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Totally equimatchable graphs

Jerzy Topp[a,*], Preben D. Vestergaard[b]

[a] *Faculty of Applied Physics and Mathematics, Gdańsk Technical University, Narutowicza 11/12, 80-952 Gdańsk, Poland*

[b] *Department of Mathematics and Computer Science, Aalborg University, Fredrik Bajers Vej 7, DK-9220 Aalborg, Denmark*

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Abstract

A subset X of vertices and edges of a graph G is totally matching if no two elements of X are adjacent or incident. In this paper we determine all graphs in which every maximal total matching is maximum.

1. Introduction

A graph G is *well-covered* if every maximal independent set of vertices in G is maximum. The edge analogue of the well-covered property for graphs is the property that every maximal matching in a graph is maximum. Such graphs are called *equimatchable*. Certainly, a graph G is equimatchable if and only if its line graph $L(G)$ is well-covered. Well-covered graphs are of interest because whereas the problem of determining the size of the largest independent set of an arbitrary graph is NP-complete, it is trivially polynomial for well-covered graphs. The concept of well-coveredness was introduced by Plummer [6] in 1970, and studied in subsequent papers. The reader is referred to [7] for a recent survey of results about well-covered graphs. In this paper we consider the total analogue of the well-covered and equimatchable properties and we characterize graphs in which every maximal total matching is maximum.

2. Notation and preliminary results

We use [1, 2] for basic terminology and notation. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The elements of the set $V(G)\cup E(G)$ are called elements of the

* Corresponding author.

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graph G. A vertex v of G is said to cover itself, all edges incident with v, and all vertices adjacent to v. Similarly, an edge e of G covers itself, the two end vertices of e, and all edges adjacent to e. Two elements of G are called *independent* if neither one covers the other. A set C of elements of G is called a *total cover* if the elements of C cover all elements of G. A set M of elements of G is called a *total matching* if the elements of M are pairwise independent. A set X of elements of G is *totally irredundant* if for every x in X, x covers an element of G which is uncovered by any element of $X - \{x\}$. Let $\alpha_2(G)$ and $\alpha'_2(G)$ denote, respectively, the smallest and the largest number of elements in a minimal total cover of G. Similarly, let $\beta'_2(G)$ and $\beta_2(G)$ denote, respectively, the smallest and the largest number of elements in a maximal total matching of G. Finally, let $\mathrm{ir}'_2(G)$ and $\mathrm{ir}_2(G)$ denote, respectively, the smallest and the largest number of elements in a maximal totally irredundant set of G. Clearly, every maximal total matching of G is a minimal total cover of G. It is also a simple matter to observe that a total cover C of G is minimal if and only if every element x belonging to C covers an element of G which is uncovered by any element of $C - \{x\}$. This implies that every minimal total cover of G is a maximal totally irredundant set of G. Therefore, the parameters defined above are related by inequalities.

$$\mathrm{ir}'_2(G) \leqslant \alpha_2(G) \leqslant \beta'_2(G) \leqslant \beta_2(G) \leqslant \alpha'_2(G) \leqslant \mathrm{ir}_2(G).$$

A graph G is *totally equimatchable* if every maximal total matching of G is maximum. Equivalently, G is totally equimatchable if $\beta'_2(G) = \beta_2(G)$. Similarly, a graph G is *totally well-covered* (*totally well-irredundant*, resp.) if $\alpha_2(G) = \alpha'_2(G)$ ($\mathrm{ir}'_2(G) = \mathrm{ir}_2(G)$, resp.). Clearly, every totally well-irredundant graph is totally well-covered and every totally well-covered graph is totally equimatchable. Our purpose in this paper is to characterize totally equimatchable graphs (or, equivalently, graphs which total graphs (see [4, p. 82]) are well-covered).

In Table 1, precise values of the above-defined parameters are given for complete graphs K_n, complete bipartite graphs $K_{n,n}$, and complete windmills $K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ where $K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ is the graph obtained from the disjoint union of K_1, $K_{2m_1}, \ldots, K_{2m_n}$ by joining the only vertex of K_1 to every vertex in $\bigcup_{i=1}^{n} V(K_{2m_i})$. (Some of these values are also given in [1] and the verification of the table, though not trivial, is left to the reader.) It follows from Table 1 that the graphs K_n, $K_{n,n}$ and $K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ are totally equimatchable. We will show that, in fact, these are the only such connected graphs. In order to simplify the proof of this result, we need additional terminology and three lemmas.

For a vertex v of a graph G, we denote the neighbourhood of v by $N_G(v)$ and the closed neighbourhood, $N_G(v) \cup \{v\}$, by $N_G[v]$. For a subset M of edges of G, let $V(M)$ denote the set of vertices which are covered by some edge belonging to M. A set M of pairwise independent edges of G is called a *matching* of G, and a matching M of G is *perfect* if $V(M) = V(G)$. A graph G is said to be *factor-critical* if $G - v$ has a perfect matching for every vertex v of G. Certainly, every factor-critical graph is a connected graph of odd order and it easily follows from the Gallai–Edmonds theorem (see [5, p. 94]) that we have the following property of factor-critical graphs.

Table 1

Parameters / Graphs G	$ir_2'(G) = \alpha_2(G) = \beta_2'(G) = \beta_2(G)$	$\alpha_2'(G) = ir_2(G)$
K_n	$\lceil n/2 \rceil$	$n - 1$ $(n \geqslant 2)$
$K_{n,n}$	n	$2n - 2$ $(n \geqslant 2)$
$K_1 + \bigcup_{i=1}^{n} K_{2m_i}$	$1 + \sum_{i=1}^{n} m_i$	$2\sum_{i=1}^{n} m_i$

Lemma 1. *A connected graph G is factor-critical if and only if every vertex of G is uncovered by at least one maximum matching of G.*

The next lemma due to Sumner [8] characterizes randomly matchable graphs, graphs in which every maximal matching is perfect.

Lemma 2 (Sumner [8]). *A connected equimatchable graph has a perfect matching if and only if it is one of the graphs K_{2n}, and $K_{n,n}$ where n is any positive integer.*

The following lemma due to Favaron [3] describes the structure of equimatchable factor-critical graphs with a cut vertex.

Lemma 3 (Favaron [3]). *A connected graph G with a cut vertex is equimatchable and factor-critical if and only if:*

(1) *G has exactly one cut vertex c, say;*

(2) *Every connected component G_i of $G - c$ is isomorphic to K_{2m_i} or to K_{m_i,m_i} for some positive integer m_i;*

(3) *c is adjacent to at least two adjacent vertices of each component G_i of $G - c$.*

3. The characterization

We can now prove our main result.

Theorem. *A connected graph is totally equimatchable if and only if it is one of the graphs K_n, $K_{n,n}$ and $K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ where n and $m_1, \dots, m_n$ are any positive integers.*

Proof. It follows from Table 1 that the graphs K_n, $K_{n,n}$ and $K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ are totally equimatchable for any positive integers n and $m_1, \dots, m_n$.

Now let G be a connected totally equimatchable graph. Observe that for any maximal matching M of G, $M \cup (V(G) - V(M))$ is a maximal (and therefore maximum) total matching of G, and certainly $|M \cup (V(G) - V(M))| = |V(G)| - |M| = \beta_2(G)$. Thus every maximal matching M of G has the same cardinality $|M| = |V(G)| - \beta_2(G)$ and this implies that G is equimatchable. In addition, if G has a perfect matching, then every maximal matching of G is perfect and it follows from Lemma 2 that $G = K_{2n}$ or $G = K_{n,n}$ for $n \geqslant 1$. Thus assume that G is equimatchable but it has no perfect matching. Then it suffices to show that $G = K_{2n-1}$ or $G = K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ for some positive integers $n, m_1, \ldots, m_n$. In the proof we frequently use the following claim.

Claim 1. *Let M be a maximum matching of G. Then for every $xy \in M$ and $t \in V(G) - V(M)$, either $\{x,y\} \subseteq N_G(t)$ or $\{x,y\} \cap N_G(t) = \emptyset$.*

Proof. Suppose to the contrary that $x \in N_G(t)$ and $y \notin N_G(t)$. Let A and I denote the sets $N_G(x) \cap N_G(y) \cap (V(G) - V(M))$ and $(M - \{xy\}) \cup \{x\} \cup (V(G) - (V(M) \cup N_G(x)))$, respectively. It is easy to observe that if $A = \emptyset$, then I is a maximal total matching of G and $|I| < \beta_2(G)$, a contradiction. Similarly, if $A \neq \emptyset$, then for any $s \in A$, $I \cup \{ys\}$ is a maximal total matching of G and $|I \cup \{ys\}| < \beta_2(G)$, a final contradiction.

Claim 2. *G is factor-critical.*

Proof. Let $D(G)$ be the set of vertices of G which are uncovered by at least one maximum matching of G. By Lemma 1, it suffices to prove that $D(G) = V(G)$. Since G is connected and $D(G) \neq \emptyset$ (as G has no perfect matching), it suffices to show that $N_G(t) \subseteq D(G)$ for every $t \in D(G)$. Take any $t \in D(G)$ and a maximum matching M of G that does not cover t. Then $t \notin V(M)$ and $N_G(t) \subseteq V(M)$. Take any $x \in N_G(t)$. Since $x \in V(M)$, there is $y \in V(M)$ such that $xy \in M$. By Claim 1, $\{x,y\} \subseteq N_G(t)$. Now $M' = (M - \{xy\}) \cup \{yt\}$ is a maximum matching avoiding x. Therefore $x \in D(G)$ and consequently $N_G(t) \subseteq D(G)$.

To complete the proof of the theorem, we consider two cases.

Case 1: *G contains a cut vertex c, say.*

Since G is equimatchable and factor-critical, Lemma 3 implies that c is the only cut vertex of G. In addition, if G_i is a component of $G - c$, then $G_i = K_{2m_i}$ or $G_i = K_{m_i,m_i}$, and c is adjacent to at least two adjacent vertices of G_i. Let n be the number of components of $G - c$. For $i = 1, \ldots, n$, let $v_1^i, \ldots, v_{m_i}^i, u_1^i, \ldots, u_{m_i}^i$ be the vertices of G_i. We may assume that v_1^i and u_1^i are neighbours of c in G and every v_l^i is adjacent to every u_k^i, $l, k = 1, \ldots, m_i$. We shall prove that $G = K_1 + (K_{2m_1} \cup \cdots \cup K_{2m_n})$.

It is obvious that $M_i = \{v_k^i u_k^i : k = 1, \ldots, m_i\}$ is a perfect matching of G_i ($i = 1, \ldots, n$) and $M = \bigcup_{i=1}^{n} M_i$ is a maximum matching of G. We shall prove that c is adjacent to every vertex of G_i, and that G_i is a complete graph, $i = 1, \ldots, n$. This is clear if $m_i = 1$. Thus, assume that $m_i \geqslant 2$. For $k = 2, \ldots, m_i$, $M_{ik} = (M - \{v_1^i u_1^i, v_k^i u_k^i\}) \cup \{v_1^i u_k^i, v_k^i u_1^i\}$ is a maximum matching of G. Since, $c \notin V(M_{ik})$ and c is adjacent to the vertex $v_1^i (u_1^i$, resp.) of the edge $v_1^i u_k^i (v_k^i u_1^i$, resp.) which belongs to M_{ik}, we conclude from Claim 1 that c is adjacent to u_k^i (v_k^i, resp.). Thus, c is adjacent to every vertex of G_i. Now for $k = 1, \ldots, m_i$, the set $M'_{ik} = (M - \{v_k^i u_k^i\}) \cup \{u_k^i c\}$ is a maximum matching of G which does not cover v_k^i. Since v_k^i is adjacent to every vertex u_l^i and $v_l^i u_l^i \in M'_{ik}$ if $l \neq k$, v_k^i is adjacent to every vertex v_l^i with $l \neq k$ (by Claim 1). Similarly, replacing M'_{ik} by

$M''_{ik} = (M - \{v_k^i u_k^i\}) \cup \{v_k^i c\}$, we observe that u_k^i is adjacent to every vertex u_l^i, $l \neq k$. Thus, G_i is a complete graph of order $2m_i$, $G_i = K_{2m_i}$. Finally, since the cut vertex c of G is adjacent to every vertex of G_i, $i = 1, \ldots, n$, we conclude that $G = K_1 + (K_{2m_1} \cup \cdots \cup K_{2m_n})$.

Case 2: G has no cut vertex.

We claim that G is a complete graph (of odd order). Suppose this is not true. Then there exists a vertex p in G for which $N_G[p] \neq V(G)$. Consequently, since G is connected, the two sets $S = \{v \in N_G(p): N_G(v) \not\subset N_G[p]\}$ and $R = \{x \in V(G) - N_G[p]: N_G(x) \cap N_G(p) \neq \emptyset\}$ are nonempty. Let M be a perfect matching of $G - p$. For a vertex w of $G - p$, let w^* denote the unique neighbour of w such that $ww^* \in M$. It is clear from Claim 1 that for every vertex w of $G - p$, either $\{w, w^*\} \subseteq N_G(p)$ or $\{w, w^*\} \subseteq V(G) - N_G[p]$. We make four additional observations.

(1) *For every $v \in S$ and $x \in R$, either $\{x, x^*\} \subset N_G(v)$ or $\{x, x^*\} \cap N_G(v) = \emptyset$.*

Assume $\{x, x^*\} \cap N_G(v) \neq \emptyset$. Because $M' = (M - \{vv^*\}) \cup \{v^*p\}$ is a maximum matching of G for which $v \notin V(M')$ and $xx^* \in M'$, we conclude from Claim 1 that $\{x, x^*\} \subset N_G(v)$.

(2) *For every $v \in S$ and $x \in R$, if $\{x, x^*\} \subset N_G(v)$, then $\{x, x^*\} \cap N_G(v^*) = \emptyset$.*

Assume $\{x, x^*\} \subset N_G(v)$ and suppose that $\{x, x^*\} \cap N_G(v^*) \neq \emptyset$. Then $\{x, x^*\} \subset N_G(v^*)$ by (1). But now $M' = (M - \{vv^*, xx^*\}) \cup \{vx, v^*x^*\}$ is a maximum matching of G. Because $p \notin V(M')$ and $vx, v^*x^* \in M'$ while $\{v, x\}$ and $\{v^*, x^*\}$ are contained neither in $N_G(p)$ nor in $V(G) - N_G[p]$, we get a contradiction to Claim 1.

(3) *For every $v \in S$ and $x \in R$, if $x \in N_G(v)$, then $N_G(x) \cap S = \{v\}$.*

Assume $x \in N_G(v)$ and suppose that there exists $u \in N_G(x) \cap S - \{v\}$. Then $u \neq v^*$ (by (2)) and $\{x, x^*\} \subset N_G(u)$ (by (1)). Now $M' = (M - \{xx^*, vv^*, uu^*\}) \cup \{vx, ux^*, pu^*\}$ is a maximum matching of G and it does not cover v^*. Since $vx \in M'$ and neither $\{v, x\} \subseteq N_G(v^*)$ nor $\{v, x\} \cap N_G(v^*) = \emptyset$, we reach a contradiction to Claim 1.

(4) *The set S has exactly one vertex.*

Suppose $|S| \geqslant 2$ and let v and u be distinct vertices of S. Let $x \in N_G(v) \cap R$ and $y \in N_G(u) \cap R$. It follows from (3) that $x \neq y$. In addition, $xy \notin M$; for otherwise (1) implies that $\{x, y\} \subset N_G(v)$ and $\{x, y\} \subset N_G(u)$ which contradicts (3). If $vu \in M$, then considering a maximum matching M' of G containing $(M - \{vu, xx^*, yy^*\}) \cup \{vx, uy\}$ (and then necessarily also x^*y^*), we get a contradiction just as in the proof of (2). If $vu \notin M$, then let M' be a maximum matching of G containing $(M - \{vv^*, uu^*, xx^*, yy^*\}) \cup \{vx, uy, pu^*\}$. Because the vertex v^* is adjacent neither to x^* (see (2)) nor to y^* (see (3)), $v^* \notin V(M')$. But now since $vx \in M'$ and neither $\{v, x\} \subset N_G(v^*)$ nor $\{v, x\} \cap N_G(v^*) = \emptyset$, we get a contradiction to Claim 1.

It is obvious from (4) and from definitions of S and R that the unique vertex of S is a cut vertex of G. This, however, contradicts the assumption that G has no cut vertex and completes the proof of the theorem. $\square$

The following corollary is immediate from Theorem, Table 1 and the observation that K_1 is a totally well-irredundant graph, and it shows that the classes of totally well-covered and totally well-irredundant graphs are quite restricted.

Corollary. *Let G be a connected graph. Then the following statements are equivalent:*

(1) *G is totally well-covered;*

(2) *G is totally well-irredundant;*

(3) *G is one of the graphs* K_1, K_2, K_3 *and* $K_{2,2}$.

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