslant g$, which gives rise to a coefficient

$$\begin{vmatrix} |v|_{\rho_{2,2i-1}} & |v|_{\rho_{1,2i}}\\ |w|_{\rho_{2,2i-1}} & |w|_{\rho_{1,2i}}\end{vmatrix},$$

and the third is given by the coefficient of $e_{k,2g-1,2g}$ via Eq. (28) for $k \in \{1,2\}$, which yields a coefficient $-d_{1,2g-1,2g}(v,w) - d_{2,2g-1,2g}(v,w)$. The sum of the first and second coefficients is equal to $a_{2i-1,2i}(v,w)$. Taking the sum of all of these coefficients leads to $|[v,w]|_B$ given in (ii), and this completes the proof of the proposition. □

Using Proposition 28, we are now in a position to prove Theorem 11, which will follow easily from Proposition 29. Consider the quotient of $\Gamma_2(G)/\Gamma_3(G)$ obtained by identifying $e_{1,i,j}$ with $e_{2,i,j}$ for all $1 \leqslant i < j \leqslant 2g$ and $i \neq 2g-1$. We denote this quotient by Q, and the image of $e_{1,i,j}$ and $e_{2,i,j}$ in Q by $e_{i,j}$. By Proposition 28, the group $\Gamma_2(G)/\Gamma_3(G)$ is the direct sum of three free Abelian subgroups H, $\langle B\rangle$ and L, where $\{[\rho_{k,2i-1},\rho_{k,2i}] \mid k=1,2,\ 1 \leqslant i < g\}$ is a basis of H, $\{B\}$ is a basis of $\langle B\rangle$, and

$$\bigl\{[\rho_{k,i},\rho_{k,j}] \bigm| k=1,2,\ 1 \leqslant i < j \leqslant 2g \text{ and } (i,j) \neq (2t-1,2t) \text{ for all } t \in \{1,\dots,g\}\bigr\}$$

is a basis of L. Moreover, H (resp. L) is the direct sum $H_1 \oplus H_2$ (resp. $L_1 \oplus L_2$) where for $k=1,2$, $\{[\rho_{k,2i-1},\rho_{k,2i}] \mid 1 \leqslant i < g\}$ is a basis of H_k, and

$$\bigl\{[\rho_{k,i},\rho_{k,j}] \bigm| 1 \leqslant i < j \leqslant 2g \text{ and } (i,j) \neq (2t-1,2t) \text{ for all } t \in \{1,\dots,g\}\bigr\}$$

is a basis of L_k. Observe that the image of H_1 (resp. L_1) in Q coincides with the image of H_2 (resp. L_2). Let $\overline{Q} = Q \otimes \mathbb{Z}_2$, and let $\overline{B}$, $\overline{H}$ and $\bar{L}$ denote the projection of B, H and L respectively in $\overline{Q}$.

Proposition 29. *Eq.* (18) *has no solution in* $P_2(S_g)$.

Proof. We saw previously that Eq. (18) is equivalent in turn to Eq. (19), and to Eq. (22), and that its projection onto $G/\Gamma_3(G)$ is given by Eq. (24). So to show that Eq. (18) has no solution in $P_2(S_g)$ it suffices to show that the projection of Eq. (24) onto the group $\overline{Q}$ has no solution. Now $\overline{H}, \langle\overline{B}\rangle$ and $\bar{L}$ are $\mathbb{Z}_2$-vector spaces of dimension equal to half the rank of H (as a free Abelian group), 1, and half the rank of L (as a free Abelian group) respectively, and we have a decomposition of $\overline{Q}$ as $\overline{H} \oplus \langle\overline{B}\rangle \oplus \bar{L}$. We have that $\Gamma_2(\overline{Q})$ is isomorphic to a sum of $\mathbb{Z}_2$'s; a basis is given by the set $\{\bar{e}_{i,j}, \overline{B} \mid 1 \leqslant i < j \leqslant 2g,\ i \neq 2g-1\}$, where $\bar{e}_{i,j}$ denotes the projection (from Q to $\overline{Q}$) of $e_{i,j}$. From now on we study the projection of Eq. (24) onto $\overline{Q}$ (apart from the basis elements of $\overline{Q}$, notationally we do not distinguish between elements of $\Gamma_2(G)/\Gamma_3(G)$ and their projection into $\overline{Q}$):

$$\overline{B} = [\tilde{v}_1,\tilde{w}_2]\bigl[\tilde{w}_1,\tilde{v}_2^{-1}\bigr](\tilde{\rho}_1\rho_2)[\tilde{v}_1,\tilde{w}_1]\bigl(\tilde{w}_1 w_2^{-1}\bigr)(\tilde{\rho}_2\rho_1)[\tilde{v}_2,\tilde{w}_2]\bigl(\tilde{w}_2 w_1^{-1}\bigr), \tag{29}$$

where each of the factors belongs to $\Gamma_2(\overline{Q})$, using Remark 27(c), and so commute pairwise. We now examine the various terms appearing in Eq. (29).

(a) We have $(\tilde{\rho}_1\rho_2)(\tilde{\rho}_2\rho_1) = [\tilde{\rho}_1,\rho_2](\rho_2\tilde{\rho}_1)(\tilde{\rho}_2\rho_1)$. We claim that $[\tilde{\rho}_1,\rho_2]=1$ in $\Gamma_2(G)/\Gamma_3(G)$, and so in $\Gamma_2(\overline{Q})$. To prove the claim, we calculate the coefficients of $[\tilde{\rho}_1,\rho_2]$ on the basis of $\Gamma_2(G)/\Gamma_3(G)$ using Proposition 28(b). First recall that $\tilde{\rho}_1,\rho_2 \in \mathbb{F}_2$, so

$$|\tilde{\rho}_1|_{\rho_{1,i}} = |\rho_2|_{\rho_{1,i}} = 0 \quad \text{for all } 1 \leqslant i \leqslant 2g \tag{30}$$

and hence $d_{1,i,j}(\tilde{\rho}_1,\rho_2) = |[\tilde{\rho}_1,\rho_2]|_{e_{1,i,j}} = 0$ for all $1 \leqslant i < j \leqslant 2g$ by Proposition 28(i) and (ii). Further,

$$\begin{aligned} d_{2,i,j}(\tilde{\rho}_1,\rho_2) &= |\tilde{\rho}_1|_{\rho_{2,i}}|\rho_2|_{\rho_{2,j}} - |\tilde{\rho}_1|_{\rho_{2,j}}|\rho_2|_{\rho_{2,i}} = |\rho_1|_{\rho_{1,i}}|\rho_2|_{\rho_{2,j}} - |\rho_1|_{\rho_{1,j}}|\rho_2|_{\rho_{2,i}} \\ &= -|\rho_2|_{\rho_{2,i}}|\rho_2|_{\rho_{2,j}} + |\rho_2|_{\rho_{2,j}}|\rho_2|_{\rho_{2,i}} \quad \text{by Eq. (23)} \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} a_{2i-1,2i}(\tilde{\rho}_1,\rho_2) &= |\tilde{\rho}_1|_{\rho_{2,2i-1}}|\rho_2|_{\rho_{1,2i}} - |\tilde{\rho}_1|_{\rho_{2,2i}}|\rho_2|_{\rho_{1,2i-1}} + |\tilde{\rho}_1|_{\rho_{1,2i-1}}|\rho_2|_{\rho_{2,2i}} - |\tilde{\rho}_1|_{\rho_{1,2i}}|\rho_2|_{\rho_{2,2i-1}} \\ &= 0, \end{aligned}$$

using Eq. (30). So $|[\tilde{\rho}_1,\rho_2]|_B = 0$, and we conclude that $[\tilde{\rho}_1,\rho_2]=1$ in $\Gamma_2(G)/\Gamma_3(G)$, which proves the claim.

(b) Consider the terms $\rho_2\tilde{\rho}_1$ and $\tilde{\rho}_2\rho_1$. As an element of G, we have that $\rho_2\tilde{\rho}_1 \in \mathbb{F}_2$, and so $\iota_\sigma(\rho_2\tilde{\rho}_1) = B\tilde{\rho}_2\rho_1 B^{-1}$ by Eq. (17). Since $\Gamma_3(G)$ is characteristic in G, ι_σ induces an automorphism of $G/\Gamma_3(G)$ which we also denote by ι_σ. But $B \in \Gamma_2(G)$, so $\iota_\sigma(\rho_2\tilde{\rho}_1) = \tilde{\rho}_2\rho_1$ in $\Gamma_2(G)/\Gamma_3(G)$. Now

$$\iota_\sigma(e_{k,i,j}) = e_{k',i,j} \quad \text{for all } 1 \leqslant i < j \leqslant 2g \text{ and } k,k' \in 1,2, \text{ where } k \neq k'. \tag{31}$$

So $|\rho_2\tilde{\rho}_1\tilde{\rho}_2\rho_1|_B = |\iota_\sigma(\tilde{\rho}_2\rho_1)\tilde{\rho}_2\rho_1|_B$, and since $\iota_\sigma(B)=B$, it follows that $|\rho_2\tilde{\rho}_1\tilde{\rho}_2\rho_1|_B$ is even. Hence the $\overline{B}$-coefficient of $\rho_2\tilde{\rho}_1\tilde{\rho}_2\rho_1$ is zero in $\overline{Q}$. Using Eq. (31), we see that

$$|\rho_2\tilde{\rho}_1\tilde{\rho}_2\rho_1|_{e_{1,i,j}} = |\rho_2\tilde{\rho}_1\tilde{\rho}_2\rho_1|_{e_{2,i,j}} \quad \text{for all } 1 \leqslant i < j \leqslant 2g,$$

hence the $\bar{e}_{i,j}$-coefficient of $\rho_2\tilde{\rho}_1\tilde{\rho}_2\rho_1$ is also zero in $\overline{Q}$, and thus $\rho_2\tilde{\rho}_1\tilde{\rho}_2\rho_1$ is trivial in $\overline{Q}$.

(c) Now consider $\tilde{w}_1 w_2^{-1}$ and $\tilde{w}_2 w_1^{-1}$. We have $\tilde{w}_1 w_2^{-1}\tilde{w}_2 w_1^{-1} = (\tilde{w}_1 w_2^{-1})^2 w_2\tilde{w}_1^{-1}\tilde{w}_2 w_1^{-1}$. Since it is a square, $(\tilde{w}_1 w_2^{-1})^2$ is certainly trivial in $\overline{Q}$. As in case (b) above, $w_2\tilde{w}_1^{-1}\tilde{w}_2 w_1^{-1}$ is also trivial in $\overline{Q}$.

Hence Eq. (29) reduces to $\overline{B} = [\tilde{v}_1, \tilde{w}_2][\tilde{w}_1, \tilde{v}_2^{-1}][\tilde{v}_1, \tilde{w}_1][\tilde{v}_2, \tilde{w}_2]$ in $\overline{Q}$. Using the results of Lemma 25, we can rewrite this as

$$\overline{B} = [\tilde{v}_1\tilde{v}_2, \tilde{w}_1\tilde{w}_2]. \tag{32}$$

First suppose that $g = 1$. In this case, the basis of $\Gamma_2(G)/\Gamma_3(G)$ is reduced to $\{B\}$. Since $\tilde{v}_2, \tilde{w}_2 \in \mathbb{F}_1$ and $\tilde{v}_1, \tilde{w}_1 \in \mathbb{F}_2$, and using Proposition 28(iii) and Eq. (23), in $\Gamma_2(G)/\Gamma_3(G)$ we have

$$\begin{aligned}
\big|[\tilde{v}_1\tilde{v}_2, \tilde{w}_1\tilde{w}_2]\big|_B &= d_{1,1,2}(\tilde{v}_1\tilde{v}_2, \tilde{w}_1\tilde{w}_2) + d_{2,1,2}(\tilde{v}_1\tilde{v}_2, \tilde{w}_1\tilde{w}_2) + c_{1,2}(\tilde{v}_1\tilde{v}_2, \tilde{w}_1\tilde{w}_2)\\
&= |\tilde{v}_2|_{\rho_{1,1}}|\tilde{w}_2|_{\rho_{1,2}} - |\tilde{v}_2|_{\rho_{1,2}}|\tilde{w}_2|_{\rho_{1,1}} + |\tilde{v}_1|_{\rho_{2,1}}|\tilde{w}_1|_{\rho_{2,2}} - |\tilde{v}_1|_{\rho_{2,2}}|\tilde{w}_1|_{\rho_{2,1}}\\
&\quad + |\tilde{v}_2|_{\rho_{1,1}}|\tilde{w}_1|_{\rho_{2,2}} - |\tilde{v}_2|_{\rho_{1,2}}|\tilde{w}_1|_{\rho_{2,1}} + |\tilde{v}_1|_{\rho_{2,1}}|\tilde{w}_2|_{\rho_{1,2}} - |\tilde{v}_1|_{\rho_{2,2}}|\tilde{w}_2|_{\rho_{1,1}}\\
&= |v_2|_{\rho_{2,1}}|w_2|_{\rho_{2,2}} - |v_2|_{\rho_{2,2}}|w_2|_{\rho_{2,1}} + |v_1|_{\rho_{1,1}}|w_1|_{\rho_{1,2}} - |v_1|_{\rho_{1,2}}|w_1|_{\rho_{1,1}}\\
&\quad + |v_2|_{\rho_{2,1}}|w_1|_{\rho_{1,2}} - |v_2|_{\rho_{2,2}}|w_1|_{\rho_{1,1}} + |v_1|_{\rho_{1,1}}|w_2|_{\rho_{2,2}} - |v_1|_{\rho_{1,2}}|w_2|_{\rho_{2,1}}\\
&= 2\big(|v_2|_{\rho_{2,1}}|w_1|_{\rho_{1,2}} - |v_2|_{\rho_{2,2}}|w_1|_{\rho_{1,1}} + |v_1|_{\rho_{1,1}}|w_1|_{\rho_{1,2}} - |v_1|_{\rho_{1,2}}|w_1|_{\rho_{1,1}}\big).
\end{aligned}$$

Thus $[\tilde{v}_1\tilde{v}_2, \tilde{w}_1\tilde{w}_2]$ is trivial in $\overline{Q}$, which contradicts Eq. (32). So let us suppose that $g > 1$. We will derive some restrictions on the element w_1 by studying Eq. (32) after projecting onto $\overline{Q}$. For $i = 1, \dots, 2g$, let $a_i = |v_1|_{\rho_{1,i}}$, $b_i = |\tilde{v}_2|_{\rho_{1,i}} = |v_2|_{\rho_{2,i}}$ and $c_i = |w_1|_{\rho_{1,i}}$, and let $d_i = a_i + b_i$. The right-hand side of Eq. (32) may be written as a product of two types of term: $[\tilde{v}_l, \tilde{w}_m]$, where $l, m \in \{1,2\}$ and $l \neq m$, and $[\tilde{v}_l, \tilde{w}_l]$, where $l \in \{1,2\}$. In the first case, considered as an element of $\Gamma_2(G)/\Gamma_3(G)$, $[\tilde{v}_l, \tilde{w}_m]$ gives rise only to terms in B by Eq. (27). In particular, in $\Gamma_2(G)/\Gamma_3(G)$ $|[\tilde{v}_l, \tilde{w}_m]|_{e_{k,i,j}} = 0$ for all $k \in \{1,2\}$ and $1 \leqslant i < j \leqslant 2g$, and so the $\bar{e}_{i,j}$-coefficient of $[\tilde{v}_l, \tilde{w}_m]$, considered as an element of $\overline{Q}$, is zero. It follows from Eq. (32) that in $\overline{Q}$, the $\bar{e}_{i,j}$-coefficient of $[\tilde{v}_1, \tilde{w}_1][\tilde{v}_2, \tilde{w}_2]$ is zero for all $(i,j) \neq (2t-1, 2t)$ and $1 \leqslant i < j \leqslant 2g$. But modulo 2, this coefficient is also given by the sum

$$\begin{aligned}
\big|[\tilde{v}_2, \tilde{w}_2]\big|_{e_{1,i,j}} + \big|[\tilde{v}_1, \tilde{w}_1]\big|_{e_{2,i,j}} &= \begin{vmatrix} |v_2|_{\rho_{2,i}} & |v_2|_{\rho_{2,j}} \\ |w_2|_{\rho_{2,i}} & |w_2|_{\rho_{2,j}} \end{vmatrix} + \begin{vmatrix} |v_1|_{\rho_{1,i}} & |v_1|_{\rho_{1,j}} \\ |w_1|_{\rho_{1,i}} & |w_1|_{\rho_{1,j}} \end{vmatrix}\\
&= \begin{vmatrix} |v_1|_{\rho_{1,i}} + |v_2|_{\rho_{2,i}} & |v_1|_{\rho_{1,j}} + |v_2|_{\rho_{2,j}} \\ |w_1|_{\rho_{1,i}} & |w_1|_{\rho_{1,j}} \end{vmatrix} = \begin{vmatrix} d_i & d_j \\ c_i & c_j \end{vmatrix},
\end{aligned} \tag{33}$$

using Eq. (23), so $\begin{vmatrix} \overline{d_i} & \overline{d_j} \\ \overline{c_i} & \overline{c_j} \end{vmatrix} = \bar{0} \pmod 2$.

Suppose that $\overline{c_i} = \bar{0} \pmod 2$ (so c_i is even) for all $i = 1, \dots, 2g$. Since $c_i = |w_1|_{\rho_{1,i}} = |w_2|_{\rho_{2,i}}$ by Eq. (23), it follows from Proposition 28(b) that $d_{k,l,m}(\tilde{v}_q, \tilde{w}_q)$ is even for all $k, q \in \{1,2\}$ and $1 \leqslant l < m \leqslant 2g$. Hence in $\overline{Q}$, the $\overline{B}$-coefficient of $[\tilde{v}_1, \tilde{w}_1][\tilde{v}_2, \tilde{w}_2]$ is zero, which contradicts Eq. (32). Thus there exists $1 \leqslant i \leqslant 2g$ such that $\overline{c_i} \neq \bar{0} \pmod 2$.

Using Proposition 28(b), a calculation similar to that of Eq. (33) shows that the $\bar{e}_{1,2}$- (resp. $\overline{B}$-) coefficient of $[\tilde{v}_1, \tilde{w}_1][\tilde{v}_2, \tilde{w}_2]$ is equal to $\begin{vmatrix} \overline{d_1} & \overline{d_2} \\ \overline{c_1} & \overline{c_2} \end{vmatrix} + \begin{vmatrix} \overline{d_{2g-1}} & \overline{d_{2g}} \\ \overline{c_{2g-1}} & \overline{c_{2g}} \end{vmatrix}$. By Eq. (32), this coefficient is equal to $\bar{0}$ (resp. $\bar{1}$), so $\begin{vmatrix} \overline{d_1} & \overline{d_2} \\ \overline{c_1} & \overline{c_2} \end{vmatrix} = \bar{1}$. Hence there exists $l \in \{1,2\}$ such that $\overline{c_l} \neq \bar{0}$. Now for all $m \in \{2g-1, 2g\}$, in $\overline{Q}$ the $\bar{e}_{l,m}$-coefficient of $[\tilde{v}_1, \tilde{w}_1][\tilde{v}_2, \tilde{w}_2]$ is zero by Eq. (32). By Eq. (33), this coefficient is equal to $\begin{vmatrix} \overline{d_l} & \overline{d_m} \\ \overline{c_l} & \overline{c_m} \end{vmatrix}$. Since $\overline{c_l} \neq \bar{0}$, this implies that $\begin{vmatrix} \overline{d_{2g-1}} & \overline{d_{2g}} \\ \overline{c_{2g-1}} & \overline{c_{2g}} \end{vmatrix} = \bar{0}$, but we know that this is the $\overline{B}$-coefficient in $\overline{Q}$ of $[\tilde{v}_1, \tilde{w}_1][\tilde{v}_2, \tilde{w}_2]$. This contradicts Eq. (32), and completes the proof of the proposition. □

Proof of Theorem 11. Consider the homomorphism $\theta_\tau : \pi_1(N_3) \longrightarrow \mathbb{Z}_2$. Up to equivalence, we may suppose that θ_τ satisfies one of the three conditions (a)–(c) given at the beginning of the discussion of this subcase (4).

In case (a), we have $\theta_\tau(v) = \bar{0}$. We thus obtain a factorisation of diagram (1) as in Theorem 5, and so by Proposition 13, the Borsuk–Ulam property does not hold for the triple (X, τ, S_g). In case (b), we have $\theta_\tau(v) = \bar{1}$ and $\theta_\tau(a_1) = \theta_\tau(a_2) = \bar{0}$, and setting $\phi(v) = \sigma$, $\phi(a_1) = \rho_{1,1}^{-1}$ and $\phi(a_2) = \rho_{2,2}$ defines a factorisation of diagram (1) by the first relation of (V) of Theorem 19. Applying once more Proposition 13, we see that the Borsuk–Ulam property does not hold for the triple (X, τ, S_g).

Finally, consider case (c), so $\theta_\tau(v) = \theta_\tau(a_2) = \bar{1}$ and $\theta_\tau(a_1) = \bar{0}$. It follows from Proposition 29 that the non-existence of a solution to Eq. (18) implies the non-existence of a solution of Eq. (13), and hence by Proposition 13, there is no factorisation of the diagram (1) by a homomorphism ϕ. This completes the proof of the theorem. □

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Appendix A

The purpose of this appendix is to reduce the number of cases to be analysed. The results presented here are known and are basically contained in [1]. For the benefit of the reader, we summarise these results and write them in a form that is more suitable for our purposes. Our problem is that of studying the existence of a solution to the algebraic factorisation problem presented in diagram (1) of Proposition 13. Using the notion of equivalence introduced at the end of Section 2, our goal is to reduce the number of surjective homomorphisms $\theta_\tau : \pi_1(X/\tau) \longrightarrow \mathbb{Z}_2$ to be analysed, where $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected surface without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$. We consider two cases, the first (resp. second) being that where the surface is orientable (resp. non-orientable). In the whole of this appendix, X will be a finite-dimensional CW-complex equipped with a free cellular involution τ.

Proposition 30. *Let $\pi_1(X/\tau)$ be isomorphic to the fundamental group of a compact, connected, orientable surface without boundary different from $\mathbb{S}^2$ of genus h, and consider the presentation of $\pi_1(X/\tau)$ given by*

$$\langle a_1, a_2, \ldots, a_{2h-1}, a_{2h} \mid [a_1, a_2] \cdots [a_{2h-1}, a_{2h}] \rangle. \tag{34}$$

The existence of a solution to the algebraic factorisation problem of diagram (1) *of Proposition* 13 *does not depend on the choice of surjective homomorphism $\theta_\tau : \pi_1(X/\tau) \longrightarrow P_2(S_g)$. In particular, it suffices to study the case $\theta_\tau(a_1) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for all $1 < i \leqslant 2h$.*

Proof. The following identities show that if $[a_1, a_2] \cdots [a_{2h-1}, a_{2h}]$ is a product of commutators as in Eq. (34) where $\theta_\tau(a_i) \neq \bar{0}$ for some $1 \leqslant i \leqslant 2h$ then $\pi_1(X/\tau)$ admits a presentation

$$\langle a'_1, a'_2, \ldots, a'_{2h-1}, a'_{2h} \mid [a'_1, a'_2] \cdots [a'_{2h-1}, a'_{2h}] \rangle,$$

where

$$[a_1, a_2] \cdots [a_{2h-1}, a_{2h}] = [a'_1, a'_2] \cdots [a'_{2h-1}, a'_{2h}], \quad \text{with } \theta_\tau(a'_1) = \bar{1} \text{ and } \theta_\tau(a'_i) = \bar{0} \text{ for all } 1 < i \leqslant 2h. \tag{35}$$

(1) Let $(a^*, b^*) = (a, ba)$. Then $[a, b] = [a^*, b^*]$, and we may assume that either $\theta_\tau(a^*)$ or $\theta_\tau(b^*)$ is zero.
(2) Let $(a^*, b^*) = (aba^{-1}, a^{-1})$. Then $[a, b] = [a^*, b^*]$, and we may assume that $\theta_\tau(a^*)$ is zero.
(3) Let $(a^*, b^*, c^*, d^*) = ([a, b]c[b, a], [a, b]d[b, a], a, b)$. Then $[a, b][c, d] = [a^*, b^*][c^*, d^*]$, and we may assume that there exists $1 \leqslant r \leqslant h$ such that $\theta_\tau(a_i)$ is zero for $i \leqslant 2r$, and for $i > r$, $\theta_\tau(a_{2i-1}) = \bar{0}$ and $\theta_\tau(a_{2i}) = \bar{1}$.
(4) Let $(a^*, b^*, c^*, d^*) = (ac, c^{-1}bc, c^{-1}bcb^{-1}c, dc^{-1}b^{-1}c)$. Then $[a, b][c, d] = [a^*, b^*][c^*, d^*]$ and if $\theta_\tau(a) = \theta_\tau(c) = \bar{0}$, $\theta_\tau(b) = \theta_\tau(d) = \bar{1}$, we obtain $\theta_\tau(a^*) = \theta_\tau(c^*) = \theta_\tau(d^*) = \bar{0}$ and $\theta_\tau(b^*) = \bar{1}$.

Applying these four identities, we see that in order to analyse the algebraic factorisation problem for an arbitrary surjective homomorphism θ_τ, it is sufficient to study the homomorphism θ_τ given by $\theta_\tau(a_i) = \bar{1}$ if $i = 1$, and $\bar{0}$ otherwise. This concludes the proof. □

Remark 31. From the above relations, in the orientable case, we deduce that any two surjective homomorphisms $\pi_1(X/\tau) \longrightarrow P_2(S_g)$ are equivalent (in the sense given at the end of Section 2).

We now study the non-orientable case.

Proposition 32. *Suppose that $\pi_1(X/\tau)$ is isomorphic to the fundamental group of a compact, connected, non-orientable surface without boundary different from $\mathbb{R}P^2$ of genus $h \geqslant 2$.*

(a) *Let h be odd, and consider the following presentation*:

$$\pi_1(X/\tau) = \langle v, a_1, a_2, \ldots, a_{h-2}, a_{h-1} \mid v^2 \cdot [a_1, a_2] \cdots [a_{h-2}, a_{h-1}] \rangle. \tag{36}$$

In order to study the algebraic problem, it suffices to consider the following three subcases:

(1) $\theta_\tau(v) = \bar{0}$, $\theta_\tau(a_1) = \bar{1}$ *and* $\theta_\tau(a_i) = \bar{0}$ *for all* $i > 1$.
(2) $\theta_\tau(v) = \bar{1}$, *and* $\theta_\tau(a_i) = \bar{0}$ *for all* $i \geqslant 1$.
(3) $\theta_\tau(v) = \bar{1}$, $\theta_\tau(a_1) = \bar{1}$ *and* $\theta_\tau(a_i) = \bar{0}$ *for all* $i > 1$.

(b) *Let h be even, and consider the following presentation*:

$$\pi_1(X/\tau) = \langle \alpha, \beta, a_1, a_2, \ldots, a_{2h-3}, a_{2h-2} \mid \alpha\beta\alpha\beta^{-1}[a_1, a_2] \cdots [a_{2h-3}, a_{2h-2}] \rangle. \tag{37}$$

(I) *If* $h = 2$ *then in order to study the algebraic problem, it suffices to consider the following subcases*:
(1) $\theta_\tau(\alpha) = \bar{0}$ *and* $\theta_\tau(\beta) = \bar{1}$.
(2) $\theta_\tau(\alpha) = \bar{1}$ *and* $\theta_\tau(\beta) = \bar{0}$.

(II) *If* $h \geqslant 4$ *then in order to study the algebraic problem, it suffices to consider the following subcases*:
(1) $\theta_\tau(\alpha) = \bar{0}$, $\theta_\tau(\beta) = \bar{1}$, *and* $\theta_\tau(a_i) = \bar{0}$ *for all* $i \geqslant 1$.
(2) $\theta_\tau(\alpha) = \bar{0}$, $\theta_\tau(\beta) = \bar{0}$, $\theta_\tau(a_1) = \bar{1}$ *and* $\theta_\tau(a_i) = \bar{0}$ *for all* $i > 1$.
(3) $\theta_\tau(\alpha) = \bar{1}$, $\theta_\tau(\beta) = \bar{0}$, *and* $\theta_\tau(a_i) = \bar{0}$ *for all* $i \geqslant 1$.

Proof. (a) Let $h \geqslant 3$ be odd. Suppose first that $\theta_\tau(v) = \bar{0}$. By the relation of the presentation given by Eq. (36), we must have $\theta_\tau(a_i) = \bar{1}$ for some i. Using the transformations of the proof of Proposition 30, we may assume that $\theta_\tau(a_i) = \bar{1}$ if $i = 1$ and zero if $i > 1$, which is case (1). Now suppose that $\theta_\tau(v) = \bar{1}$. One possibility is that $\theta_\tau(a_i) = \bar{0}$ for all $i \geqslant 1$, which is case (2). Now suppose that for some $1 \leqslant i \leqslant h-1$, we have $\theta_\tau(a_i) = \bar{1}$. Again using the transformations of the proof of Proposition 30, we may assume that $\theta_\tau(a_i) = \bar{1}$ if $i = 1$ and zero if $i > 1$, which is case (3). This completes the proof of part (a).

(b) If $a, b \in \pi_1(X/\tau)$, let $[a, b]' = abab^{-1}$ denote their twisted commutator.

(I) Let $h = 2$. Then there are three surjective homomorphisms:

(i) $\theta_\tau(\alpha) = \bar{0}$ and $\theta_\tau(\beta) = \bar{1}$, which is case (1).
(ii) $\theta_\tau(\alpha) = \bar{1}$ and $\theta_\tau(\beta) = \bar{0}$, which is case (2).
(iii) $\theta_\tau(\alpha) = \theta_\tau(\beta) = \bar{1}$.

Now if we let $(\alpha^*, \beta^*) = (\alpha, \beta\alpha)$, then we have $[\alpha, \beta]' = [\alpha^*, \beta^*]'$. This shows that the second and third homomorphisms are equivalent, and this completes the proof of part (I).

(II) Let $h \geqslant 4$. First we reduce the number of cases to five. Arguing as in the case $h = 2$ on the values of θ_τ on α, β, we see that we may reduce to the following cases:

(i) $\theta_\tau(\alpha) = \bar{0} = \theta_\tau(\beta) = \bar{0}$.
(ii) $\theta_\tau(\alpha) = \bar{0}$ and $\theta_\tau(\beta) = \bar{1}$.
(iii) $\theta_\tau(\alpha) = \bar{1}$ and $\theta_\tau(\beta) = \bar{0}$.

For the first case $\theta_\tau(\alpha) = \bar{0} = \theta_\tau(\beta) = \bar{0}$, we must have $\theta_\tau(a_i) = \bar{1}$ for some $1 \leqslant i \leqslant 2h-2$. It then follows from the proof of Proposition 30 that we may assume that $\theta_\tau(a_i) = \bar{1}$ if $i = 1$ and zero if $i > 1$. For the second case, $\theta_\tau(\alpha) = \bar{0}$ and $\theta_\tau(\beta) = \bar{1}$, we can either have $\theta_\tau(a_i) = \bar{0}$ for all $1 \leqslant i \leqslant 2h-2$, or $\theta_\tau(a_i) = \bar{1}$ for some $1 \leqslant i \leqslant 2h-2$. In the latter case, again by the proof of Proposition 30, we may assume that $\theta_\tau(a_i) = \bar{1}$ if $i = 1$ and zero if $i > 1$. The third case $\theta_\tau(\alpha) = \bar{1}$ and $\theta_\tau(\beta) = \bar{0}$ is completely analogous to the second case, and so the three cases above yield a total of five subcases:

(i) $\theta_\tau(\alpha) = \bar{0}$, $\theta_\tau(\beta) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for all $i \geqslant 1$, which is case (1).
(ii) $\theta_\tau(\alpha) = \theta_\tau(\beta) = \bar{0}$, $\theta_\tau(a_1) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for all $i > 1$, which is case (2).
(iii) $\theta_\tau(\alpha) = \bar{0}$, $\theta_\tau(\beta) = \bar{1}$, $\theta_\tau(a_1) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for all $i > 1$.
(iv) $\theta_\tau(\alpha) = \bar{1}$, $\theta_\tau(\beta) = \bar{0}$, and $\theta_\tau(a_i) = \bar{0}$ for all $i \geqslant 1$, which is case (3).
(v) $\theta_\tau(\alpha) = \bar{1}$, $\theta_\tau(\beta) = \bar{0}$, $\theta_\tau(a_1) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for all $i > 1$.

We now reduce these five cases to three.

$$\left(a^*, b^*, c^*, d^*\right) = \left(acac^{-1}a^{-1}, aca^{-1}c^{-1}bac^{-1}a^{-1}, aca^{-1}, da^{-1}\right). \tag{38}$$

Then

$$\begin{aligned}\left[a^*, b^*\right]'\left[c^*, d^*\right] &= acac^{-1}a^{-1}aca^{-1}c^{-1}bac^{-1}a^{-1}acac^{-1}a^{-1}aca^{-1}b^{-1}cac^{-1}a^{-1}aca^{-1}da^{-1}ac^{-1}a^{-1}ad^{-1} \\ &= [a, b]'[c, d].\end{aligned}$$

The substitution (38) shows that among the above five subcases, the second subcase is equivalent to the third, and the fourth is equivalent to $\theta_\tau(\alpha) = \bar{1}$, $\theta_\tau(\beta) = \bar{0}$, and $\theta_\tau(a_2) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for $i \neq 2$. But from the proof of Proposition 30, this is equivalent to $\theta_\tau(\alpha) = \bar{1}$, $\theta_\tau(\beta) = \bar{0}$, and $\theta_\tau(a_1) = \bar{1}$ and $\theta_\tau(a_i) = \bar{0}$ for $i > 1$, which is the fifth subcase. This completes the proof of part (II), and thus that of the proposition. □

Remark 33. For each of the three cases (h odd, $h = 2$ and $h \geqslant 4$ even) listed above in Proposition 32, the corresponding subcases are not equivalent. To see this, let us first consider the case $h = 2$. Using a set of generators for $\mathrm{Out}(N_2)$, it follows that the two subcases are not equivalent. For the case h odd we use the following observations. It is a general fact that an automorphism of $\pi_1(N_h)$ maps orientable loops to orientable loops and non-orientable loops to non-orientable loops. Moreover, consider the induced automorphism on the Abelianisation of $\pi_1(N_h)$. Since the class of the generator v given in the presentation of $\pi_1(N_h)$ generates the torsion part of the Abelianisation of $\pi_1(N_h)$, the subgroup generated by the class of v is invariant under any homomorphism. These two facts tell us that the class of v in the Abelianisation is mapped into itself, and that the subgroup generated by the classes of the elements $a_1, \dots, a_{h-1}$ is also invariant. A straightforward analysis using these two properties shows that the three subcases cannot be equivalent. The last case, $h \geqslant 4$ even, can be obtained by arguing in a similar way, and is left to the reader.

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