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# Fundamental solutions for the super Laplace and Dirac operators and all their natural powers

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## Abstract

The fundamental solutions of the super Dirac and Laplace operators and their natural powers are determined within the framework of Clifford analysis.

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*Keywords:* Clifford analysis; Dirac operator; Superspace; Fundamental solution

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## 1. Introduction

In a previous set of papers (see [1–4]) we have developed the basic framework for Clifford analysis in superspace. Clifford analysis in standard Euclidean space is a function theory of the so-called Dirac operator and as such a generalization of the theory of holomorphic functions of one complex variable. Basic references are [5–7]. A superspace on the other hand is a generalization of the classical concept of space, where not only commuting variables are considered, but also a set of anti-commuting ones. These superspaces find their use in theoretical physics. We refer the reader e.g. to [8–12].

We have first established the basic algebraic framework necessary for developing a theory of Clifford analysis in superspace (see [1] and [2]). We have constructed the fundamental differential operators such as the Dirac and Laplace operators, the Euler and Gamma operators, etc. Next we have constructed a theory of spherical monogenics in superspace (see [3]), which was then used to introduce an integration over the supersphere and over superspace (see [4]), inspired by similarities with the classical theory of harmonic analysis. Moreover it turned out that this integration is equivalent with the one introduced in the work of Berezin (see [8,9]), although our approach offers better insight. Indeed, the definition given by Berezin is not motivated by any connection with classical types of integration, whereas our approach connects the integral over superspace with the well-known theory of integration in Euclidean space (see the discussion in [4]).

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All previous work is situated on the level of polynomial functions. A next step is to consider larger algebras and to start the study of the function-theoretical properties of our differential operators. Thus is the aim of the present paper. More precisely we will determine the fundamental solutions for all natural powers of the super Dirac operator.

The paper is organized as follows. We start with a short introduction to Clifford analysis on superspace. Then we recapitulate the results on polyharmonic functions needed later on. In the next section we derive the fundamental solutions for the super Laplace and Dirac operators and compare them with an ad hoc approach, inspired by the theory of radial algebra. In the following section we extend this technique to construct fundamental solutions for all natural powers. Finally we discuss how the present technique is valid for a still larger class of differential operators.

## 2. Clifford analysis in superspace

We first consider the real algebra  $\mathcal{P} = \text{Alg}(x_i, e_i; \hat{x}_j, \hat{e}_j), i = 1, \dots, m, j = 1, \dots, 2n$ , generated by

- $m$  commuting variables  $x_i$  and  $m$  orthogonal Clifford generators  $e_i$ ,
- $2n$  anti-commuting variables  $\hat{x}_i$  and  $2n$  symplectic Clifford generators  $\hat{e}_i$

subject to the multiplication relations

$$\begin{cases} x_i x_j = x_j x_i, \\ \hat{x}_i \hat{x}_j = -\hat{x}_j \hat{x}_i, \\ x_i \hat{x}_j = \hat{x}_j x_i \end{cases} \quad \text{and} \quad \begin{cases} e_j e_k + e_k e_j = -2\delta_{jk}, \\ \hat{e}_{2j} \hat{e}_{2k} - \hat{e}_{2k} \hat{e}_{2j} = 0, \\ \hat{e}_{2j-1} \hat{e}_{2k-1} - \hat{e}_{2k-1} \hat{e}_{2j-1} = 0, \\ \hat{e}_{2j-1} \hat{e}_{2k} - \hat{e}_{2k} \hat{e}_{2j-1} = \delta_{jk}, \\ e_j \hat{e}_k + \hat{e}_k e_j = 0 \end{cases}$$

and where moreover all elements  $e_i, \hat{e}_j$  commute with all elements  $x_i, \hat{x}_j$ .

If we denote by  $\Lambda_{2n}$  the Grassmann algebra generated by the anti-commuting variables  $\hat{x}_j$  and by  $\mathcal{C}$  the algebra generated by all the Clifford numbers  $e_i, \hat{e}_j$ , then we clearly have that

$$\mathcal{P} = \mathbb{R}[x_1, \dots, x_m] \otimes \Lambda_{2n} \otimes \mathcal{C}.$$

The most important element of the algebra  $\mathcal{P}$  is the vector variable  $x = \underline{x} + \underline{\hat{x}}$  with

$$\underline{x} = \sum_{i=1}^m x_i e_i, \quad \underline{\hat{x}} = \sum_{j=1}^{2n} \hat{x}_j \hat{e}_j.$$

One calculates that

$$x^2 = \underline{x}^2 + \underline{\hat{x}}^2 = \sum_{j=1}^n \hat{x}_{2j-1} \hat{x}_{2j} - \sum_{j=1}^m x_j^2.$$

The super Dirac operator is defined as

$$\partial_x = \partial_{\underline{x}} - \partial_{\underline{\hat{x}}} = 2 \sum_{j=1}^n (\hat{e}_{2j} \partial_{\hat{x}_{2j-1}} - \hat{e}_{2j-1} \partial_{\hat{x}_{2j}}) - \sum_{j=1}^m e_j \partial_{x_j}.$$

Its square is the super Laplace operator:

$$\Delta = \partial_x^2 = 4 \sum_{j=1}^n \partial_{\hat{x}_{2j-1}} \partial_{\hat{x}_{2j}} - \sum_{j=1}^m \partial_{x_j}^2.$$

The bosonic part of this operator is  $\Delta_b = -\sum_{j=1}^m \partial_{x_j}^2$ , which is the classical Laplace operator. The fermionic part is  $\Delta_f = 4 \sum_{j=1}^n \partial_{\hat{x}_{2j-1}} \partial_{\hat{x}_{2j}}$ . For the other important operators in super Clifford analysis and their commutation relations we refer the reader to [1]. If we let  $\partial_x$  act on  $x$  we find that

$$\partial_x x = m - 2n = M$$

where  $M$  is the so-called super-dimension. This numerical parameter gives a global characterization of our superspace and will be used in Remark 1. We will also need the following basic formulae (see [1])

$$\partial_x(x^{2s}) = 2sx^{2s-1}, \tag{1}$$

$$\partial_x(x^{2s+1}) = (M + 2s)x^{2s}. \tag{2}$$

In the case where  $m = 1, n = 0$  this reduces to the familiar formula  $\frac{d}{dx}x^k = kx^{k-1}$ .

Now we can consider several generalizations of the algebra  $\mathcal{P}$ . This leads to the introduction of the function-spaces:

$$\mathcal{F}(\Omega)_{m|2n} = \mathcal{F}(\Omega) \otimes \Lambda_{2n} \otimes \mathcal{C}$$

where  $\mathcal{F}(\Omega)$  stands for  $\mathcal{D}(\Omega), \mathcal{C}^k(\Omega), L_p(\Omega), L_1^{\text{loc}}(\Omega), \dots$  with  $\Omega$  an open domain in  $\mathbb{R}^m$ . Finally the space of harmonic functions, i.e. null-solutions of the super Laplace operator, will be denoted by  $\mathcal{H}(\Omega)_{m|2n} \subseteq \mathcal{C}^2(\Omega) \otimes \Lambda_{2n}$ . Similarly we denote by  $\mathcal{M}(\Omega)_{m|2n} \subseteq \mathcal{C}^2(\Omega)_{m|2n}$  the space of monogenic functions, i.e. null-solutions of the super Dirac operator. We have that  $\mathcal{M}(\Omega)_{m|2n} \subseteq \mathcal{H}(\Omega)_{m|2n} \otimes \mathcal{C}$ .

Now we have the following theorem, which generalizes a classical result in harmonic analysis.

**Theorem 1.** *Null-solutions of the super Laplace and the super Dirac operator are  $\mathcal{C}^\infty$ -functions, i.e.*

$$\mathcal{H}(\Omega)_{m|2n} \otimes \mathcal{C} \subseteq \mathcal{C}^\infty(\Omega)_{m|2n}, \quad \mathcal{M}(\Omega)_{m|2n} \subseteq \mathcal{C}^\infty(\Omega)_{m|2n}.$$

**Proof.** It suffices to give the proof for harmonic functions, as monogenic functions are also harmonic. So we consider a function  $f \in \mathcal{H}(\Omega)_{m|2n}$ . Such a function can be written as

$$f = \sum_{(\alpha)} f_{(\alpha)} \hat{x}_1^{\alpha_1} \dots \hat{x}_{2n}^{\alpha_{2n}}$$

with  $(\alpha) = (\alpha_1, \dots, \alpha_{2n}), \alpha_i \in \{0, 1\}$  and  $f_{(\alpha)} \in \mathcal{C}^2(\Omega)$ .

Expressing that  $\Delta f = \Delta_b f + \Delta_f f = 0$  leads to a set of equations of the following type:

$$\Delta_b f_{(\alpha)} = \sum_{(\beta)} c_{(\beta)} f_{(\beta)}, \quad |(\beta)| = |(\alpha)| + 2,$$

with  $c_{(\beta)} \in \mathbb{R}$ . We conclude that for every  $(\alpha)$  there exists a  $k \in \mathbb{N} (k \leq n + 1)$  such that  $\Delta_b^k f_{(\alpha)} = 0$ . Hence  $f_{(\alpha)}$  is polyharmonic and thus an element of  $\mathcal{C}^\infty(\Omega)$ .  $\square$

We end this section with a few words on integration in superspace. The proper integral to consider is the so-called Berezin integral  $\int_B$  (see [8,9] and [4]) which has the following formal definition:

$$\int_B = \int_{\mathbb{R}^m} dV(\underline{x}) \partial_{\hat{x}_{2n}} \dots \partial_{\hat{x}_1}$$

with  $dV(\underline{x})$  the Lebesgue measure in  $\mathbb{R}^m$ .

One can also define a super Dirac distribution as

$$\delta(x) = \delta(\underline{x}) \hat{x}_1 \dots \hat{x}_{2n} = \delta(\underline{x}) \frac{\hat{x}^{2n}}{n!}$$

with  $\delta(\underline{x})$  the classical Dirac distribution in  $\mathbb{R}^m$ . We clearly have that

$$\langle \delta(x - y), f(x) \rangle = \int_B \delta(\underline{x} - \underline{y})(\hat{x}_1 - \hat{y}_1) \dots (\hat{x}_{2n} - \hat{y}_{2n}) f(x) = f(y)$$

with  $f \in \mathcal{D}(\Omega)_{m|2n}$ .

### 3. Fundamental solutions in $\mathbb{R}^m$

The fundamental solutions for the natural powers of the classical Laplace operator  $\Delta_b$  are very well-known, e.g. see [13].

We denote by  $v_{2l}^{m|0}$ ,  $l = 1, 2, \dots$ , a sequence of such fundamental solutions, satisfying

$$\begin{aligned} \Delta_b^j v_{2l}^{m|0} &= v_{2l-2j}^{m|0}, \quad j < l, \\ \Delta_b^l v_{2l}^{m|0} &= \delta(\underline{x}). \end{aligned}$$

Their explicit form depends both on the dimension  $m$  and on  $l$ . More specifically, in the case where  $m$  is odd we have that

$$v_{2l}^{m|0} = \frac{r^{2l-m}}{\gamma_{l-1}}, \quad \gamma_l = (-1)^{l+1} (2-m) 4^l l! \frac{\Gamma(l+2-m/2)}{\Gamma(2-m/2)} \frac{2\pi^{m/2}}{\Gamma(m/2)} \tag{3}$$

with  $r = \sqrt{-x^2}$ . The formulae for  $m$  even are more complicated and can be found in [13].

Concerning the refinement to Clifford analysis, we clearly have that  $v_{2l+1}^{m|0} = \partial_{\underline{x}} v_{2l+2}^{m|0}$  is a fundamental solution of  $\Delta_b^l \partial_{\underline{x}}$ .

### 4. Fundamental solution of $\Delta$ and $\partial_x$

From now on we restrict ourselves to the case where  $m \neq 0$ . The purely fermionic case will be discussed briefly in Section 7.

Our aim is to construct a function  $\rho$  such that in distributional sense

$$\Delta \rho = \delta(x).$$

We propose the following form for the fundamental solution:

$$\rho = \sum_{k=0}^n a_k (\Delta_b^{n-k} \phi) \underline{x}^{2n-2k},$$

with  $\phi$  and  $a_k \in \mathbb{R}$  to be determined.

Now let us calculate  $\Delta \rho$

$$\begin{aligned} \Delta \rho &= (\Delta_b + \Delta_f) \rho \\ &= \sum_{k=0}^n a_k (\Delta_b^{n-k+1} \phi) \underline{x}^{2n-2k} + \sum_{k=1}^n a_k (2n-2k)(2n-2k-2-2n) (\Delta_b^{n-k} \phi) \underline{x}^{2n-2k-2} \\ &= a_0 (\Delta_b^{n+1} \phi) \underline{x}^{2n} + \sum_{k=1}^n [a_k - 2k(2n-2k+2)a_{k-1}] (\Delta_b^{n-k+1} \phi) \underline{x}^{2n-2k}. \end{aligned}$$

So  $\rho$  is a fundamental solution if and only if

$$a_0 (\Delta_b^{n+1} \phi) = \delta(\underline{x}) \frac{1}{n!}$$

and  $a_k$  satisfies the recurrence relation

$$a_k = 4k(n-k+1)a_{k-1}.$$

The first equation leads to

$$\phi = v_{2n+2}^{m|0}, \quad a_0 = \frac{1}{n!}.$$

We then immediately find the following expression for the  $a_k$

$$a_k = \frac{4^k k!}{(n - k)!}, \quad k = 0, \dots, n.$$

Summarizing we obtain the following theorem.

**Theorem 2.** *The function  $v_2^{m|2n}$  defined by*

$$v_2^{m|2n} = \sum_{k=0}^n \frac{4^k k!}{(n - k)!} v_{2k+2}^{m|0} \underline{x}^{2n-2k},$$

with  $v_{2k+2}^{m|0}$  as in Section 3, is a fundamental solution for the operator  $\Delta$ .

**Proof.** It is clear that  $v_2^{m|2n} \in L_1^{\text{loc}}(\mathbb{R}^m)_{m|2n}$ . Moreover, we have that  $v_2^{m|2n} \in \mathcal{H}(\mathbb{R}^m - \{0\})_{m|2n}$  and that  $\Delta v_2^{m|2n} = \delta(x)$  in distributional sense.  $\square$

Now suppose that  $m$  is odd, then the previous formula simplifies to

$$v_2^{m|2n} = \frac{\Gamma(m/2)}{2(2 - m)\pi^{m/2}} \sum_{k=0}^n \frac{(-1)^{k+1}}{(n - k)!} \frac{\Gamma(2 - m/2)}{\Gamma(k + 2 - m/2)} r^{2k+2-m} \underline{x}^{2n-2k}, \tag{4}$$

where we have used formula (3).

As we have that  $\Delta v_2^{m|2n} = \delta(x)$ , a fundamental solution for the Dirac operator  $\partial_x$  is given by  $\partial_x v_2^{m|2n}$ . This leads to the following

**Theorem 3.** *The function  $v_1^{m|2n}$  defined by*

$$v_1^{m|2n} = \sum_{k=0}^{n-1} 2 \frac{4^k k!}{(n - k - 1)!} v_{2k+2}^{m|0} \underline{x}^{2n-2k-1} - \sum_{k=0}^n \frac{4^k k!}{(n - k)!} v_{2k+1}^{m|0} \underline{x}^{2n-2k}$$

is a fundamental solution for the operator  $\partial_x$ .

**Remark 1.** We could propose the following form:

$$g = \frac{1}{(x^2)^{\frac{M-2}{2}}}$$

for the fundamental solution of  $\Delta$ , where we have replaced  $m$  by the super-dimension  $M$  in the classical expression. This technique is inspired by radial algebra (see [14]), which gives a very general framework for constructing theories of Clifford analysis, based on the introduction of an abstract dimension parameter (in this case the super-dimension). This leads partially to the correct result (see also [15]). Indeed, formally we can expand this as

$$\begin{aligned} g &= \frac{1}{(x^2)^{\frac{M-2}{2}}} = \frac{1}{(\underline{x}^2 + \underline{x}^2)^{\frac{M-2}{2}}} \\ &= \frac{1}{(x^2)^{\frac{M-2}{2}}} \left(1 + \frac{\underline{x}^2}{x^2}\right)^{1-\frac{M}{2}} = \sum_{k=0}^n \binom{1-\frac{M}{2}}{k} \frac{\underline{x}^{2k}}{(x^2)^{\frac{M}{2}-1+k}}. \end{aligned}$$

The coefficients in this development are proportional to the ones obtained in Theorem 2 (see also formula (4)), so this yields the correct result. This expansion is however only valid if  $m$  is odd.

The fundamental solution can of course be used to determine solutions of the inhomogeneous equation  $\Delta f = \rho$ . We have for example the following

**Proposition 1.** Let  $\rho \in \mathcal{D}(\Omega)_{m|2n}$ , then a solution of  $\Delta f = \rho$  is given by

$$f(x) = v_2^{m|2n} * \rho = \int_B v_2^{m|2n}(x - y)\rho(y).$$

**5. Fundamental solution of  $\Delta^k$  and  $\Delta^k \partial_x$**

A similar technique as in Section 4 can be used for the polyharmonic case. First we expand  $\Delta^k$  as

$$\Delta^k = \sum_{j=0}^k \binom{k}{j} \Delta_b^{k-j} \Delta_f^j.$$

This expansion is valid as  $\Delta_b$  commutes with  $\Delta_f$ .

Now we propose the following form for its fundamental solution:

$$\rho = \sum_{l=0}^n a_l (\Delta_b^{n-l} \phi) \hat{x}^{2n-2l}$$

with  $\phi$  and  $a_l \in \mathbb{R}$  still to be determined. We calculate that

$$\Delta^k \rho = \sum_{l=0}^n a_l \sum_{j=0}^k \binom{k}{j} (\Delta_b^{n-l+k-j} \phi) \Delta_f^j \hat{x}^{2n-2l}.$$

As we have that, using formulae (1) and (2) in the case where  $m = 0$ ,  $M = -2n$ ,

$$\Delta_f^j \hat{x}^{2n-2l} = 4^j (-1)^j \frac{(n-l)!}{(n-l-j)!} \frac{(l+j)!}{l!} \hat{x}^{2n-2l-2j}, \quad j \leq n-l,$$

this yields

$$\Delta^k \rho = \sum_{l=0}^n a_l \sum_{j=0}^k \binom{k}{j} 4^j (-1)^j \frac{(n-l)!}{(n-l-j)!} \frac{(l+j)!}{l!} (\Delta_b^{n-l+k-j} \phi) \hat{x}^{2n-2l-2j}.$$

Putting  $\Delta^k \rho = \delta(x)$  leads to the following set of equations:

$$a_0 \Delta_b^{n+k} \phi = \delta(\underline{x}) \frac{1}{n!}, \tag{5}$$

$$\sum_{j=0}^k a_{l-j} \binom{k}{j} 4^j (-1)^j \frac{(n-l+j)!}{(n-l)!} \frac{l!}{(l-j)!} = 0, \tag{6}$$

$$a_{-1} = a_{-2} = a_{-3} = \dots = 0. \tag{7}$$

We immediately have that  $a_0 = 1/n!$  and that  $\phi = v_{2n+2k}^{m|0}$ .

Equation (6) can be simplified by the substitution

$$a_l = 4^l \frac{l!}{(n-l)!} b_l$$

to

$$\sum_{j=0}^k b_{l-j} \binom{k}{j} (-1)^j = 0, \quad b_0 = 1,$$

which has the solution (see the subsequent Lemma 1)

$$b_l = \binom{l+k-1}{l}.$$

We conclude that

$$a_l = 4^l \frac{(l+k-1)!}{(n-l)!(k-1)!}, \quad l = 0, \dots, n.$$

We can summarize the previous results in the following theorem.

**Theorem 4.** The function  $v_{2k}^{m|2n}$  defined by

$$v_{2k}^{m|2n} = \sum_{l=0}^n 4^l \frac{(l+k-1)!}{(n-l)!(k-1)!} v_{2l+2k}^{m|0} x^{2n-2l},$$

is a fundamental solution for the operator  $\Delta^k$ .

In a similar vein we obtain the fundamental solution  $v_{2k+1}^{m|2n}$  for the operator  $\Delta^k \partial_x = \partial_x^{2k+1}$  by calculating  $\partial_x v_{2k+2}^{m|2n}$ . This leads to

**Theorem 5.** The function  $v_{2k+1}^{m|2n}$  defined by

$$v_{2k+1}^{m|2n} = \sum_{l=0}^{n-1} 2 \frac{4^l (l+k)!}{(n-l-1)!k!} v_{2l+2k+2}^{m|0} x^{2n-2l-1} - \sum_{l=0}^n \frac{4^l (l+k)!}{(n-l)!k!} v_{2l+2k+1}^{m|0} x^{2n-2l},$$

is a fundamental solution for the operator  $\Delta^k \partial_x$ .

We still have to prove the technical lemma we used in the derivation of Theorem 4.

**Lemma 1.** The sequence  $(b_l)$ ,  $l = 0, 1, \dots$ , recursively defined by

$$\sum_{j=0}^{\min(k,l)} b_{l-j} \binom{k}{j} (-1)^j = 0, \quad b_0 = 1,$$

is given explicitly by

$$b_l = \binom{l+k-1}{l}.$$

**Proof.** Define the polynomial  $R_l(x)$  by

$$\begin{aligned} R_l(x) &= \sum_{j=0}^{\min(k,l)} (-1)^j \binom{k}{j} \frac{(x+k-j-1)!}{(x-j)!} \\ &= \sum_{j=0}^{\min(k,l)} (-1)^j \binom{k}{j} (x+k-1-j) \cdots (x+1-j). \end{aligned}$$

We then have to prove that  $R_l(l) = 0$ .

We distinguish between three cases.

(1)  $l \leq k-2$ .

We claim that for all  $t \leq k-2$

$$R_t(x) = (-1)^t \binom{k-1}{t} (x+k-t-1) \cdots (x+1)(x-1) \cdots (x-t).$$

This can be proven using induction. The case where  $t = 1$  is easily checked. So we suppose the formula holds for  $t-1$ , then we calculate

$$\begin{aligned}
 &R_{t-1}(x) + (-1)^t \binom{k}{t} (x+k-1-t) \cdots (x+1-t) \\
 &= (-1)^t (x+k-t-1) \cdots (x+1)(x-1) \cdots (x-t+1) \left( \binom{k}{t} x - \binom{k-1}{t-1} (x+k-t) \right) \\
 &= (-1)^t \binom{k-1}{t} (x+k-t-1) \cdots (x+1)(x-1) \cdots (x-t) \\
 &= R_t(x)
 \end{aligned}$$

which proves the hypothesis. Now clearly  $R_l(l) = 0$ .

(2)  $l = k - 1$ .

Using the previous results, it is shown that in this case  $R_{k-1}(x)$  equals

$$R_{k-1}(x) = -(-1)^k (x-1) \cdots (x+1-k)$$

so  $R_{k-1}(k-1) = 0$ .

(3)  $l \geq k$ .

Now we have that

$$R_l(x) = R_k(x) = 0$$

so this case is also proven.  $\square$

### 6. A larger class of differential operators

The technique used above can be extended to a larger class of differential operators. Suppose we consider an operator of the following form:

$$P = L(x, \partial_x) + \Delta_f$$

with  $L(x, \partial_x)$  an elliptic operator in  $\mathbb{R}^m$  and  $\Delta_f$  the fermionic Laplace operator. Note that  $L(x, \partial_x)$  and  $\Delta_f$  clearly commute. An interesting operator in this class is the super Helmholtz operator

$$\Delta - \lambda^2, \quad \lambda \in \mathbb{R},$$

with  $L(x, \partial_x) = \Delta_b - \lambda^2$ .

Denoting by  $\mu_{2k}^{m|0}$  ( $k = 1, 2, \dots$ ) a set of fundamental solutions for the operators  $L(x, \partial_x)^k$  such that

$$L(x, \partial_x)^j \mu_{2k}^{m|0} = \mu_{2k-2j}^{m|0}, \quad j < k,$$

$$L(x, \partial_x)^k \mu_{2k}^{m|0} = \delta(\underline{x})$$

we can now use the same technique as in Section 5 to obtain a fundamental solution  $\mu_{2k}^{m|2n}$  for the operator  $P^k$ . This leads to

$$\mu_{2k}^{m|2n} = \sum_{l=0}^n 4^l \frac{(l+k-1)!}{(n-l)!(k-1)!} \mu_{2l+2k}^{m|0} \hat{x}^{2n-2l}.$$

### 7. The purely fermionic case

In this case there is no fundamental solution. Indeed, determining the fundamental solution of  $\Delta_f$  requires solving the algebraic equation

$$\Delta_f v_2^{0|2n} = \hat{x}_1 \dots \hat{x}_{2n}$$

which clearly has no solution, since there are no polynomials of degree higher than  $2n$ .



## 8. Conclusions

In this paper we have developed a technique to construct fundamental solutions for certain differential operators in superspace. In particular we have constructed the fundamental solutions of the natural powers of the super Dirac operator  $\partial_x^k$ .

We envisage to use these fundamental solutions in a further development of the function theory of Clifford analysis in superspace. There is no doubt that they will play an important role in the generalization of the Cauchy and Hilbert transform to superspace.

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