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# A fractional porous medium equation ☆

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## Abstract

We develop a theory of existence, uniqueness and regularity for the following porous medium equation with fractional diffusion,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{1/2}\big(|u|^{m-1}u\big) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

with $m > m_* = (N-1)/N$, $N \geqslant 1$ and $f \in L^1(\mathbb{R}^N)$. An $L^1$-contraction semigroup is constructed and the continuous dependence on data and exponent is established. Nonnegative solutions are proved to be continuous and strictly positive for all $x \in \mathbb{R}^N$, $t > 0$.


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*Keywords:* Porous medium; Fractional diffusion

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## 1. Introduction

This paper is concerned with the existence, uniqueness and properties of solutions $u = u(x,t)$ to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{1/2}\big(|u|^{m-1}u\big) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

for exponents $m > 0$, in space dimension $N \geqslant 1$, and with initial value $f \in L^1(\mathbb{R}^N)$. By a solution it is meant a suitable concept of weak or strong solution. In particular, we prove that $u \in C([0,\infty) : L^1(\mathbb{R}^N))$ and that the equation is satisfied a.e. in $Q = \mathbb{R}^N \times (0,\infty)$. The sign requirement $u \geqslant 0$ is not strictly needed but when enforced some additional properties hold.

We recall that the nonlocal operator $(-\Delta)^{1/2}$ is defined for any function $g$ in the Schwartz class through the Fourier transform,

$$\widehat{(-\Delta)^{1/2}g}(\xi) = |\xi|\hat{g}(\xi), \tag{1.2}$$

or via the Riesz potential,

$$(-\Delta)^{1/2}g(x) = C_N \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(y)}{|x-y|^{N+1}}\,dy, \tag{1.3}$$

where $C_N = \pi^{-\frac{N+1}{2}}\Gamma(\frac{N+1}{2})$ is a normalization constant, see for example [29,35].

Equations of this form can be considered as nonlinear variations of the linear fractional diffusion equation obtained for $m = 1$, which is a model of so-called anomalous diffusion, a much studied topic in physics, probability and finance, see for instance [1,26,28,30,39,40] and the references therein. We recall that fractional Laplacian operators of the form $(-\Delta)^{\sigma/2}$, $\sigma \in (0,2)$, are infinitesimal generators of stable Lévy processes [4,12]. The analysis of the linear equation in the whole space is easy since an integral representation can be used for the solutions, see below. Such a representation is not available in the nonlinear case.

Interest in studying the nonlinear model we propose is two-fold: on the one hand, experts in the mathematics of diffusion want to understand the combination of fractional operators with porous medium type propagation, and on the other hand models of this kind arise in statistical mechanics [27] and heat control [6]. The rigorous study of such nonlinear models has been delayed by mathematical difficulties in treating at the same time the nonlinearity and fractional diffusion.

Observe that the above equation becomes the well-known Porous Medium Equation (PME) when replacing the nonlocal diffusion operator $(-\Delta)^{1/2}$ by the classical Laplacian $-\Delta$. A number of techniques in dealing with the present nonlinear fractional diffusion model will be borrowed from the experience obtained with the PME, see for instance [37]. Our original purpose was to study Problem (1.1) for every $m > 1$, to examine the existence and properties of “fractional slow diffusion”. But the development of the theory allows to cover with a reasonable additional effort the “fast diffusion cases”, $m < 1$, on the condition that we restrict the exponent to be larger than a critical value, $m > m_* \equiv (N-1)/N$. This critical value is intrinsic to the equation, it appears in various contexts of the theory. It corresponds to the classical critical value

$m_* = (N-2)_+/N$ in the PME case, see [10]. Existence of a weak solution is however proved for every $m > 0$, for data which are moreover bounded.

HARMONIC EXTENSIONS. Besides formulae (1.2) and (1.3), there is another way of computing the half Laplacian, through the so-called Dirichlet to Neumann operator. If $g = g(x)$ is a smooth bounded function defined in $\mathbb{R}^N$, we consider its harmonic extension $v = v(x, y)$ to the upper half-space $\mathbb{R}^{N+1}_+$, $v = \mathrm{E}(g)$, i.e., the unique smooth bounded solution to

$$\begin{cases} \Delta_{x,y} v = 0, & x \in \mathbb{R}^N,\ y > 0, \\ v(x,0) = g(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

Then,

$$-\frac{\partial v}{\partial y}(x,0) = (-\Delta_x)^{1/2} g(x), \tag{1.5}$$

where $\Delta_{x,y}$ is the Laplacian in all $(x, y)$ variables and $\Delta_x$ acts only on the $x$ variables (in the sequel we will drop the subscripts when no confusion arises). In order to check (1.5), just apply the operator in the right-hand side twice. The choice of sign for the normal derivative makes the operator positive. Observe that the extension operator is well defined in $H^{1/2}(\mathbb{R}^N)$, and so is the Dirichlet to Neumann operator, which coincides with $(-\Delta)^{1/2}$ in this more general setting. This well-known technique has been recently used in several situations, see for instance [17,18,33].

PROBLEM-SETTING. By means of the above-mentioned harmonic extension we rewrite, for smooth solutions, the nonlocal Problem (1.1) in a "local way" (i.e., using local differential operators) as a quasi-stationary problem with a dynamical boundary condition. Indeed, $w = |u|^{m-1}u$ satisfies

$$\begin{cases} \Delta w = 0 & \text{for } x \in \mathbb{R}^N,\ y > 0,\ t > 0, \\ \dfrac{\partial w}{\partial y} - \dfrac{\partial(|w|^{\frac{1}{m}-1} w)}{\partial t} = 0 & \text{for } x \in \mathbb{R}^N,\ y = 0,\ t > 0, \\ w(x,0,0) = f^m(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

This problem has been recently considered by Athanasopoulos and Caffarelli [6]. They prove that bounded weak energy solutions to (1.6) are Hölder continuous if $m > 1$. The existence and uniqueness of that kind of solutions is one of the outcomes of the present paper. We also quote the work [3], where a more general problem is considered, though in a bounded domain, instead of in the half space.

The connection between problems with dynamical boundary conditions and nonlocal equations has already been exploited in [38] in the case of a bounded domain, and in [2] for a semilinear problem in the half-space $\mathbb{R}^{N+1}_+$. However, in those works the study of the nonlocal equation is used to obtain properties of the local one. Here, our approach is exactly the opposite.

RESULTS AND ORGANIZATION. Our purpose is to establish a theory of existence, uniqueness, comparison and regularity for suitable weak solutions of Problem (1.1) with initial data $f \in L^1(\mathbb{R}^N)$. The full theory works for values of $m$ larger than the critical value $m_*$ mentioned above, but basic existence and uniqueness holds for all $m > 0$, for data which are moreover bounded.

Section 2 contains preliminaries, the basic definitions of solutions, and a list of main results. We define the concept of weak solution to Problem (1.1) through the standard concept of weak solution to the associated local Problem (1.6). We also define the concept of strong solution.

We establish the existence of weak solutions in Section 3 by means of Semigroup Theory, solving first some associated elliptic problem, under the condition that the initial data $f$ are both integrable and bounded. Actually, the obtained solution is strong and the equation is satisfied almost everywhere. We also prove in this context an $L^1$–$L^\infty$ estimate that will be basic for the so-called smoothing effect.

Uniqueness is studied in Section 4. Section 5 deals with further properties of the constructed solution. It includes conservation of mass, positivity and regularity. At this point we use the continuity result from [6] to show that solutions to (1.1) corresponding to nonnegative initial data become immediately strictly positive if $m > 1$. This is a remarkable property since it departs from the well-known properties of the standard PME, cf. [37]. On the other hand, for $m_* < m < 1$ we are able to prove the expected positivity property using a different approach. This is used later, in combination with boundedness and a result in [6] to prove Hölder continuity also in this case. Let us notice that, unlike in the local case, there is still no general regularity result for linear nonlocal equations (with reasonable coefficients) guaranteeing that positive bounded solutions to Problem (1.1) are in fact $C^\infty$, though this is expected to be true.

After such a work, we are able to treat general solutions with data in $L^1(\mathbb{R}^N)$ in Section 6. Here we complete the proof of uniform boundedness of the solutions with integrable data for positive times, the $L^1$–$L^\infty$ smoothing effect.

In Section 7 we study the continuous dependence of the solution in terms of the exponent $m$ and the data $f$, in the case $m > m_*$. In particular, we show that the linear case $m = 1$ can be obtained as a limit of the nonlinear case both from above and below.

Section 8 contains a brief description of alternative approaches to the existence theory and an announcement of extensions. Finally, Appendix A gathers some technical lemmas.

Notice on the linear case. For the value of the parameter $m = 1$ we obtain the equation

$$\frac{\partial u}{\partial t} + (-\Delta)^{1/2} u = 0. \tag{1.7}$$

This is a linear fractional heat equation where the fractional derivatives act only on the space variable. It is explicitly solvable in terms of the initial value, $u(x,0) = f(x)$, through convolution with the explicit Poisson kernel in $\mathbb{R}^{N+1}_+$,

$$u(x,t) = C_N \int_{\mathbb{R}^N} \frac{t f(z)}{(|x-z|^2 + t^2)^{(N+1)/2}}\, dz, \tag{1.8}$$

where $C_N$ is the constant in (1.3). Note that this corresponds to an anomalous diffusion law of the form $\langle x\rangle \sim t^\alpha$ with $\alpha = 1$ instead of the standard $\alpha = 1/2$ of the Brownian case.

Notations. In dealing with extended functions, we denote the upper half-space, $\mathbb{R}^{N+1}_+$, by $\Omega$, and write its points as $\bar{x} = (x, y)$, $x \in \mathbb{R}^N$, $y > 0$. We denote by $\Gamma$ the boundary of $\Omega$, i.e., $\Gamma = \mathbb{R}^N \times \{0\}$, which is identified to the original $\mathbb{R}^N$ with variable $x$. We consider also the extension and trace operators, E, Tr: for a function $v \in H^{1/2}(\Gamma)$, we denote its harmonic extension to $\Omega$ as $\mathrm{E}(v)$; notice that $\mathrm{E}(v) \in H^1(\Omega)$; on the other hand, given a function $z \in H^1(\Omega)$, we denote its trace on $\Gamma$, which belongs to $H^{1/2}(\Gamma)$, as $\mathrm{Tr}(z)$.

As in the PME theory, we will be mostly interested in nonnegative data and solutions. However, the basic theory can be developed for data of any sign, and in that case we will use the simplified notation $u^m$ instead of the “odd power” $|u|^{m-1}u$, and we will also use such a notation when $m$ is replaced by $1/m$. In fact, we will show that if the initial value is nonnegative, then the weak solution we construct is also nonnegative, $u \geqslant 0$, which helps justify our abbreviated notations.

## 2. Preliminaries and main results

As mentioned above, we define the concept of weak solution to Problem (1.1) through the standard concept of weak solution to an associated local problem, which we write here again by convenience.

$$\begin{cases} \Delta w = 0 & \text{for } \bar{x} \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial y} - \dfrac{\partial w^{1/m}}{\partial t} = 0 & \text{on } \Gamma,\ t > 0, \\ w(x,0,0) = f^m(x) & \text{on } \Gamma. \end{cases} \tag{2.1}$$

In order to define a weak solution of this problem we multiply formally the equation in (2.1) by a test function $\varphi$ and integrate by parts to obtain

$$-\int_0^T \int_\Omega \langle \nabla w, \nabla \varphi \rangle \, d\bar{x}\, ds + \int_0^T \int_\Gamma u \frac{\partial \varphi}{\partial t} \, dx\, ds = 0, \tag{2.2}$$

with $u = (\mathrm{Tr}(w))^{1/m}$, on the condition that $\varphi$ vanishes for $t = 0$ and $t = T$, and also for large $|x|$ and $y$.

**Definition 2.1.** We say that a pair of functions $(u, w)$ is a *weak solution* to Problem (2.1) if $w \in L^1((0,T); W^{1,1}_{loc}(\Omega))$, $u = (\mathrm{Tr}(w))^{1/m} \in L^1(\Gamma \times (0,T))$ and equality (2.2) holds for every $\varphi \in C^1_0(\overline{\Omega} \times [0,T))$. Finally, for every $t > 0$ we have $u(\cdot,t) \in L^1(\Gamma)$ and $\lim_{t\to 0} u(\cdot,t) = f$ in $L^1(\Gamma)$.

An alternative form of equality (2.2), including the initial value in it, is

$$\begin{aligned} &-\int_0^T \int_\Omega \langle \nabla w, \nabla \varphi \rangle \, d\bar{x}\, ds + \int_0^T \int_\Gamma u \frac{\partial \varphi}{\partial t} \, dx\, ds \\ &\quad = \int_\Gamma u(x,T)\varphi(x,0,T)\, dx - \int_\Gamma f(x)\varphi(x,0,0)\, dx. \end{aligned} \tag{2.3}$$

As is usual, more general test functions can be considered by approximation, whenever the integrals make sense. Note that the trace $u = (\mathrm{Tr}(w))^{1/m}$ is well defined. For brevity we will refer sometimes to the solution as only $u$, or even only $w$, when no confusion arises, since it is clear how to complete the pair from one of the components, $u = (\mathrm{Tr}(w))^{1/m}$, $w = \mathrm{E}(u^m)$. By a weak solution of our original Problem (1.1) we understand $u$, the first element of the solution to Problem (2.1).

Observe that the definite advantage of working with the local version is compensated in some sense by the difficulty of having integrals in (2.2) defined in spaces of different dimensions.

This definition is a very general notion of solution: in this framework we can construct a weak solution to Problem (2.1) provided the initial value $f$ is integrable and bounded. We restrict ourselves in the next results to such data. However, weak solutions are sometimes difficult to work with, and we are not able to prove uniqueness. Hence, a class of solutions with better properties is welcome. A quite convenient choice is the class of so-called *weak energy solutions*, cf. [37] for the standard PME.

**Definition 2.2.** A weak solution pair $(u, w)$ to Problem (1.1) is said to be a weak energy solution if moreover $w \in L^2([0,T]; H^1(\Omega))$.

**Theorem 2.1.** *Let $m > 0$. For every $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there exists a unique weak energy solution to Problem* (2.1). *Moreover $u \in C([0,\infty) : L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times [0,\infty))$.*

The importance of this class of solutions, besides having uniqueness, is that, if we restrict to nonnegative data and exponents $m > m_* = (N-1)/N$, we can obtain regularity and positivity.

**Theorem 2.2.** *Let $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be nonnegative, and assume $m > m_*$. Then the weak energy solution $(u, w)$ to Problem* (2.1) *satisfies*:

(i) *Conservation of mass*: *for every $t > 0$ we have*

$$\int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} f(x)\,dx. \tag{2.4}$$

(ii) *Positivity*: *$u(\cdot,t) > 0$ in $\mathbb{R}^N$ for every $t > 0$.*
(iii) *Regularity*: *there exists some $0 < \alpha < 1$ such that $u \in C^\alpha(\mathbb{R}^N \times (0,T))$.*
(iv) *Maximum Principle*: *if $u_1, u_2$ are solutions with data $u_{01}, u_{02}$ and $u_{01} \leqslant u_{02}$ a.e. in $\mathbb{R}^N$, then $u_1 \leqslant u_2$ a.e., in $Q = \mathbb{R}^N \times (0,\infty)$.*
(v) *Contraction*: *for any two solutions $u_1, u_2$ with data $u_{01}, u_{02}$ we have*

$$\left\| u_1(\cdot,t) - u_2(\cdot,t) \right\|_{L^1(\mathbb{R}^N)} \leqslant \|u_{01} - u_{02}\|_{L^1(\mathbb{R}^N)}. \tag{2.5}$$

The restriction $m > m_*$ is not technical: positivity and conservation of mass are not true if $m < m_*$, see Proposition 5.1. On the other hand, conservation of mass holds also for solutions with changing sign if $m > m_*$. If $m > 1$ the $C^\alpha$ regularity result is true also for any changing sign solution, [6].

A further interesting property is that the weak energy solutions are *strong solutions*, which means that the terms (in principle only distributions) involved in Eq. (2.1) are in fact functions, and equalities hold almost everywhere. The main technical difficulty is to prove that $\partial_t u$ is a function.

**Theorem 2.3.** *In the hypotheses of Theorem* 2.2 *we have $\partial_t u \in L^1(\mathbb{R}^N)$.*

We observe that for strong solutions we can multiply (2.1) by any integrable function to get, instead of (2.2), the following identity

$$\int_{\Omega} \langle \nabla w, \nabla \varphi \rangle \, d\bar{x} + \int_{\Gamma} \frac{\partial u}{\partial t} \varphi \, dx = 0. \tag{2.6}$$

Working with strong solutions we can use the solution itself as a test function in formula (2.6). In particular this allows us to obtain a universal bound for all solutions with the same mass.

**Theorem 2.4.** *Let $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and assume $m > m_*$. Then, there exists a positive constant $C$ such that the weak energy solution to Problem* (2.1) *satisfies*

$$\sup_{x \in \mathbb{R}^N} \left| u(x,t) \right| \leqslant C t^{-\gamma} \| f \|_{L^1(\mathbb{R}^N)}^{\gamma/N} \tag{2.7}$$

*with $\gamma = (m - 1 + 1/N)^{-1}$. The constant $C$ depends only on $N$ and $m$.*

General integrable data. Once this theory is settled, we are interested in considering all integrable functions $f$ as possible data in Problem (1.1). As we have advanced, this can be managed by approximation by bounded initial data, and this is possible if we have an $L^1$-contraction at hand. We thus introduce the concept of $L^1$ energy solution: a weak solution, continuous in $L^1$, which is also an energy solution for positive times.

**Definition 2.3.** We say that a weak solution $(u, w)$ to Problem (2.1) is an $L^1$-energy solution if $u \in C([0, \infty) : L^1(\mathbb{R}^N))$ and $|\nabla w| \in L^2(\Omega \times [\tau, \infty))$, for every $\tau > 0$.

The $L^1$-contraction property for $L^1$-energy solutions is as follows.

**Theorem 2.5.** *Let $(u, w)$ and $(\tilde{u}, \tilde{w})$ be two $L^1$-energy solutions to Problem* (2.1). *Then, for every $0 \leqslant t_1 < t_2$,*

$$\int_{\mathbb{R}^N} \left[ u(x,t_2) - \tilde{u}(x,t_2) \right]_+ dx \leqslant \int_{\mathbb{R}^N} \left[ u(x,t_1) - \tilde{u}(x,t_1) \right]_+ dx. \tag{2.8}$$

We have that, when performing the approximation by problems with bounded data, the limit function obtained is an $L^1$-energy solution. Now, since estimate (2.7) does not depend on the $L^\infty$ norm of the data, it is also true for the limit solution (for changing sign solutions it holds by comparison). In particular this represents an $L^1$–$L^\infty$ smoothing effect that allows to obtain the same properties of Theorem 2.2 for positive times.

**Theorem 2.6.** *Let $m > m_*$. Then for every $f \in L^1(\mathbb{R}^N)$ there exists a unique $L^1$-energy solution to Problem* (2.1). *It satisfies estimate* (2.7) *and the conservation of mass* (2.4). *If moreover $f \geqslant 0$, positivity and regularity also hold, and the solution is strong. The maps $S_t : f \mapsto u(t)$ generate a nonlinear semigroup of order-preserving contractions in $L^1(\mathbb{R}^N)$.*

The next sections of the paper are devoted to treat the case of bounded data. In Section 6 we drop this restriction and deal by approximation with general $L^1$ data. We point out that the continuous dependence of the solutions constructed above with respect to the initial data and the exponent is stated and proved in Section 7.

## 3. Weak solutions

We set out to construct a weak solution to the extended local Problem (2.1) taking initial values $f \in L^1(\Gamma) \cap L^\infty(\Gamma)$. We point out that the construction of a weak solution can be made for every $m > 0$ and data not necessarily signed.

A well-known method of construction of solutions of evolution equations, and also of generating a semigroup in a convenient functional space, is the so-called Implicit Time Discretization. It runs as follows: if the evolution equation is $dv/dt + A(v) = 0$, where $A$ is a linear or nonlinear, bounded or unbounded operator acting on a Banach space $\mathcal{X}$, and given initial data $v(0) = f \in \mathcal{X}$, then the construction of an approximate solution of the problem in a time interval $[0, T]$ proceeds dividing the time interval $[0, T]$ in $n$ subintervals of length $\varepsilon = T/n$ and then defining the approximate solution $v_\varepsilon$ constant on each subinterval in the following way: in each interval $(t_{k-1}, t_k]$, $t_k = k\varepsilon$, $k = 1, \dots, n$, we consider the solution $v_{\varepsilon,k}$ to the discretized problem

$$\frac{1}{\varepsilon}(v_{\varepsilon,k} - v_{\varepsilon,k-1}) + A(v_{\varepsilon,k}) = 0. \tag{3.1}$$

We take as starting condition $v_{\varepsilon,0} = f_\varepsilon$, an approximation of $f$. In the case of linear operators, a variant of the Hille–Yosida Theorem ensures the convergence of these approximate solutions to the so-called *mild solution* of the evolution problem when the operator $A$ satisfies some properties, like being maximal monotone, cf. [14]. The convergence result in the case of nonlinear and possibly unbounded operators is given by the famous Crandall–Liggett Theorem [21] under the assumption that $A$ must be accretive and satisfy a certain rank condition. (Reminder: a possibly nonlinear and unbounded operator $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ is called accretive if for every $\varepsilon > 0$ the map $I + \varepsilon A$ is one-to-one onto a subspace $R_\varepsilon(A) \subset \mathcal{X}$ and the inverse $\mathcal{R}(\varepsilon, A) := (I + \varepsilon A)^{-1} : R_\varepsilon(A) \to \mathcal{X}$ is a contraction in the $\mathcal{X}$-norm. The precise rank condition that we will use is $R_\varepsilon(A) \supset \overline{D(A)}$ for every $\varepsilon > 0$.)

One of the typical examples of such theory is the standard PME posed on the whole space or on a bounded domain with homogeneous boundary conditions. The early work due to Bénilan and collaborators, [8], drew attention to this important results, as well as the application to more general nonlinear diffusion-convection models.

We will apply such a strategy to our evolution Problem (2.1). The discretized problem is:

Given $f \in L^1(\Gamma) \cap L^\infty(\Gamma)$ and $\varepsilon > 0$, to find $u_\varepsilon = \{u_{\varepsilon,1}, \dots, u_{\varepsilon,n}\}$ by solving for $k = 1, \dots, n$ the problem

$$\begin{cases} \Delta w_{\varepsilon,k} = 0 & \text{in } \Omega, \\ \varepsilon \dfrac{\partial w_{\varepsilon,k}}{\partial y} = u_{\varepsilon,k} - u_{\varepsilon,k-1} & \text{on } \Gamma, \end{cases} \tag{3.2}$$

with initial value $u_{\varepsilon,0} = f$ on $\Gamma$. In each such step, $u_{\varepsilon,k-1} = (\mathrm{Tr}(w_{\varepsilon,k-1}))^{1/m}$ is known and $u_{\varepsilon,k}$ and $w_{\varepsilon,k} = \mathrm{E}(u^m_{\varepsilon,k})$ are the unknowns.

The second equation in (3.2) can be written as

$$u_{\varepsilon,k} + \varepsilon A(u_{\varepsilon,k}) = u_{\varepsilon,k-1}, \tag{3.3}$$

where the operator $A : D(A) \subset L^1(\Gamma) \to L^1(\Gamma)$ is defined as

$$A(v) = -\operatorname{Tr}\left(\frac{\partial \mathrm{E}(v^m)}{\partial y}\right), \tag{3.4}$$

with domain

$$D(A) = \left\{v \in L^1(\Gamma) \cap L^\infty(\Gamma)\colon\ A(v) \in L^1(\Gamma),\ \|v\|_{L^\infty(\Gamma)} \leqslant \|f\|_{L^\infty(\Gamma)}\right\}. \tag{3.5}$$

This operator is nothing but the half-Laplacian of the power $m$, $A(v) = (-\Delta)^{1/2} v^m$.

### 3.1. The elliptic problem

Therefore, in order to perform the plan we need to establish the solvability and properties of the elliptic problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ -\dfrac{\partial w}{\partial y} + w^{1/m} = g & \text{on } \Gamma. \end{cases} \tag{3.6}$$

for all $g \in L^1(\Gamma) \cap L^\infty(\Gamma)$. As we have said before, the power $w^{1/m}$ in the boundary condition means $|w|^{1/m-1} w$ if $w$ takes on some negative values. We will also prove that if $g \geqslant 0$ then $w \geqslant 0$. A weak solution to this problem is a function $w \in W^{1,1}_{loc}(\Omega)$, such that $(\operatorname{Tr}(w))^{1/m} \in L^1(\Gamma)$, verifying

$$\int_\Omega \langle \nabla w, \nabla \varphi \rangle + \int_\Gamma w^{1/m} \varphi - \int_\Gamma g\varphi = 0 \tag{3.7}$$

for any $\varphi \in C_0^1(\overline{\Omega})$. We have to prove existence of the solution $w$ and contractivity of the map $g \mapsto (\operatorname{Tr}(w))^{1/m}$ in the norm of $L^1(\Gamma)$, which plays the role of $\mathcal{X}$ in the definition of accretivity.

To prove this we perform an approximation substituting the unbounded domain $\Omega$ by an increasing sequence of bounded domains $\Omega_R$ (half balls), imposing zero Dirichlet condition on the part of the boundary of the domain which does not lie on the hyperplane $y = 0$. The approximate problems are

$$\begin{cases} \Delta w = 0 & \text{in } \Omega_R = \Omega \cap B_R, \\ \dfrac{\partial w}{\partial y} = w^{1/m} - g & \text{on } \Gamma_R = \partial\Omega_R \cap \{y = 0\}, \\ w = 0 & \text{on } \Sigma_R = \partial\Omega_R \cap \{y > 0\}, \end{cases} \tag{3.8}$$

where $B_R = B_R(0)$. The concept of weak solution for a given datum $g \in L^1(\Gamma_R) \cap L^\infty(\Gamma_R)$ is analogous to the one given above (3.7), after changing the domains of the integrals into the corresponding bounded domains.

**Theorem 3.1.** *For every $g \in L^1(\Gamma) \cap L^\infty(\Gamma)$ there exists a unique weak solution $w \in H^1(\Omega)$ to Problem* (3.6) *such that* $(\mathrm{Tr}(w))^{1/m} \in L^1(\Gamma) \cap L^\infty(\Gamma)$. *Moreover, if $w$ and $\tilde{w}$ are the solutions corresponding to data $g$ and $\tilde{g}$, then*

$$\int_\Gamma \left[w^{1/m}(x,0) - \tilde{w}^{1/m}(x,0)\right]_+ dx \leqslant \int_\Gamma \left[g(x) - \tilde{g}(x)\right]_+ dx. \tag{3.9}$$

*This in turn implies that if $g \geqslant 0$ in $\Gamma$ then $w \geqslant 0$ in $\overline{\Omega}$. Moreover, $\|u\|_{L^\infty(\Gamma)} \leqslant \|g\|_{L^\infty(\Gamma)}$.*

**Proof.** Step 1. We first prove that there exists a weak solution $w \in H^1(\Omega_R)$ to Problem (3.8). This is done by solving the following minimization problem:

To find a function $w \in H^1(\Omega_R)$ minimizing

$$J(w) = \frac{1}{2}\int_{\Omega_R} |\nabla w|^2 + \frac{m}{m+1}\int_{\Gamma_R} |w|^{\frac{m+1}{m}} - \int_{\Gamma_R} wg.$$

This functional is coercive, since

$$J(w) \geqslant C_1\|w\|^2_{H^1(\Omega_R)} - C_2\|w\|_{H^1(\Omega_R)},$$

which follows by using the Poincaré inequality, Cauchy–Schwartz and the trace embedding. Moreover, coercivity then provides a bound for $\|w\|_{H^1(\Omega_R)}$, though it depends on $R$.

Step 2. We now establish contractivity of solutions to Problem (3.8) in $L^1(\Gamma_R)$. Let $w$ and $\tilde{w}$ be the solutions corresponding to data $g$ and $\tilde{g}$. We claim that

$$\int_{\Gamma_R} \left[w^{1/m}(x,0) - \tilde{w}^{1/m}(x,0)\right]_+ dx \leqslant \int_{\Gamma_R} \left[g(x) - \tilde{g}(x)\right]_+ dx. \tag{3.10}$$

This inequality follows easily if we consider in the weak formulation the test function $\varphi = p(w - \tilde{w})$, where $p$ is any smooth monotone approximation of the sign function, $0 \leqslant p(s) \leqslant 1$, $p'(s) \geqslant 0$. We get

$$\int_{\Omega_R} p'(w-\tilde{w})\left|\nabla(w-\tilde{w})\right|^2 + \int_{\Gamma_R} \left(w^{1/m} - \tilde{w}^{1/m}\right)p(w-\tilde{w}) - \int_{\Gamma_R} (g-\tilde{g})p(w-\tilde{w}) = 0.$$

Passing to the limit, we obtain

$$\int_{\Gamma_R} \left(w^{1/m} - \tilde{w}^{1/m}\right)_+ dx \leqslant \int_{\Gamma_R} (g-\tilde{g})\,\mathrm{sign}(w-\tilde{w})\,dx \leqslant \int_{\Gamma_R} (g-\tilde{g})_+\,dx.$$

In particular, under the assumption $g \geqslant 0$ we have $w(\cdot,0) \geqslant 0$. Moreover, $w(\cdot,0) \in L^1(\Gamma_R) \cap L^\infty(\Gamma_R)$. Finally, since the Poisson kernel in the half-ball is nonnegative, we also conclude that $w \geqslant 0$ in $\Omega_R$.

STEP 3. In order to pass to the limit $R \to \infty$ in the case of nonnegative data, we use a monotonicity property of the family of approximate solutions, denoted here by $w_R$. Namely, $R < R'$ implies $w_R \leqslant w_{R'}$ in $\Omega_{R'}$. This follows from the ordering of the restrictions, using again that the Poisson kernel is nonnegative. The ordering of the restrictions results from comparison in $\Gamma_R$, since $w_{R'} \geqslant 0$ in $\Sigma_R$ (there is a contraction property analogous to (3.10) for problems with non-homogeneous boundary data).

Monotonicity implies that there exists the pointwise (and also in the sense of distributions) limit $w = \lim_{R\to\infty} w_R$. This limit satisfies $w \geqslant 0$ in $\Omega$, $(\mathrm{Tr}(w))^{1/m} \in L^1(\Gamma) \cap L^\infty(\Gamma)$. Since $|\nabla w_R|$ is uniformly bounded in $L^2(\Omega_R)$,

$$\int_{\Omega_R} |\nabla w_R|^2 \leqslant \int_{\Gamma_R} g w_R \leqslant \|g\|_{L^1(\mathbb{R}^N)} \|g\|^m_{L^\infty(\mathbb{R}^N)},$$

we conclude that $\nabla w_R \rightharpoonup \nabla w$ in $L^2(\Omega)$. This is enough to pass to the limit in the identity

$$\int_{\Omega_R} \langle \nabla w_R, \nabla\varphi\rangle + \int_{\Gamma_R} w_R^{1/m}\varphi - \int_{\Gamma_R} g\varphi = 0,$$

to show that $w$ satisfies (3.7). Also the estimate of the $L^2$ norm of the gradients passes to the limit, and leads to

$$\int_{\Omega} |\nabla w|^2 \leqslant \|g\|_{L^1(\mathbb{R}^N)} \|g\|^m_{L^\infty(\mathbb{R}^N)}. \tag{3.11}$$

STEP 4. The pass to the limit in the case of non-positive data uses a similar argument. Finally, in the case of data $g$ of both signs, we use comparison with the solutions with data $g_1 = g^+ \geqslant 0$ and $g_2 = -g^- \leqslant 0$ and compactness to pass to the limit.

STEP 5. Contractivity for the limit problem is proved exactly in the same way as for the approximate problems. This gives uniqueness. We also have that the $L^1$ norm and the $L^\infty$ norm of the function $w^{1/m}(\cdot, 0)$ are bounded respectively by the $L^1$ norm and the $L^\infty$ norm of the datum. □

### *3.2. Existence of solution for the evolution problem*

We now use the previously mentioned procedure to construct the solution to the evolution problem (2.1). We recall that the Crandall–Liggett result only provides us in principle with an abstract type of solution called *mild solution.*

**Theorem 3.2.** *For every $f \in L^1(\Gamma) \cap L^\infty(\Gamma)$ there exists a weak solution $(u, w)$ to Problem (2.1) with $u(\cdot, t) \in L^1(\Gamma) \cap L^\infty(\Gamma)$ for every $t > 0$ and $w \in L^2([0, T]; H^1(\Omega))$. Moreover, the following contractivity property holds: if $(u, w)$, $(\tilde u, \tilde w)$ are the constructed weak solutions corresponding to initial data $f$, $\tilde f$, then*

$$\int_{\Gamma} \bigl[u(x,t) - \tilde u(x,t)\bigr]_+ \, dx \leqslant \int_{\Gamma} \bigl[f(x) - \tilde f(x)\bigr]_+ \, dx. \tag{3.12}$$

*In particular a comparison principle for constructed solutions is obtained.*

**Proof.** For each time $T > 0$ we divide the time interval $[0, T]$ in $n$ subintervals. Letting $\varepsilon = T/n$, we construct the function $w_\varepsilon$ piecewise constant in each interval $(t_{k-1}, t_k]$, where $t_k = k\varepsilon$, $k = 1, \dots, n$, as the solutions to the discretized Problems (3.2). For convenience we write here again the problems: $w_{\varepsilon,k}$ solves

$$\begin{cases} \Delta w_{\varepsilon,k} = 0 & \text{in } \Omega, \\ \varepsilon \dfrac{\partial w_{\varepsilon,k}}{\partial y} = u_{\varepsilon,k} - u_{\varepsilon,k-1} & \text{on } \Gamma, \end{cases}$$

with $u_{\varepsilon,0} = f$. Our solution is the (uniform in $[0, T]$) limit

$$(u, w) = \lim_{\varepsilon \to 0} (u_\varepsilon, w_\varepsilon)$$

in $L^1_{loc}(\Omega)$. It is a *mild solution*, whose existence is guaranteed by the classical semigroup approach, $u \in C([0, \infty) : L^1(\Gamma))$. In fact we obtain first the function $u$ by Crandall–Ligget's Theorem, and the harmonic extension $w$ of $u^m$ coincides with $\lim_{\varepsilon \to 0} w_\varepsilon$. By construction we have $w \in L^\infty(\Omega \times [0, T])$. We must now show that we have obtained in fact a weak solution.

Multiplying the equation by $w_{\varepsilon,k}$, integrating by parts, and applying Young's inequality, we obtain

$$\varepsilon \int_\Omega |\nabla w_{\varepsilon,k}|^2 \, d\bar{x} \leqslant \frac{1}{(m+1)} \left( \int_\Gamma |u_{\varepsilon,k-1}|^{m+1} \, dx - \int_\Gamma |u_{\varepsilon,k}|^{m+1} \, dx \right). \tag{3.13}$$

Adding from $k = 1$ to $k = n$ we get

$$\int_0^T \int_\Omega \left|\nabla w_\varepsilon(\bar{x}, t)\right|^2 d\bar{x} \, dt \leqslant \frac{1}{(m+1)} \int_\Gamma \left|f(x)\right|^{m+1} dx.$$

Passing to the limit, the same estimate is obtained for $|\nabla w|$, and therefore $w \in L^2([0, T]; H^1(\Omega))$. On the other hand, (3.13) yields

$$\int_\Gamma |u_{\varepsilon,k}|^{m+1} \, dx \leqslant \int_\Gamma |u_{\varepsilon,k-1}|^{m+1} \, dx \leqslant \int_\Gamma |f|^{m+1} \, dx,$$

which, after passing to the limit, gives

$$\int_\Gamma \left|u(x, t)\right|^{m+1} dx \leqslant \int_\Gamma \left|f(x)\right|^{m+1} dx,$$

for every $t \in [0, T]$. Now, choosing appropriate test functions, as in [32], it follows that we can pass to the limit in the elliptic weak formulation to get the identity of the parabolic weak formulation. We first have

$$\int_\Omega \langle \nabla w_{\varepsilon,k}, \nabla \varphi \rangle = \int_\Gamma \frac{1}{\varepsilon} (u_{\varepsilon,k-1} - u_{\varepsilon,k}) \varphi.$$

Integrating in $(t_{k-1}, t_k)$ and adding for $k = 1, \dots, n$, we get that the right-hand side becomes

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{\Gamma} \frac{1}{\varepsilon}\big(u_{\varepsilon,k-1}(x,t) - u_{\varepsilon,k}(x,t)\big)\varphi(x,0,t)\,dx\,dt$$

$$= \int_0^T \int_{\Gamma} u_\varepsilon(x,t)\frac{1}{\varepsilon}\big(\varphi(x,0,t+\varepsilon) - \varphi(x,0,t)\big)\,dx\,dt$$

$$+ \frac{1}{\varepsilon}\int_0^{\varepsilon}\int_{\Gamma} f(x)\varphi(x,0,t)\,dx\,dt - \frac{1}{\varepsilon}\int_{T-\varepsilon}^{T}\int_{\Gamma} u_\varepsilon(x,T)\varphi(x,0,t)\,dx\,dt.$$

Passing to the limit $\varepsilon \to 0$ we get (2.2).

The contractivity (3.12) obtained in Theorem 3.1 in each step is inherited in the limit. In fact, if $w_{\varepsilon,k}$ and $\tilde{w}_{\varepsilon,k}$ are the discretized approximations of $w$ and $\tilde{w}$, then we have

$$\int_{\Gamma} \big[u_{\varepsilon,k+1}(x) - \tilde{u}_{\varepsilon,k+1}(x)\big]_+\,dx \leqslant \int_{\Gamma} \big[u_{\varepsilon,k}(x) - \tilde{u}_{\varepsilon,k}(x)\big]_+\,dx,$$

which easily implies (3.12). Comparison is a trivial consequence of contractivity. □

**Remark.** This contractivity property also implies the following estimates for the weak solution to Problem (1.1) just constructed

$$\big\|u(\cdot,t)\big\|_{L^1(\mathbb{R}^N)} \leqslant \|f\|_{L^1(\mathbb{R}^N)}, \qquad \big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R}^N)} \leqslant \|f\|_{L^\infty(\mathbb{R}^N)}. \tag{3.14}$$

Using now the Poisson kernel of the half-space and Young's inequality, we have that for every $y > 0$ and every $1/m \leqslant p \leqslant \infty$, it holds

$$\big\|\mathrm{E}\big(u^m\big)(\cdot,y,t)\big\|_{L^p(\mathbb{R}^N)} \leqslant \big\|u^m(\cdot,0,t)\big\|_{L^p(\mathbb{R}^N)} \leqslant \|f\|_{L^\infty(\mathbb{R}^N)}^{m-1/p}\|f\|_{L^1(\mathbb{R}^N)}^{1/p}, \tag{3.15}$$

with the first inequality replaced by equality if $p = 1$, $m > 1$.

We end this section with a property satisfied by the weak solutions just constructed which is very useful in the proofs to come, with a number of other applications.

**Proposition 3.1.** *Assume $f \geqslant 0$. If $w$ is the nonnegative weak solution to Problem* (2.1) *constructed in Theorem* 3.2*, then the inequality*

$$(m-1)t\frac{\partial w}{\partial t} + mw \geqslant 0 \tag{3.16}$$

*holds in the sense of distributions.*

**Proof.** We use a simple scaling argument based on the homogeneity of the problem, (see [9], and also [5]). Assume first $m > 1$. We have that for every $\lambda > 1$ the function $w_\lambda(x,t) = \lambda w(x, \lambda^{\frac{m-1}{m}} t)$ is also a solution with initial value bigger than $w(\cdot, 0)$. Then by the comparison principle we get,

$$\lim_{\lambda\to 1^+} \frac{w_\lambda(x,t) - w(x,t)}{\lambda - 1} \geqslant 0,$$

which gives (3.16). For $m < 1$ the sign is reversed. Comparison can be easily justified in the discretized approximations. Since the function $w_\lambda$ is the limit of the rescaled approximations of the function $w$, we can compare $w$ and $w_\lambda$. $\square$

We observe for future reference that at the nonlocal level of function $u$ we have the "monotonicity formulae"

$$\begin{aligned} &\frac{\partial u}{\partial t} \geqslant -\frac{u}{(m-1)t} \qquad \text{if } m > 1, \\ &\frac{\partial u}{\partial t} \leqslant \frac{u}{(1-m)t} \qquad \text{if } m < 1. \end{aligned} \tag{3.17}$$

Formula (3.16) is empty for $m = 1$, but in this case it is easy to derive from the explicit representation of the solution that $t\partial_t u + Nu \geqslant 0$.

**Remark.** In the PME model (the local analogue), a similar lower estimate of $\partial_t u$ is also available in the case $m < 1$. The proof uses in an essential way a second variable called the pressure, which is a potential for the velocity. It is not clear which could be the corresponding pressure for the nonlocal problem.

## 4. Uniqueness of weak energy solutions

In the previous section we have constructed a weak solution to the local Problem (2.1). As we have said, the construction itself shows that this weak solution is in fact a weak energy solution. We prove uniqueness using an argument taken from Oleinik et al. [31].

**Lemma 4.1.** *Assume $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. There is at most one weak energy solution to Problem* (1.1).

**Proof.** Let $(u, w)$ and $(\tilde{u}, \tilde{w})$ be two weak solutions to Problem (2.1). We take as test, in the weak formulation, the following function

$$\varphi(\bar{x}, t) = \int_t^T (w - \tilde{w})(\bar{x}, s)\, ds, \quad 0 \leqslant t \leqslant T,$$

with $\varphi \equiv 0$ for $t \geqslant T$. Observe that this is a good test function when $w$ and $\tilde{w}$ are weak energy solutions. We have

$$\int_0^T \int_\Omega \langle \nabla(w-\tilde{w})(\bar{x},t), \int_t^T \nabla(w-\tilde{w})(\bar{x},s)\,ds\rangle\,d\bar{x}\,dt$$
$$+\int_0^T \int_\Gamma (u-\tilde{u})(x,t)\big(u^m-\tilde{u}^m\big)(x,t)\,dx\,dt = 0.$$

Integration of the first term gives

$$\frac{1}{2}\int_\Omega \left| \int_0^T \nabla(w-\tilde{w})(\bar{x},s)\,ds \right|^2 d\bar{x} + \int_0^T \int_\Gamma (u-\tilde{u})(x,t)\big(u^m-\tilde{u}^m\big)(x,t)\,dx\,dt = 0.$$

Since both terms are nonnegative, they must be zero. Therefore, $u=\tilde{u}$ in $\Gamma$. Obviously this also gives, as a byproduct, $w=\tilde{w}$ in $\Omega$. $\square$

**Remark.** Observe that this proof only requires $u(\cdot,t), \tilde{u}(\cdot,t) \in L^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$.

## 5. Properties of weak energy solutions

In the next sections we restrict ourselves to the range $m > m_*$.

### *5.1. Conservation of mass*

We establish next a property that is typical of diffusive processes.

**Theorem 5.1.** *If* $(u,w)$ *is a weak energy solution to Problem* (2.1) *with initial datum* $f$*, then for every* $t>0$ *we have*

$$\int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} f(x)\,dx. \tag{5.1}$$

**Proof.** The case $m=1$ follows from the explicit representation (1.8). For general $m$ we use the integral identity (2.2) with a particular test function. Consider a nonnegative nonincreasing cut-off function $\psi(s)$ such that $\psi(s)=1$ for $0 \leqslant s \leqslant 1$, $\psi(s)=0$ for $s \geqslant 2$, and define $\varphi_R(\bar{x}) = \psi(|\bar{x}|/R)$. We obtain, for every $t_2 > t_1 \geqslant 0$,

$$\int_\Gamma \big(u(x,t_2)-u(x,t_1)\big)\varphi_R(x,0)\,dx = -\int_{t_1}^{t_2} \int_\Omega \big\langle \nabla w(\bar{x},t), \nabla\varphi_R(\bar{x})\big\rangle\,d\bar{x}\,dt.$$

Integrating by parts, noting that $\frac{\partial\varphi_R}{\partial y}(x,0)=0$, we have that the space integral inside the right-hand side is

$$I = \int_\Omega w(\bar{x},t)\Delta\varphi_R(\bar{x})\,d\bar{x}.$$

With the test function chosen we have

$$|I| \leqslant cR^{-2} \int\limits_{R<|\bar{x}|<2R} w(\bar{x},t)\,d\bar{x}.$$

We now estimate this integral for every $t_1 < t < t_2$. If $m > 1$ we just observe that

$$\begin{aligned}|I| &\leqslant cR^{-2} \int\limits_0^{2R} \int\limits_{\mathbb{R}^N} \left|w(x,y,t)\right| dx\,dy \leqslant cR^{-1} \left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}^{m-1} \left\|u(\cdot,t)\right\|_{L^1(\mathbb{R}^N)} \\ &\leqslant cR^{-1} \to 0,\end{aligned}$$

where we have used (3.15). In the case $m < 1$, applying Hölder's inequality with some exponent $p > 1/m$ we have

$$\begin{aligned}|I| &\leqslant cR^{-2} \left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}^{m-1/p} \Biggl( \int\limits_{R<|\bar{x}|<2R} \left|w(\bar{x},t)\right|^{1/m} d\bar{x} \Biggr)^{1/p} \left|\left\{R < |\bar{x}| < 2R\right\}\right|^{(p-1)/p} \\ &\leqslant cR^{-2+(N+1)((p-1)/p)} \left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}^{m-1/p} \Biggl( \int\limits_0^{2R} \int\limits_{\mathbb{R}^N} \left|w(x,y,t)\right|^{1/m} dx\,dy \Biggr)^{1/p} \\ &\leqslant cR^{-2+(N+1)((p-1)/p)+1/p} \left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}^{m-1/p} \left\|u(\cdot,t)\right\|_{L^1(\mathbb{R}^N)}^{1/p} \leqslant cR^{N(p-1)/p-1}.\end{aligned}$$

Finally observe that if $m > m_*$ we can choose $1/m < p < N/(N-1)$ to force the last term to go to zero as $R \to \infty$. $\square$

SOLUTIONS THAT LOSE MASS. We now present a result that shows the necessity of the condition $m > m_*$. In fact if $0 < m < m_*$ there is a phenomenon of extinction in finite time, which makes impossible to have conservation of mass. The proof is almost exactly the same as the one in [10] for the PME model, where the corresponding condition on $m$ is $0 < m < (N-2)/N$ instead of $0 < m < m_* = (N-1)/N$.

**Proposition 5.1.** *Let $N > 1$ and $0 < m < (N-1)/N$, and let $f \in L^1(\mathbb{R}^N) \cap L^{(1-m)N}(\mathbb{R}^N)$. Then there is a finite time $T > 0$ such that the solution $u$ to Problem* (1.1) *satisfies $u(x,T) \equiv 0$ in $\mathbb{R}^N$.*

**Proof.** As we have said, the proof follows the argument in [10]. Therefore we leave the details to be consulted there. Assume also for simplicity $u \geqslant 0$.

If we consider $\varphi = w^p$ in the equality (2.6), we get

$$\frac{4p}{(p+1)^2} \int\limits_\Omega \left|\nabla w^{\frac{p+1}{2}}\right|^2 d\bar{x} + \frac{1}{1+pm} \int\limits_\Gamma \frac{\partial(u^{pm+1})}{\partial t}\,dx = 0.$$

The use of this test function is justified in [10]. By the trace inequality we obtain that there exists a positive constant $C = C(p, N)$ such that

$$C\bigg(\int_\Gamma u^{\frac{(p+1)mN}{N-1}}\,dx\bigg)^{\frac{N-1}{N}} + \int_\Gamma \frac{\partial(u^{pm+1})}{\partial t}\,dx \leqslant 0.$$

If we now choose $p = (N(1-m)-1)/m > 0$, we get $((p+1)mN)/(N-1) = pm+1$. Therefore, the function $J(t) = \int_\Gamma u^{pm+1}\,dx$ satisfies the inequality

$$J'(t) + CJ^{\frac{N-1}{N}}(t) \leqslant 0.$$

This implies extinction in finite time for $J(t)$ and thus for $u(x,t)$, provided $J(0)$ is finite. $\square$

**Example.** There exists an explicit example of the above extinction property for a particular $m < m_*$. It has the form (separated variables),

$$u(x,t) = G(x)H(t).$$

Substituting this expression in (1.1), we have $H(t) = c(T-t)^{1/(1-m)}$, and $G$ solves the nonlocal equation

$$(-\Delta)^{1/2}G^m = G.$$

In the special case $m = (N-1)/(N+1)$, there exists an explicit family of solutions

$$G_{\delta,\tau} = A(\tau)\big[\tau^2 + |x-\delta|^2\big]^{-(N+1)/2},$$

with $\tau > 0$, $\delta \in \mathbb{R}^N$ and where $A(\tau) > 0$ has an explicit expression, see [13] and also [19]. Observe that $m < m_*$ and $G \in L^1(\mathbb{R}^N)$, and thus $u(\cdot,t) \in L^1(\mathbb{R}^N)$ for any $0 \leqslant t < T$, while $u(x,T) \equiv 0$.

### *5.2. Positivity and regularity*

We show in this subsection that nonnegative bounded weak energy solutions are in fact positive everywhere in $\overline{\Omega}$. This is true for every $m > m_*$. The result is in sharp contrast with what happens for the local analog, the PME $\partial_t u = \Delta u^m$ in the case $m > 1$, for which initial values with compact support produce solutions that develop a free boundary. Free boundaries are a main feature of the standard PME theory, but they are not available here.

The idea behind our positivity result is as follows: if $u \geqslant 0$ is a classical solution and $u(x_0,t) = 0$ for some $x_0$ and $t$, then formula (1.3) gives $(-\Delta)^{1/2}u^m(x_0,t) < 0$, and hence $\partial_t u(x_0,t) > 0$. If $m > 1$ we only have that the bounded solution $u$ is Hölder continuous by [6]. In the case $m < 1$ we do not even have that regularity. Now we perform rigorously the proof of positivity for weak solutions using the extension Problem (2.1).

**Theorem 5.2.** *Assume $m > m_*$. The weak energy solution $u$ to Problem* (1.1) *with a bounded nonnegative initial datum is positive for positive times. Even more, the corresponding pair $(u, w)$ to Problem* (2.1) *satisfies $w(x, y, t) > 0$ for every $x \in \mathbb{R}^N$, $y \geqslant 0$ and every $t > 0$.*

**Proof.** The case $m = 1$ follows from formula (1.8).

CASE $m > 1$: We already know that for every $t > 0$, function $w$ is Hölder continuous, $w \geqslant 0$ in $\overline{\Omega}$, and $w > 0$ in $\Omega$. We have to prove that $w > 0$ also on the boundary, $\Gamma$. By the comparison result, we only need to consider compactly supported initial data.

In a first step we show that if $w$ is not strictly positive on $\Gamma$ at some time $T$, then the supports of $w(\cdot, 0, t)$ form an expanding (in time) family of compact sets for $0 \leqslant t \leqslant T$. This follows from estimate (3.17) and Alexandrov's reflection principle. In fact (3.17) implies that

$$w(\bar{x}, t_2) \geqslant w(\bar{x}, t_1)\left(\frac{t_1}{t_2}\right)^{m/(m-1)},$$

for every $\bar{x} \in \overline{\Omega}$, $t_2 \geqslant t_1 > 0$, which is called retention property. Next we claim that if the support of the initial value $f$ is contained in the ball $B_R$, and $w(x_0, 0, t_0) = 0$ for some $x_0 \in \mathbb{R}^N$, $t_0 > 0$, then the support of $w(\cdot, 0, t)$ is also compact for any $0 < t \leqslant t_0$, and contained in a ball of radius depending on $|x_0|$ and $R$.

To prove the claim we reflect, for any given point $(x_1, y) \in \overline{\Omega}$, around the hyperplane $\pi \equiv (x_1 - x_0) \cdot (x - (x_0 + x_1)/2) = 0$. It is clear that if $|x_0 - (x_0 + x_1)/2| > |x_0| + R$, then the hyperplane $\pi$ divides the half-space $\overline{\Omega}$ in two parts, $\overline{\Omega} = M_1 \cup M_2$ with $B_R \times [0, \infty) \subset M_1$, $(x_0, 0) \in M_2$. In this way, by the comparison principle, we obtain that the function $z(x, y, t) = w(x, y, t) - w(x_0 + x_1 - x, y, t)$ satisfies $z(x, y, t) \geqslant 0$ for every $(x, y) \in M_1$, $t > 0$. The comparison principle holds on the half-space $M_1$. Thus $w(x_1, 0, t) \leqslant w(x_0, 0, t) = 0$ for every $0 < t \leqslant t_0$. A sufficient condition for $x_1$ to get this argument work is $|x_1| \geqslant 3|x_0| + 2R$.

In a second step we assume (thanks to the previous argument) that $w(x, 0, t) \equiv 0$ in some ball $B \subset \Gamma$ for $0 \leqslant t \leqslant t_1$. Then we have, for every test function $\varphi$ that vanishes on $\partial B \times (0, \infty)$, that

$$0 = \int_0^{t_1}\int_B u\frac{\partial\varphi}{\partial t}\,dx\,dt = \int_0^{t_1}\int_0^{\infty}\int_B \langle\nabla w, \nabla\varphi\rangle\,dx\,dy\,dt = -\int_0^{t_1}\int_B \frac{\partial w}{\partial y}\varphi\,dx\,dt.$$

This gives $\partial_y w(x, 0, t) \equiv 0$ for $x \in B$, $0 \leqslant t \leqslant t_1$. But $w$ is a continuous nonnegative harmonic function in the half cylinder which vanishes on the part of the boundary $y = 0$. Hopf's Lemma implies $\partial_y w(x, 0, t) < 0$ for $x \in B$. This is a contradiction. Therefore, $w(\cdot, 0, t)$ is positive everywhere.

CASE $m < 1$: In this case the proof is different, based on a weak Harnack inequality. Using estimate (3.17) and the fact that the solution is bounded, we know that for every $t > 0$ there exists a constant $A > 0$ such that

$$\int_\Omega \langle\nabla w, \nabla\varphi\rangle + A\int_\Gamma w\varphi \geqslant 0,$$

for every nonnegative test function $\varphi$. Once we have this, we can use part of the proof of Lemma 2.4 in [16] to get a weak Harnack inequality in each large ball $B_R = \{|x - x_0|^2 + y^2 <$

$R^2\} \subset \mathbb{R}^{N+1}$ with center on $\Gamma$. First of all, conservation of mass plus (3.15), together with the fact that $w \geqslant 0$, $w \not\equiv 0$, imply that there exists some $R > 0$ large such that $\int_{B_{R/2}} w\, d\bar{x} > 0$.

If we consider the function

$$z(x, y) = e^{-A|y|} w\big(x, |y|\big),$$

then we have that $z$ satisfies

$$\int\limits_{B_R} \langle \nabla z, \nabla \varphi \rangle - 2A \int\limits_{B_R} \operatorname{sign}(y) \frac{\partial z}{\partial y} \varphi - A^2 \int\limits_{B_R} z\varphi \geqslant 0,$$

i.e., it is a weak supersolution to an equation for which we can apply Theorem 8.18 in [25], to get

$$\inf_{B_{R/2}} z \geqslant c R^{-N-1} \|z\|_{L^1(B_{R/2})}.$$

This means $z > 0$, (and thus $w > 0$) in $\overline{\Omega}$. □

As a corollary of this result, we can establish also regularity for the case $m < 1$, provided $m > m_*$.

**Theorem 5.3.** *Let $m_* < m < 1$. Then any bounded weak energy nonnegative solution $u$ to Problem* (1.1) *satisfies $u \in C^\alpha(\mathbb{R}^N \times (0, \infty))$ for some $0 < \alpha < 1$.*

**Proof.** The above-mentioned regularity result of [6] applies to the equation

$$\frac{\partial \beta(v)}{\partial t} + (-\Delta)^{1/2}(v) = 0$$

in some ball $x \in B \subset \mathbb{R}^N$, $t > 0$, with some nondegeneracy condition on the constitutive monotone function $\beta$. Once we know that in such a ball the solution is essentially bounded below away from zero, the requirements on the function $\beta$ in [6] are fulfilled. □

### *5.3. Strong solutions*

We prove here that every nonnegative bounded weak energy solution is in fact a strong solution. We need to show that the time partial derivative of $u$ is an $L^1$ function and that the second equality in (2.1) holds almost everywhere.

As a first step we show that the time-increment quotients are bounded in $L^1(\Gamma)$, and thus the limit $\partial_t u$ must be a Radon measure. Our purpose is to prove that the limit is still in $L^1$, and this is proved later. Observe that the result is clear in the case $m = 1$, from (1.8). We therefore assume $m \neq 1$.

**Proposition 5.2.** *If $u$ is the weak solution to Problem* (1.1) *constructed in Theorem* 2.1*, then $h^{-1}(u(\cdot, t+h) - u(\cdot, t)) \in L^1(\mathbb{R}^N)$ for every $t, h > 0$.*

The proof is exactly the same as in the PME case, see [9]. In fact, the following estimate holds

$$\int_{\mathbb{R}^N} \frac{1}{h}\big|u(x,t+h)-u(x,t)\big|\,dx \leqslant \frac{2}{|m-1|t}\|f\|_{L^1(\mathbb{R}^N)}. \tag{5.2}$$

An analogous result for more general nonlinearities is given in [22].

Observe that by the Mean Value Theorem, the same type of estimate can be obtained for the time-increment quotients of $w=u^m$ if $m>1$ since $u\in L^\infty$. If $m<1$ we obtain a bound locally for nonnegative solutions, since in each ball of $\mathbb{R}^N$ we have $u\geqslant c>0$.

We now prove a result which turns out to be fundamental in the proof of Theorem 2.6.

**Proposition 5.3.** *Let $f\geqslant 0$ and let $u$ be the weak solution to Problem* (1.1) *constructed in Theorem* 2.1. *Then $\partial_t(u^{(m+1)/2})\in L^2_{loc}(\mathbb{R}^N\times[0,T])$.*

**Proof.** The formal proof is simple, using the function $\partial_t w$ as test function in the weak formulation, much as in the PME case, cf. [37, Section 5.5], of course after some obvious changes. The problem is the justification of the calculations, since we have not established the existence of any kind of differentiability for the solutions or suitable approximations.

Here we do by brute force as follows:

• We use the weak formulation with the test function $\varphi=\delta^h(w*\rho)$, where $\rho=\eta*\eta$ and $\eta$ is a convolution kernel, acting only on the time variable. We fix the notation $\hat f=f*\eta$, $\tilde f=f*\rho$, applied to functions of $t$. We also make use of the following calculus identity

$$\int \tilde f(t)g(t)\,dt=\int \hat f(t)\hat g(t)\,dt. \tag{5.3}$$

We prove this identity in Lemma A.1 of Appendix A for the reader's convenience. In addition, we take $\eta(t)=\eta_h(t)=(1/h)\eta_1(t/h)$, where $\eta_1$ is a smooth, symmetric, compactly supported nonnegative function, with support $[-1,1]$. Thus, $\rho=\rho_h$ inherits the same properties, with $\operatorname{supp}(\rho_1)=[-2,2]$. We also use the notation

$$\delta^h w(t)=\frac{1}{2h}\big(w(t+h)-w(t-h)\big)$$

for a discrete time derivative, omitting the spatial variable. Observe that since $|\nabla w|$ is in $L^2$, this is a good test function.

• Going to the weak formulation (2.3) with the test function $\delta^h\tilde w$, we have

$$-\int_{t_1}^{t_2}\int_{\Gamma}\delta^h\big(w*\rho'\big)u\,dx\,dt+\int_{\Gamma}\delta^h(w*\rho)u\big|_{t_1}^{t_2}\,dx=-\int_{t_1}^{t_2}\int_{\Omega}\big\langle\nabla\delta^h\tilde w,\nabla w\big\rangle\,d\bar x\,dt, \tag{5.4}$$

where $\rho'=d\rho/dt$.

The left-hand side mimicks $\iint(\partial_t w)(\partial_t u)\,dx\,dt=c\iint(\partial_t(u^{(m+1)/2}))^2\,dx\,dt$, while the right-hand side mimicks $-\iint\langle\nabla(\partial w/\partial t),\nabla w\rangle\,d\bar x\,dt$. Here the times $t_1$ and $t_2$ are subject to be moved slightly on the condition that $t_1<T_1<T_2<t_2$ for some fixed $0<T_1<T_2$. We now analyze the different integrals.

• Using (5.3), the integral in the right-hand side can be written as

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\int_{\Omega}\langle\delta^h\nabla\hat{w},\nabla\hat{w}\rangle\,d\bar{x}\,dt\\
&\quad=\frac{1}{2h}\int_{t_1}^{t_2}\int_{\Omega}\big\langle\nabla\hat{w}(t-h),\nabla\hat{w}(t)\big\rangle\,d\bar{x}\,dt-\frac{1}{2h}\int_{t_1}^{t_2}\int_{\Omega}\big\langle\nabla\hat{w}(t+h),\nabla\hat{w}(t)\big\rangle\,d\bar{x}\,dt\\
&\quad=\frac{1}{2h}\int_{t_1}^{t_1+h}\int_{\Omega}\big\langle\nabla\hat{w}(t-h),\nabla\hat{w}(t)\big\rangle\,d\bar{x}\,dt-\frac{1}{2h}\int_{t_2-h}^{t_2}\int_{\Omega}\big\langle\nabla\hat{w}(t+h),\nabla\hat{w}(t)\big\rangle\,d\bar{x}\,dt\\
&\quad=\frac{1}{2h}\int_{t_1-h}^{t_1}\int_{\Omega}\big\langle\nabla\hat{w}(t+h),\nabla\hat{w}(t)\big\rangle\,d\bar{x}\,dt-\frac{1}{2h}\int_{t_2}^{t_2+h}\int_{\Omega}\big\langle\nabla\hat{w}(t-h),\nabla\hat{w}(t)\big\rangle\,d\bar{x}\,dt.
\end{aligned}
$$

These two last integrals are bounded, in absolute value, by

$$
\frac{1}{2h}\int_{t_1-h}^{t_1+h}\int_{\Omega}\big|\nabla\hat{w}(t)\big|^2\,d\bar{x}\,dt+\frac{1}{2h}\int_{t_2-h}^{t_2+h}\int_{\Omega}\big|\nabla\hat{w}(t)\big|^2\,d\bar{x}\,dt=Y_1.
$$

Since $|\nabla w|^2\in L^1$, by picking good times $t_1$ and $t_2$ from a dyadic division of intervals around $T_1$ and $T_2$, the quantity $Y_1$ is bounded uniformly in $h$ (though $t_1$ and $t_2$ depend on $h$).

• We turn to the left-hand side terms in (5.4). Assume first $m>1$. The second integral in (5.4) is bounded using Proposition 5.3.

As to the first integral in (5.4) we have

$$
\begin{aligned}
-\int_{t_1}^{t_2}\int_{\Gamma}\delta^h\big(w*\rho'\big)u\,dx\,dt&=\int_{t_1+h}^{t_2-h}\int_{\Gamma}\big(w*\rho'\big)\delta^h u\,dx\,dt\\
&\quad+\frac{1}{2h}\int_{t_1-h}^{t_1+h}\int_{\Gamma}\big(w*\rho'\big)u(t+h)\,dx\,dt\\
&\quad-\frac{1}{2h}\int_{t_2-h}^{t_2+h}\int_{\Gamma}\big(w*\rho'\big)u(t-h)\,dx\,dt.
\end{aligned}
$$

The last two integrals are bounded in absolute value, in the same way as the second integral in (5.4). The first term in the right-hand side above is the one we want to estimate carefully. By Lemma A.2 and under the extra assumption of monotonicity in time, $\partial_t u\geqslant 0$, we have

$$
I=\big(w*\rho'\big)(x,t)\geqslant c\delta^h w(x,t), \tag{5.5}
$$

with a positive constant $c$ that depends on $m$, $N$ and the smoothing kernel $\eta_1$. It only remains to observe that

$$\left(\delta^h w\right)\left(\delta^h u\right) = \left(\delta^h u^m\right)\left(\delta^h u\right) \geqslant c\left(\delta^h u^{(m+1)/2}\right)^2.$$

This turns out to be an easy calculus problem, using the technical Lemma A.3.

• Summing up, we get a uniform estimate

$$\int_{T_1}^{T_2}\int_{\Gamma} \left(\delta^h u^{(m+1)/2}\right)^2 \, dx \, dt \leqslant C.$$

Letting $h \to 0$, we get $\partial_t(u^{\frac{m+1}{2}}) \in L^2(\mathbb{R}^N \times [T_1, T_2])$ for every $0 < T_1 < T_2$.

• General situation when $m > 1$: we want to apply Lemma A.2 to $I = (w * \rho')(x,t)$ without the extra assumption of monotonicity in time, only using that if $m > 1$, $\partial_t u \geqslant -cu$. Then we use the second version of Lemma A.2. An extra term appears but it is controllable.

• In the case $m < 1$ we need to perform some little extra calculations. First, in order to use a bound of the form (5.2) for $w$ we take advantage of the fact that $w \geqslant c > 0$ in every compact set of $\overline{\Omega}$. Thus we consider a new test function by multiplying $\varphi$ by a cutoff function $\psi(\bar{x})$. The extra terms obtained are easily bounded.

On the other hand, formula (5.5) holds with reverse inequality,

$$I = \left(w * \rho'\right)(x,t) \leqslant c\delta^h w(x,t), \tag{5.6}$$

provided $\partial_t w \leqslant 0$, as before. Care has to be taken in the general case, where we use the estimate $\partial_t w \leqslant cw$. □

We now prove the main result.

**Theorem 5.4.** *The weak* (*nonnegative*) *solution $u$ to Problem* (1.1) *constructed in Theorem* 2.1 *is a strong solution. Moreover,*

$$\left\| \frac{\partial u}{\partial t}(\cdot,t) \right\|_{L^1(\mathbb{R}^N)} \leqslant \frac{2}{|m-1|t} \|f\|_{L^1(\mathbb{R}^N)}. \tag{5.7}$$

**Proof.** Let $w$ be the weak solution to Problem (2.1) associated to $u$. We want to prove that the time derivative of $u$ is actually an integrable function and that the normal derivative of $w$ on $\Gamma$ is a distribution. Thus the second equation in (2.1), i.e. Eq. (1.1), holds almost everywhere.

To deal with the normal derivative we only have to take into account that, thanks to the trace embedding, since for every $t > 0$ we have $w \in H^1(\Omega)$, then $\partial_y w(\cdot,0,t)$ is a distribution in $H^{-1/2}(\Gamma)$.

We now look at $\partial_t u(\cdot,t)$. We use a technical result by Bénilan [7], see also [37]. As in the previous proof, we may assume $\partial_t u \geqslant 0$. Also we know that $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for every $t > 0$ and finally, from Proposition 5.3 we have $\partial_t(u^{(m+1)/2}) \in L^2(\mathbb{R}^N)$.

All these estimates allow us to apply Lemma 8.2 in [37] to get $\partial_t u \in L^p_{loc}(\mathbb{R}^N)$ for every $p \in [1, p_1)$, where $p_1 = \min\{(m+1)/m), 2\}$. This gives $\partial_t u \in L^1_{loc}(\mathbb{R}^N)$ for every $t > 0$. Formula (5.7) follows from (5.2), and in fact $\partial_t u \in L^1(\mathbb{R}^N)$ for every $t > 0$. □

Once we know that our solution is a strong solution, we can perform the above-mentioned formal calculation, analogous to the case of the PME.

**Proposition 5.4.** *In the above hypotheses,*

$$\int_{t_1}^{t_2}\int_{\Gamma}\left|\frac{\partial u^{(m+1)/2}}{\partial t}\right|^2(x,t)\,dx\,dt \leqslant \frac{m+1}{8mt_1}\int_{\Gamma} u^{m+1}(x,t_1)\,dx. \tag{5.8}$$

We end this section with two more estimates that will be useful in the sequel. By comparison assume $f \geqslant 0$. Putting $\varphi = w$ as test function and obtain

$$\int_0^t\int_{\Omega}|\nabla w|^2\,d\bar{x}\,ds + \frac{1}{m+1}\int_{\Gamma} u^{m+1}\,dx = \frac{1}{m+1}\int_{\Gamma} f^{m+1}\,dx. \tag{5.9}$$

We have thus a control of the $L^2$ norm of the gradient in terms of the initial data.

**Proposition 5.5.** *In the above hypotheses*

$$\|\nabla w\|_{L^2(\Omega\times(0,\infty))} \leqslant c\|f\|_{L^{m+1}(\Gamma)}^{(m+1)/2}. \tag{5.10}$$

Another consequence of (5.9) is that the norm $\|u(\cdot,t)\|_{L^{m+1}(\mathbb{R}^N)}$ is nonincreasing in time. In fact this also follows from the elliptic estimates of Section 3.2.

An easy generalization shows the following property.

**Proposition 5.6.** *In the above hypotheses any $L^p$ norm of the solution is nonincreasing in time for every $p \geqslant 1$. Even more, if $\Psi$ is any convex nonnegative real function, then $\int_{\mathbb{R}^N}\Psi(u(x,t))\,dx$ is a nonincreasing function.*

**Proof.** We have

$$\int_{\Gamma}\Psi\big(u(x,t_2)\big)\,dx - \int_{\Gamma}\Psi\big(u(x,t_1)\big)\,dx$$
$$= -\int_{t_1}^{t_2}\frac{1}{m}\int_{\Omega} w^{(1-m)/m}\Psi''\big(w^{1/m}\big)|\nabla w|^2\,d\bar{x}\,ds \leqslant 0. \qquad \square$$

## 6. Solutions with data in $L^1$

In order to construct solutions with initial data in $L^1$ we approximate by problems with data in $L^1 \cap L^\infty$ and use the $L^1$ contractivity to pass to the limit (together with the estimates of the gradients in $L^2$).

We begin by proving the bound in $L^\infty$ in terms of the $L^1$ norm of the initial datum, estimate (2.7).

### 6.1. Smoothing effect

**Proof of Theorem 2.4.** We use a classical parabolic Moser iterative technique, together with the trace immersion $H^1(\Omega) \hookrightarrow L^{\frac{2N}{N-1}}(\Gamma)$. Let $t > 0$ be fixed, and consider the sequence of times $t_k = (1 - 2^{-k})t$. Using the integral formulation of the problem in $[t_{k-1}, t_k]$ with the test function $\varphi = w^q$, for some $q > 0$, we get

$$(qm+1)\int_{t_{k-1}}^{t_k}\int_{\Omega}\langle\nabla w, \nabla w^q\rangle(\bar{x},\tau)\,d\bar{x}\,d\tau = -\int_{\Gamma} u^{qm+1}(x,t_k)\,dx + \int_{\Gamma} u^{qm+1}(x,t_{k-1})\,dx.$$

Manipulating the integral in $\Omega$ we obtain

$$\begin{aligned}
\int_{\Gamma} u^{mq+1}(x,t_{k-1})\,dx &\geqslant \frac{4q(qm+1)}{(q+1)^2}\int_{t_{k-1}}^{t_k}\int_{\Omega}\big|\nabla w^{\frac{q+1}{2}}\big|^2(\bar{x},\tau)\,d\bar{x}\,d\tau \\
&\geqslant c(q)\int_{t_{k-1}}^{t_k}\bigg(\int_{\Gamma} u^{\frac{Nm(q+1)}{N-1}}(x,\tau)\,dx\bigg)^{\frac{N-1}{N}}d\tau \\
&\geqslant c(q)2^{-k}t\bigg(\int_{\Gamma} u^{\frac{Nm(q+1)}{N-1}}(x,t_k)\,dx\bigg)^{\frac{N-1}{N}}.
\end{aligned}$$

We have also used that any $L^r$-norm of the solution is nonincreasing in time. Therefore we have

$$\big\|u(\cdot,t_k)\big\|_{sm(q+1)} \leqslant \big(c(q)t\big)^{-\frac{1}{m(q+1)}}2^{\frac{k}{m(q+1)}}\big\|u(\cdot,t_{k-1})\big\|_{mq+1}^{\frac{mq+1}{m(q+1)}},$$

where $s = N/(N-1)$. The constant $c(q)$ can be bounded from below by a positive constant depending on the minimum value taken by $q$, as well as on $m$ and $N$. In order to iterate this estimate, we put $p_k = sm(q+1)$, and $p_{k-1} = mq+1$, i.e., $p_k = s(p_{k-1}+m-1)$. If we start with $q = 1/m$, thus taking $p_0 = 2$, we easily have $p_k > p_{k-1} \geqslant 2$, and therefore the constant in the above estimate can be taken independent of $k$. Thus, if we denote $U_k = \|u(\cdot,t_k)\|_{p_k}$, we have

$$U_k \leqslant (ct)^{-\frac{s}{p_k}}2^{\frac{ks}{p_k}}U_{k-1}^{\frac{sp_{k-1}}{p_k}}.$$

This implies

$$U_k \leqslant (ct)^{-a_k}2^{b_k}U_0^{c_k},$$

with exponents

$$a_k = \frac{1}{p_k}\sum_{j=1}^{k}s^j, \qquad b_k = \frac{1}{p_k}\sum_{j=1}^{k}(k+1-j)s^j, \qquad c_k = \frac{2s^k}{p_k}.$$

It is then easy to see that

$$p_k = As^k + B, \quad B = \frac{s(1-m)}{s-1} = N(1-m), \quad A = 2 - B,$$

and then, using that $s > 1$, the following limits hold

$$\lim_{k\to\infty} p_k = \infty, \qquad \lim_{k\to\infty} a_k = \frac{N}{A}, \qquad \lim_{k\to\infty} b_k = \frac{N^2}{A}, \qquad \lim_{k\to\infty} c_k = \frac{2}{A}.$$

We obtain in this way the following $L^2$–$L^\infty$ regularity result,

$$\|u(\cdot,t)\|_\infty = \lim_{k\to\infty} U_k \leqslant ct^{-\frac{N}{A}} U_0^{\frac{2}{A}} = ct^{-\gamma'} \|f\|_2^{\frac{2\gamma'}{N}},$$

where $\gamma' = (m-1+2/N)^{-1}$.

To pass from this estimate to the desired $L^1$–$L^\infty$ smoothing effect, we use an iterative interpolation argument. Putting $\tau_k = 2^{-k}t$, the above applied in the interval $[\tau_1, \tau_0]$ gives,

$$\|u(\cdot,t)\|_\infty \leqslant c(t/2)^{-\gamma'} \|u(\cdot,\tau_1)\|_2^{\frac{2\gamma'}{N}} \leqslant c(t/2)^{-\gamma'} \|u(\cdot,\tau_1)\|_1^{\frac{\gamma'}{N}} \|u(\cdot,\tau_1)\|_\infty^{\frac{\gamma'}{N}}.$$

We now apply the same estimate in the interval $[\tau_2, \tau_1]$, thus getting

$$\|u(\cdot,t)\|_\infty \leqslant c(t/2)^{-\gamma'} \|u(\cdot,\tau_1)\|_1^{\frac{\gamma'}{N}} \Big(c(t/4)^{-\gamma'} \|u(\cdot,\tau_2)\|_2^{\frac{2\gamma'}{N}}\Big)^{\frac{\gamma'}{N}}.$$

Iterating this calculation in $[\tau_k, \tau_{k-1}]$ and using again that the norms are nonincreasing in time, we obtain

$$\|u(\cdot,t)\|_\infty \leqslant c^{a_k} 2^{b_k} t^{-d_k} \|u(\cdot,0)\|_1^{e_k} \|u(\cdot,\tau_k)\|_2^{f_k}.$$

Using the fact that $m > m_*$ implies $\frac{\gamma'}{N} = \frac{1}{(m-1)N+2} < 1$, we see that the exponents satisfy, in the limit $k \to \infty$,

$$\begin{aligned}
a_k &= \sum_{j=0}^{k-1} \left(\frac{\gamma'}{N}\right)^j \to \frac{(m-1)N+2}{(m-1)N+1} = \frac{\gamma}{N} + 1 < \infty, \\
b_k &= \sum_{j=0}^{k-1} \gamma'(j+1) \left(\frac{\gamma'}{N}\right)^j \to \frac{(m-1)N+2}{((m-1)N+1)^2} < \infty, \\
d_k &= \gamma' a_k \to \gamma, \\
e_k &= a_k - 1 \to \frac{\gamma}{N}, \\
f_k &= 2\left(\frac{\gamma'}{N}\right)^k \to 0. \qquad \square
\end{aligned}$$

**Remark.** Once we assert that the constant $c$ in (2.7) is universal, then the values of the exponents $\gamma$ and $\gamma/N$ are given as an immediate consequence of the invariance of the equation under the two-parameter scaling group. This is similar to what happens in the Sobolev inequalities, cf. [24, p. 262], or what happens in the PME, cf. [36, p. 29].

### 6.2. $L^1$ contraction

**Proof of Theorem 2.5.** We first recall that $L^1$-energy solutions are weak energy solutions for every $\tau > 0$. Therefore, by the trace embedding we have $u(\cdot,\tau) \in L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant 2N/(N-1)$ if $N > 1$ (for every $p \geqslant 1$ if $N = 1$). Now condition $m > m_*$ implies $u(\cdot,\tau) \in L^{m+1}(\mathbb{R}^N)$. This allows us to use the proof of uniqueness for weak energy solutions, cf. Lemma 4.1 and the Remark after it, for every $\tau > 0$. Continuity in $L^1$ gives uniqueness up to $\tau = 0$. We now approximate the initial datum $f \in L^1(\mathbb{R}^N)$ by functions $f_k \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f_k \to f$ in $L^1(\mathbb{R}^N)$. The above implies that corresponding solutions $u_k$ converge in $L^1(\mathbb{R}^N)$ to our solution $u$, and the contractivity result in (3.2) passes to the limit. □

### 6.3. End of the proof of Theorem 2.6

Once we have the $L^1$ contraction (2.8) at our disposal, and using the above approximation of $u$ by $u_k$, we observe that the estimate (2.7) does not depend on the $L^\infty$ norm of the approximations, and thus it is true for the limit.

As to the conservation of mass, we recall that the proof of Theorem 5.1 relies on bounds for the $L^p$ norms of $u(\cdot,t)$ for $t > 0$. This is handled with the smoothing effect just proved. We thus have, repeating the proof of Theorem 5.1,

$$\left|\int_{\mathbb{R}^N} \big(u(x,t) - f(x)\big)\varphi_R(x)\,dx\right| \leqslant cR^{-2}\int_0^t \int_{R<|\bar{x}|<2R} \big|w(\bar{x},s)\big|\,d\bar{x}\,ds.$$

We now use (2.7) to estimate the right-hand side $I$. If $m > 1$ we get

$$I \leqslant cR^{-1}\int_0^t \big\|u(\cdot,s)\big\|_{L^\infty(\mathbb{R}^N)}^{m-1}\big\|u(\cdot,s)\big\|_{L^1(\mathbb{R}^N)}\,ds \leqslant cR^{-1}\|f\|_1^{\frac{\gamma(m-1)}{N}+1}\int_0^t s^{-\gamma(m-1)}\,ds \to 0,$$

since $\gamma(m-1) < 1$. In the case $m < 1$ we obtain, instead,

$$I \leqslant cR^{N(p-1)/p-1}\int_0^t \big\|u(\cdot,s)\big\|_{L^\infty(\mathbb{R}^N)}^{m-1/p}\big\|u(\cdot,s)\big\|_{L^1(\mathbb{R}^N)}^{1/p}\,ds$$

$$\leqslant cR^{N(p-1)/p-1}\|f\|_1^{\frac{\gamma(m-1/p)}{N}+\frac{1}{p}}\int_0^t s^{-\gamma(m-1/p)}\,ds \to 0,$$

by choosing $1/m < p < N/(N-1)$, which is possible whenever $m > m_*$.

Finally, the proofs of positivity and regularity in Theorems 5.2 and 5.3 also apply using again the smoothing effect.

## 7. Continuous dependence

The aim of this section is to prove the continuous dependence of the solutions constructed in this paper with respect to the initial data $f$ and exponent $m$. This is true for $m > m_*$. Let us introduce the notation: if $u = u_{m,f}$ is the solution corresponding to $m$ and $f$, we write $S(m, f) = u_{m,f}$.

**Theorem 7.1.** *The map* $S : (m_*, \infty) \times L^1(\mathbb{R}^N) \to C([0, T] : L^1(\mathbb{R}^N))$ *is continuous.*

This will follow from a result of nonlinear Semigroup Theory which states that if each of $A_n$, $n = 1, 2, \ldots, \infty$ is an $m$-accretive operator in a Banach space $\mathcal{X}$, $f_n \in \overline{D(A_n)}$ and $u_n$ is the solution of

$$\frac{du_n}{dt} + A_n u_n = 0, \quad u_n(0) = f_n,$$

then $A_n \to A_\infty$, $f_n \to f_\infty$ implies $u_n \to u_\infty$ in $C([0, \infty) : \mathcal{X})$, where $A_n \to A$ means

$$\lim_{n\to\infty} (I + A_n)^{-1} g = (I + A_\infty)^{-1} g \quad \text{for all } g \in \mathcal{X}.$$

See, e.g, [20,24] for statements and references. Hence, the theorem will be a corollary of the convergence of $(I + A_{m_n})^{-1}$, where $A_m(u) = (-\Delta)^{1/2} u^m = -\partial_y w$, $w = E(u^m)$. This is what we prove next.

**Proposition 7.1.** *Let* $\{m_n\}_{n=1}^{\infty}$ *be a bounded sequence of numbers in* $(m_*, \infty)$ *such that* $\lim_{n\to\infty} m_n = \overline{m} > m_*$. *Then* $\lim_{n\to\infty} (I + A_{m_n})^{-1} g = (I + A_{\overline{m}})^{-1} g$ *for all* $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

**Proof.** We borrow ideas from [11]. Note that an analogous result for the PME was proved in [10]. Let $u_m = (I + A_m)^{-1} g$. The $L^1$-contraction estimate (3.9) implies the bounds

$$\begin{aligned} \|u_{m_n}\|_{L^1(\mathbb{R}^N)} &\leqslant \|g\|_{L^1(\mathbb{R}^N)}, \\ \|u_{m_n} - \tau_h u_{m_n}\|_{L^1(\mathbb{R}^N)} &\leqslant \|g - \tau_h g\|_{L^1(\mathbb{R}^N)}, \end{aligned} \tag{7.1}$$

for each $h \in \mathbb{R}^N$, where $(\tau_h v)(x) = v(x + h)$. This is enough, thanks to Fréchet–Kolmogorov's compactness criterium [15], to prove that $\{u_{m_n}\}$ is precompact in $L^1(K)$ for each compact set $K \subset \mathbb{R}^N$.

To extend compactness to the whole $\mathbb{R}^N$ we need to control the tails of the solutions uniformly. More precisely, we need to prove that, given $\epsilon > 0$, there exists some $R > 0$ such that $\|u_{m_n}\|_{L^1(\mathbb{R}^N \setminus B_R(0))} < \epsilon$. This follows from a computation which is very similar to the one in the proof of Theorem 5.1, but now taking as test function $1 - \varphi_R$ instead of $\varphi_R$. First observe that $u_{m_n} = (I + A_{m_n})^{-1} g$ means

$$\int_\Gamma u_{m_n} \varphi_R \, dx = \int_\Gamma g \varphi_R \, dx - \int_\Omega \langle \nabla w_{m_n}, \nabla \varphi_R \rangle \, d\bar{x}.$$

Therefore, with the above mentioned test function we obtain

$$\int\limits_{\{|x|>R\}} u_{m_n}\,dx \leqslant \int\limits_{\{|x|>R\}} g\,dx + cR^{-2} \int\limits_{\{R<|\bar{x}|<2R\}} w_{m_n}\,d\bar{x} \to 0$$

as $R \to \infty$ uniformly in $n$, see the proof of Theorem 5.1.

We have obtained that along some subsequence, which we also call $\{m_n\}$, the following convergence holds

$$u_{m_n} \to u_*, \quad \text{in } L^1(\Gamma),$$

for some function $u_*$. Using the Poisson kernel, we also have $w_{m_n} \to w_*$ in $L^1(\Omega)$, where $w_*$ is the harmonic extension of $u_*^m$ to the upper half-space. On the other hand, we have a uniform control in $L^2(\Omega)$ of the gradients of $w$ in terms of the data $g$, see (3.11). Thus, there is weak convergence in $L^2(\Omega)$ of the gradients $\nabla w_{m_n}$ along some subsequence towards $\nabla w_*$. All this is enough to pass to the limit in (7.1) to show that the limit $u_*$ is indeed $u_{\overline{m}}$. □

**Remark.** If $m > 1$, an easier alternative proof can be performed using the compactness results of [34].

## 8. Comments and extensions

ALTERNATIVE APPROACHES. It is not difficult to prove a posteriori that the constructed semigroup is also contractive with respect to the norm $H^{-1/2}$. This property could be used as a starting point in the existence and uniqueness theory by using results of the theory of monotone operators in Hilbert spaces, as developed in [14] for the PME case. We have chosen our present formulation because we have found a number of advantages in proceeding in this manner.

On the other hand, Crandall and Pierre developed in [22] an abstract approach to study evolution equations of the form $\partial_t u + A\varphi(u) = 0$ when $A$ is an $m$-accretive operator in $L^1$ and $\varphi$ is a monotone increasing real function. This allows to obtain a *mild* solution using the Crandall–Liggett Theorem. Our problem falls within this framework. Let us point out that such an abstract construction does not give enough information to prove that the mild solution is in fact a weak solution, in other words, to identify the solutions in a differential sense.

EXTENSION. As a natural extension of this work we can consider the more general model based on the equation

$$\frac{\partial u}{\partial t} + (-\Delta)^{\sigma/2}\big(|u|^{m-1}u\big) = 0, \tag{8.1}$$

where the fractional Laplacian has an exponent $\sigma \in (0,2)$, and $m > m_*$ for some $m_*(N,\sigma) \in [0,1)$. Though the main qualitative results are similar to the ones presented here, the theory of these fractional operators with $\sigma \neq 1$ has some technical difficulties that make it convenient to be treated at a second stage. We recall that Caffarelli and Silvestre [18] have recently characterized the *Laplacian of order* $\sigma$, $(-\Delta)^{\sigma/2}$, by means of another auxiliary extension approach. We will use such an extension in a separate paper, [23], to treat in detail the more general fractional diffusion model.

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## Appendix A

We recall some technical results that we have needed in the proof of the property of strong solutions. The first seems to be well-known.

**Lemma A.1.** *On the condition that $\eta$ is a smooth convolution kernel with $\eta(-x)=\eta(x)$ we have for every pair of $L^2$ functions in $\mathbb{R}$*

$$\int \bigl(h*(\eta*\eta)\bigr)(t)g(t)\,dt=\int (h*\eta)(t)(g*\eta)(t)\,dt. \tag{A.1}$$

**Lemma A.2.** (i) *Let $g$ be a positive nondecreasing function. If $\rho_h(t)=(1/h)\rho_1(t/h)$, where $\rho_1$ is a smooth, symmetric, compactly supported nonnegative function, with support $[-2,2]$, and $\delta^h g(t)=(g(t+h)-g(t-h))/(2h)$, then*

$$\bigl(g*\rho_h'\bigr)(t)\geqslant c\delta^h g(t). \tag{A.2}$$

(ii) *If instead of nondecreasing $g$ we have $g'(t)\geqslant -Ag(t)$, the conclusion is*

$$\bigl(g*\rho_h'\bigr)(t)\geqslant c\delta^h g(t)-cAg(t). \tag{A.3}$$

**Proof.** (i) In the case of nondecreasing $g$ we do as follows:

$$\begin{aligned}\bigl(g*\rho_h'\bigr)(t)&=\int_0^{2h}\bigl(g(t+s)-g(t-s)\bigr)\bigl(-\rho'(s)\bigr)\,ds\\&\geqslant\int_h^{2h}\bigl(g(t+h)-g(t-h)\bigr)\bigl(-\rho'(s)\bigr)\,ds\\&\geqslant\delta^h g(t)\int_h^{2h}\bigl(-h\rho'(s)\bigr)\,ds=\delta^h g(t)\rho_1(1).\end{aligned}$$

We pass from the fist to the second line using the positivity of the integrand and the fact that for $s\in(h,2h)$ we have $g(t+s)\geqslant g(t+h)\geqslant g(t-h)\geqslant g(t-s)$.

(ii) The difference is now that we have to use the weaker inequality $g(t+s)\geqslant g(t)e^{-As}$ if $s>0$. We have again

$$\bigl(g*\rho_h'\bigr)(t)=\int_0^{2h}\bigl(g(t+s)-g(t-s)\bigr)\bigl(-\rho'(s)\bigr)\,ds.$$

Since $g(t+s) - g(t-s) \geqslant -g(t)(e^{As} - e^{-As})$ we get

$$\int_0^h \big(g(t+s) - g(t-s)\big)\big(-\rho'(s)\big)\, ds \geqslant -g(t) \int_0^h \big(e^{As} - e^{-As}\big)\big(-\rho'(s)\big)\, ds$$

and the last integral is bounded uniformly for $h$ small in the form $cA$. For the other part of the integral we have

$$\begin{aligned}\int_h^{2h} \big(g(t+s) - g(t-s)\big)\big(-\rho'(s)\big)\, ds &\geqslant \int_h^{2h} \big(g(t+h)e^{-A(s-h)} - g(t-h)e^{A(s-h)}\big)\big(-\rho'(s)\big)\, ds \\ &\geqslant \delta^h g(t) \int_h^{2h} \big(-h\rho'(s)\big)\, ds - I_1 = \delta^h g(t)\rho_1(1) - I_1,\end{aligned}$$

where $I_1 = \int_h^{2h} (g(t+h)(1 - e^{-A(s-h)}) + g(t-h)(e^{A(s-h)} - 1))(-\rho'(s))\, ds$ can also be estimated as $cAg(t)$. $\square$

We end this list of results with an easy but useful calculus lemma.

**Lemma A.3.** *There exists a positive constant c depending on $m > 0$ such that*

$$\big(x^m - 1\big)(x - 1) \geqslant c\big(x^{\frac{m+1}{2}} - 1\big)^2, \quad \forall x \geqslant 1. \tag{A.4}$$

*and also*

$$\big(x^m + 1\big)(x + 1) \geqslant c\big(x^{\frac{m+1}{2}} + 1\big)^2, \quad \forall x \geqslant 1. \tag{A.5}$$

**Proof.** The quotient of the two positive functions $F(x) = (x^m - 1)(x - 1)$ and $G(x) = (x^{\frac{m+1}{2}} - 1)^2$ is bounded below away from zero in the interval $[1, \infty)$ unless the limit at $x = 1$ or $x \to \infty$ is zero. It is clear that at infinity the limit is 1, whereas at $x = 1$ we can use L'Hopital to get

$$\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{4m}{(m+1)^2}.$$

This number is positive and equal or less than 1. The other inequality is similar. $\square$

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