Interaction between Weak Discontinuities and Shocks in a Dusty Gas

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We consider an axi-symmetric dusty gas flow and we study the interaction of a weak discontinuity with shock wave. If the shock curve is a similarity line, after the interaction it loses this property if the result of the interaction must be uniquely determined. © 2001 Academic Press

1. INTRODUCTION

The problem of interaction between discontinuity waves and shock waves has been considered by many authors both from a theoretical point of view and applications in different physical contexts [1–4].

Usually, for the sake of simplicity, it is assumed that the propagation of the incident discontinuity, travelling faster than the shock wave, occurs in a constant state. Such an assumption is no longer valid for non-autonomous systems, like in the case of an axi-symmetric flow of a dusty gas, which is the one considered in this paper. In a recent paper [5], this model’s equations have been studied in two different physical contexts and, among other results, it has been shown that the governing system can be reduced to autonomous form using its invariance properties.

This paper is devoted to studying the interaction between a discontinuity, propagating in a non-constant state characterized by a similarity solution, and a shock wave by determining the reflected and transmitted waves, as well the jump in the shock acceleration along the shock line.
We consider, essentially, two cases:

(i) the characteristic shock corresponding to the case considered in [2] but where the propagation occurs in a non-constant state;

(ii) the strong shock when the shock line is a similarity curve.

In this last case, we show that, in order for the result of the interaction to be uniquely determined, after the interaction the shock curve cannot be a similarity curve any more.

2. BASIC EQUATIONS AND RANKINE–HUGONIOT CONDITIONS

The motion equations for a one-dimensional axi-symmetric flow in conservative form are given by [6]

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) + \frac{mpu}{x} &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + p) + \frac{mpu^2}{x} &= 0, \\
\partial_t (\varepsilon + \frac{u^2}{2}) + \partial_x (\varepsilon + \frac{u^2}{2} + \frac{p}{\rho}) + \frac{mpu}{x} \left(\varepsilon + \frac{u^2}{2} + \frac{p}{\rho}\right) &= 0.
\end{align*}
\]

Let us assume the flow to be a gas mixture obeying to the equation of state of Mie–Grüneisen type [7, 8]

\[
p = \frac{(1 - k_p) \rho RT}{1 - \theta \rho},
\]

which implies the following expressions, respectively, for the sound speed \(c\), the internal energy \(\varepsilon\), and the enthalpy \(i\),

\[
\begin{align*}
c^2 &= \frac{\Gamma p}{\rho (1 - \theta \rho)}, \\
\varepsilon &= \frac{p(1 - \theta \rho)}{\rho (\Gamma - 1)}, \\
i &= \frac{p(\Gamma - \theta \rho)}{\rho (\Gamma - 1)}.
\end{align*}
\]

where \(u\) is the particle velocity along the \(x\)-axis, \(t\) the time, \(\rho\) the density, \(p\) the pressure, \(T\) the temperature, \(R\) the gas constant, \(Z\) and \(k_p\) are, respectively, the volume fraction and the mass concentration of the solid particles in the medium related by the expression \(Z = \theta \rho\) where
\[ \theta = k_p/\rho_{sp}, \text{ with } \rho_{sp} \text{ the species density of the solid particles, and } m = 0, m = 1, m = 2 \text{ correspond, respectively, to planar, cylindrical, and spherical motion; } \Gamma = \gamma(1 + \lambda \beta)/(1 + \lambda \beta \gamma), \lambda = k_p/(1 - k_p), \beta = c_{sp}/c_p, \gamma = c_p/c_v, \text{ where } c_{sp} \text{ is the specific heat of the solid particles, } c_p \text{ the specific heat of the gas at constant pressure, and } c_v \text{ the specific heat of the gas at constant volume.} \]

For \( \theta = 0 \) the previous relations reduce to the well known ones for an ideal isentropic gas (ideal in the sense that there are no particle interactions).

Let us consider a shock wave propagating in the gas and denote by \( \psi(x, t) = 0 \) the shock line. The state where the shock propagates is characterized by a known solution \((\rho^*, u^*, p^*)\) so that, across the shock line, the following Rankine–Hugoniot relations must be satisfied,

\[
[\rho(u - s)] = 0,
[\rho u(u - s) + p] = 0,
\left[ \rho(u - s)(\varepsilon + \frac{u^2}{2}) + pu \right] = 0,
\]

where the square brackets denote the jump across \( \psi = 0 \), i.e., \( [\cdot] = (\cdot)_{\psi=0^+} - (\cdot)_{\psi=0^-} \) and \( s \) is the shock velocity defined by \( s = -\psi_t/\psi_x \). From relations (6), after some manipulations, we obtain [6]

\[
\varepsilon - \varepsilon^* + \frac{1}{2} \left( \frac{1}{\rho} - \frac{1}{\rho^*} \right) (p + p^*) = 0,
\]

which represents, in the plane \((\frac{1}{\rho}, p)\), a curve called a Hugoniot adiabat.

For the mixture we are considering Eq. (7) becomes

\[
(p + h p^*) \left( \frac{1}{\rho} - \frac{h}{\rho^*} - 2\theta \right) = (1 - h)\frac{p^*}{\rho^*} (1 + h + 2\theta \rho^*); \quad h = \frac{\Gamma - 1}{\Gamma + 1},
\]

that is, a hyperbole with asymptotes

\[
\frac{1}{\rho} = \frac{h}{\rho^*} + 2\theta \quad \text{and} \quad p = -h p^*.
\]

The curve on which the entropy \( S \) remains constant is called a Poisson adiabat, and, in our case, is given by

\[
p \left( \frac{1}{\rho} - \theta \right)^\Gamma = a^2(S^*),
\]

with asymptotes \( p = 0 \) and \( \frac{1}{\rho} = \theta \).
Relation (8) may also be written in the form
\[ \frac{1}{\rho} - \frac{h}{\rho^*} - 2\theta = \frac{1}{\rho^*} - \frac{h}{\rho} - 2\theta \] (9)
and, after some manipulations, allows us to write
\[ p + hp^* = \frac{p - p^*}{1/\rho^* - 1/\rho} \left[ (1 - h) \frac{1}{\rho^*} - 2\theta \right] \] (10)
But, from the Rankine–Hugoniot conditions, it follows
\[ \frac{p - p^*}{1/\rho^* - 1/\rho} = \rho^2 (u^* - s)^2 \] (11)
and, introducing a Mach number defined by
\[ M_0 = \frac{|u^* - s|}{c_0^*}, \quad (c_0^*)^2 = \frac{\Gamma p_0}{\rho_0}, \] (12)
where \( c_0^* \) is the sound velocity when the particles interactions are absent, we get
\[ p = \left[ 2\Gamma M_0^2 \left( \frac{1}{\Gamma + 1} - \theta \rho^* \right) - \frac{\Gamma - 1}{\Gamma + 1} \right] p^*. \] (13)
In the case of a strong shock, \( M_0 \to \infty, \ p^* \to 0, \) and \( u^* \to 0, \) we obtain the following expression for \( p: \)
\[ p = 2s^2 \rho^* \left( \frac{1}{\Gamma + 1} - \theta \rho^* \right). \] (14)
Again, from (9), we deduce
\[ \frac{\rho}{\rho^*} = \frac{p + hp^*}{p^* + hp + 2\theta \rho^* (p - p^*)}, \] (15)
specifying into
\[ \frac{\rho}{\rho^*} = \frac{(\Gamma + 1)M_0^2}{[\Gamma - 1 + 2\theta(\Gamma + 1)\rho^*]M_0^2 + 2} \] (16)
and for a strong shock
\[ \frac{\rho}{\rho^*} = \frac{\Gamma + 1}{\Gamma - 1 + 2\theta(\Gamma + 1)\rho^*}. \] (17)
Finally, taking into account that, from the Rankine–Hugoniot conditions, we have
\[ u - u^* = \rho^* (u^* - s) \left( \frac{1}{\rho} - \frac{1}{\rho^*} \right), \] (18)
after some calculations, we obtain
\[ u - u^* = 2M_0c_0^* \left[ \frac{1}{(\Gamma + 1)M_0^2} - \frac{1}{\Gamma + 1} + \theta p^* \right] \]  
(19)

which, in the case of a strong shock, yields
\[ u = 2s \left[ \frac{1}{\Gamma + 1} - \theta p^* \right]. \]  
(20)

When \( \theta = 0 \) we recover the well known relations for an ideal gas [6, 9].

In closing this section, it’s worth noting how the so-called “generating function” of shocks, introduced in [10] and computed by D. Fusco in [11], defined as
\[ \eta = \rho_0(s - u^*)(S - S^*), \]  
(21)

modifies in the case under consideration.

Taking into account that the entropy is given by
\[ S = \frac{1 - k_p R}{\Gamma - 1} p \ln \left( \frac{1}{\rho - \theta} \right)^\Gamma \]  
(22)

and making use of the above determined expressions of the Rankine–Hugoniot conditions, we easily obtain
\[ \eta = \frac{1 - k_p R \rho^* c_0^* M_0}{\Gamma - 1} \times \ln \frac{((\Gamma + 1)M_0^2(1 - \theta p^*)/[(\Gamma - 1 + \theta(\Gamma + 1)p^*)M_0^2 + 2])^{\Gamma/2}}{2\Gamma M_0^2(1/(\Gamma + 1) - \theta p^*) - (\Gamma - 1)/(\Gamma + 1)} \]  
(23)

which is defined for
\[ M_0^2 > \frac{\Gamma - 1}{2\Gamma[1 - \theta(\Gamma + 1)p^*]}. \]

Moreover, \( \eta = 0 \) for \( M_0 \to 0 \), that is, \( s = u^* \) which represents an isolated point, and also for
\[ M_0^2 = \frac{1}{1 - \theta(\Gamma + 1)p^*}. \]

Consequently, the growth of the entropy is accomplished when
\[ s - u^* > \frac{c_0^* \sqrt{1 - \theta(\Gamma + 1)p^*}}, \]

or
\[ s - u^* < -\frac{c_0^* \sqrt{1 - \theta(\Gamma + 1)p^*}}. \]
3. INTERACTION BETWEEN WEAK DISCONTINUITIES AND SHOCKS

Let us suppose that the initial conditions \( \rho(x, t_0), u(x, t_0), \) and \( p(x, t_0) \) associated to system (1) suffer a jump in the first order derivatives at the point \( x_0 > 0 \) and, moreover, a strong discontinuity at a point \( x_1 > x_0 \) so that, at the initial time \( t_0 > 0 \), a weak discontinuity wave and a strong one both originate. If the two perturbations have different velocities it may occur that they meet each other; consider, for instance, that the velocity \( \lambda^+ = u + c \) of the fastest weak discontinuities originated in \( x_0 \) is greater than the shock velocity or that the shock wave moves against the discontinuity wave. Now, from the interaction occurring at a certain time \( t_p \), reflected and transmitted weak discontinuities may originate issuing from the collision point denoted by \( P \).

A general theory allowing us to determine the amplitude of the transmitted and reflected waves has been developed in [1, 12] and the effective computation of these amplitudes can be worked out only if the evolution law of the incident discontinuity and the evolution law of the strong discontinuity in the field variables are known.

It’s interesting to note that the interaction generates a jump in the acceleration of the shock wave along the shock line which can be easily determined in the case of weak or characteristic shocks. As it’s well known [13–15] characteristic shocks are the ones propagating with characteristic velocities.

Moreover, the possibility of solving the problem is based on the validity of the evolutionary conditions of Lax [16].

In our case, as the characteristic velocities are \( \lambda_1 = u - c, \lambda_2 = u, \) and \( \lambda_3 = u + c, \) we have only three possibilities:

(i) The 1-shock,

\[
\lambda_1^* \leq s < \lambda_2^*, \quad s < \lambda_1 < \lambda_2 < \lambda_3; \tag{24}
\]

we have two transmitted waves when the fastest characteristic propagating with velocity \( \lambda_3 \) meets the shock line, with \( \lambda^* = \lambda(U^*) \) and \( \lambda = \lambda(U) \) where \( U^* \) and \( U \) denote the field variables, respectively, to the right and to the left of the shock line issuing from the point \( x_1 \) towards \( t > t_0 \). In this case, the algebraic system to be solved is [1]

\[
\overline{s}[U] - \alpha_2(s - u^*)^2d_2^* - \alpha_3(s - u^* - c^*)^2d_3^* = -\pi (s - u - c)^2d_3, \tag{25}
\]

where

\[
\overline{s} = \dot{s}_p^* - \dot{s}_p^* \tag{26}
\]
is the jump in the acceleration of the shock generated by the collision,

$$[U] = U - U^*$$  \hspace{1cm} (27)

is the jump of the field variables across $$\psi(x, t) = 0$$, where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho (e + \frac{u^2}{2}) \end{bmatrix}$$, \hspace{1cm} (28)

$$\pi$$ is the amplitude of the incident discontinuity wave satisfying a Bernoulli type equation, $$\alpha_2$$ and $$\alpha_3$$ are the amplitudes of the transmitted waves, and we denote by $$d_k$$ the right eigenvectors of the matrix $$A = \nabla_U F$$, with

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u (e + \frac{u^2}{2} + \frac{p}{\rho}) \end{bmatrix}$$ \hspace{1cm} (29)

with respect to the eigenvalue $$\lambda_k$$.

In our case, after some tedious but simple calculations, we find:

1. for $$\lambda_1 = u - c$$

$$d_1 = \begin{bmatrix} 1 \\ \frac{1}{2} u^2 - uc + \frac{c(1-\gamma p)}{1-1} - \frac{\gamma p}{1-1} \end{bmatrix}$$; \hspace{1cm} (30)

2. for $$\lambda_2 = u$$

$$d_2 = \begin{bmatrix} 1 \\ \frac{1}{2} u^2 - \gamma p \end{bmatrix}$$; \hspace{1cm} (31)

3. for $$\lambda_3 = u + c$$

$$d_3 = \begin{bmatrix} 1 \\ \frac{1}{2} u^2 + uc + \frac{c(1-\gamma p)}{1-1} - \frac{\gamma p}{1-1} \end{bmatrix}$$. \hspace{1cm} (32)

Therefore system (25) may be solved to give an unique solution of $$\dot{s}$$, $$\alpha_2$$, $$\alpha_3$$ in terms of $$[U]$$, $$\pi$$, and $$s$$. Of course the two discontinuity vectors across the reflected discontinuities satisfy the relations

$$\pi^*_i = \alpha_i d^*_i \hspace{1cm} (i = 2, 3)$$. \hspace{1cm} (33)

(ii) The 2-shock,

$$\lambda^*_1 < \lambda^*_2 \leq s \leq \lambda^*_3, \hspace{1cm} \lambda_1 \leq s \leq \lambda_2 < \lambda_3$$; \hspace{1cm} (34)

we have one transmitted wave and one reflected wave which can be determined, together with $$\overline{s}$$, solving the algebraic system

$$\overline{s}[U] + \beta_1 (s - u + c)^2 d^*_1 - \alpha_3 (s - u^* - c^*)^2 d^*_3 = -\pi (s - u - c)^2 d_3$$. \hspace{1cm} (35)
(iii) The 3-shock,
\[ \lambda_1^* < \lambda_2^* < \lambda_3^* \leq s, \quad \lambda_1 < \lambda_2 \leq s \leq \lambda_3; \] (36)
we have two reflected waves which, together with \( \dot{s} \), result from the system
\[ \dot{s} [\mathbf{U}] + \beta_1 (s - u + c)^2 \mathbf{d}_1 + \beta_2 (s - u)^2 \mathbf{d}_2 = -\pi (s - u - c)^2 \mathbf{d}_3. \] (37)

Obviously, the discontinuity vectors across the transmitted waves are given by
\[ \pi_i = \beta_i \mathbf{d}_i \quad (i = 1, 2). \] (38)

4. REDUCTION TO AUTONOMOUS FORM AND INVARIANCE OF THE RANKINE–HUGONIOT CONDITIONS

In [2] the interaction between a discontinuity wave and a contact shock in a polytropic fluid has been studied under the assumption that the flow is plane, i.e., \( m = 0 \), so that system (1) takes the form
\[ \partial_t \mathbf{U} + \nabla \mathbf{U} \partial_x \mathbf{U} = 0, \] (39)
with \( \mathbf{U} \) and \( \mathbf{F} \) given by (28) and (29). In this case, the governing system has a constant solution and it is possible to assume that the incident discontinuity propagates in a constant state. For axi-symmetric flows (\( m = 1 \) or \( m = 2 \)) this assumption is no longer valid but the governing system, because of its invariance properties [5, 17], may be written in autonomous form which admits constant solutions that are not constant in the original variables, or more general, the system has similarity solutions. It may be easily seen [5] that for \( \theta \neq 0 \) one can look only for particular exact non-constant solutions or for similarity solutions as travelling waves admitted by the transformed autonomous system. This is not the case for \( \theta = 0 \) which is compatible with both constant and non-constant solution [18].

The procedure of reduction of system (1) to the autonomous form is useful to study shock waves or the interaction problem provided that the Rankine–Hugoniot relations, Eqs. (24), (34), and (36) allowing us to determine transmitted and reflected waves, as well as the jump in the shock acceleration, are all invariant with respect to the same transformation leaving the governing system invariant. We will see that the transformation of variables we deal with has the above requirements.

In the sequel we are interested in considering two problems: (a) for \( \theta = 0 \) with the propagation occurring in a medium at rest with respect to the transformed variables) with an initial density of mass of the form \( \rho = \rho_0 x^\nu \), we consider the same problem as in [2] but for an axi-symmetric flow;
(b) for $\theta \neq 0$ with a strong shock propagating in a state characterized by a similarity solution, the shock curve being a similarity, line like in the case of imploding or exploding shocks.

By a similarity solution we mean a solution depending on

$$\xi = \frac{x}{t^{1/\gamma}}$$

which satisfies a system of ordinary differential equations obtained from the governing system. As a consequence, a similarity line is a plane curve given by $x = \xi t^{1/\gamma}$ with $\xi = \text{const}$.

We consider first the case $\theta = 0$ and $m \neq 0$.

In order to reduce system (1) to the autonomous form we consider the following transformation of variables

$$\tau = \frac{1}{\gamma} \ln t, \quad \eta = \ln x - \frac{1}{\gamma} \ln t = \ln \xi, \quad \rho = t^{\alpha/\gamma} R(\eta, \tau),$$

$$u = x \gamma t U(\eta, \tau), \quad p = \frac{\gamma^2 t^{\alpha/\gamma}}{t^{2}} P(\eta, \tau),$$

where $\alpha$ and $\gamma$ are arbitrary constants to be determined by suitable initial or boundary conditions related to specific problems.

Taking into account that

$$\partial_t = \frac{1}{\gamma t} (\partial_x - \partial_\eta), \quad \partial_x = \frac{1}{x} \partial_\eta,$$

system (1) may be written under the autonomous form

$$\partial_x \mathbb{I} + \partial_\eta \mathbb{\Xi} + \mathbb{\Psi} = 0,$$

where

$$\mathbb{I} = \begin{bmatrix} R & RU \\ RU & R(E + \frac{U^2}{2}) \end{bmatrix}; \quad \mathbb{\Xi} = \begin{bmatrix} RU - R \\ RU^2 + P - RU \end{bmatrix};$$

$$\mathbb{\Psi} = \begin{bmatrix} aR + (m + 1)RU \\ (a - \gamma)RU + (m + 2)RU^2 + 2P \\
(a - 2\gamma)R(E + \frac{U^2}{2}) + (m + 3)RU(E + \frac{P}{R} + \frac{U^2}{2}) \end{bmatrix};$$

here $E$ has the same expression of the internal energy $\varepsilon$ with $R$ and $P$ instead of $\rho$ and $p$. For $\theta \neq 0$ the procedure of reduction to the autonomous form is still valid but we must require $a = 0$ in order to have the requested invariance [5, 19].

System (43) has a constant solution $\mathbb{I} = \mathbb{I}_0$ solution of

$$\mathbb{\Psi}(\mathbb{I}) = 0.$$
It is worth noting that other constant solutions may be obtained by the substitution
\[ R = e^{bn\eta}, \quad U = U, \quad P = e^{bn\bar{R}}, \] (47)
as the vectors \( \mathbf{l} \) and \( \bar{R} \) maintain the same structure while \( \bar{R} \) modifies with extra terms of the same type of its components. Of course to these constant solutions there correspond non-constant solutions of the original system because of (41).

For what concerns the Rankine–Hugoniot relations we first note that the shock velocity, under the transformation (41), becomes
\[ s = -\frac{\psi_t}{\psi_x} = \frac{x}{\gamma t}(\Sigma + 1), \quad \Sigma = -\frac{\psi_x}{\psi_t}; \] (48)
it is now clear that, when we assume the shock line to be a similarity line \( x = \xi t^{1/\gamma} \), it follows \( \Sigma = 0 \).

Moreover, the substitution of relations (41) into (6) produces
\[ - (\Sigma + 1)[\mathbf{l}] + [\bar{\mathbf{n}}] = 0 \] (49)
which are, in fact, the Rankine–Hugoniot relations that can be obtained from system (43). For \( \Sigma = 0 \) in (49) we read the Rankine–Hugoniot relations across a similarity shock curve. Taking into account that the time derivative along the shock line satisfies the condition
\[ \frac{d}{dt} = \frac{1}{\gamma t} \frac{d}{d\tau}; \quad \frac{d}{d\tau} = \partial_t + \Sigma \partial_\eta, \quad \frac{d}{dt} = \partial_t + s \partial_x, \] (50)
one obtains, by direct calculations using (43) and (49) or by substitution of (41), (48) and (50) in (35) and (37), where it must be taken into account that \( \phi_\xi = \frac{1}{\gamma} \phi_\eta \) for any \( \phi(x, t) = \phi(\eta, \tau) \), the following relations allowing us to determine the transmitted and reflected waves as well the jump in the shock acceleration:

for \( k = 2 \) we have
\[ \Sigma [\mathbf{l}] + \beta_1(\Sigma - \Lambda_1)^2 \mathbf{r}_1 - \alpha_3(\Sigma - \Lambda_3)^2 \mathbf{r}_3 = -\Pi(\Sigma - \Lambda_3)^2 \mathbf{r}_3; \] (51)

for \( k = 3 \) we have
\[ \Sigma [\mathbf{l}] + \beta_1(\Sigma - \Lambda_1)^2 \mathbf{r}_1 + \beta_2(\Sigma - \Lambda_2)^2 \mathbf{r}_2 = -\Pi(\Sigma - \Lambda_3)^2 \mathbf{r}_3; \] (52)

here \( \mathbf{r}_1, \mathbf{r}_2, \) and \( \mathbf{r}_3 \) are the same as \( \mathbf{d}_1, \mathbf{d}_2, \) and \( \mathbf{d}_3 \) evaluated with \( \mathbf{l} \) instead of \( U \). Moreover, the link between the characteristic velocities is
\[ \lambda = \frac{x}{\gamma t}(\Lambda + 1), \] (53)
so that
\[ \Lambda_1 = U - 1 - C, \quad \Lambda_2 = U - 1, \]
\[ \Lambda_3 = U - 1 + C, \quad C^2 = \frac{\Gamma P}{R(1 - \theta R)}. \] (54)
5. DETERMINATION OF REFLECTED AND TRANSMITTED WAVES

In the $x, t$ plane the fastest discontinuity originated at the time $t_0$ in the point $x_0$ propagates with velocity $\lambda_3 = u + c$ in a non-constant state characterized, say, by a similarity solution.

In details, we consider first the case $\theta = 0$ and $m \neq 0$ when, using the transformation (47) with $b = -1$, the governing system has the following non-constant solution, called the Sedov solution [17],

$$\rho = xt^{(a-1)/\gamma}R_0, \quad u = \frac{x}{\gamma t^{1/\Gamma + 1}}, \quad p = \frac{x^3t^{((a-1)/\gamma)-2}}{\gamma^2} \frac{2R_0}{\Gamma + 1}, \quad (55)$$

with $a = \frac{\Gamma - 7}{\Gamma + 1}, \quad \gamma = \frac{3\Gamma - 1}{\Gamma + 1}, \quad R_0 = \frac{\Gamma + 1}{\Gamma - 1}$. The incident discontinuity as well as the time of occurrence of a break-down in the solution has been determined in [17]. Precisely, as in the $\eta, \tau$ plane we have $\lambda_{03} = \Lambda_0 = U_0 + C_0 - 1$, the characteristic rays are

$$\tau = \sigma, \quad \eta = \eta_0 + \Lambda_0(\tau - \tau_0). \quad (56)$$

The incident discontinuity, as well as the time of occurrence of the break-down of the solution has been determined in [17] and results, respectively,

$$\pi = \frac{\pi_0(t/t_0)^{-h_0/\gamma}}{1 - \pi_0(a_0/h_0)[(t/t_0)^{-h_0/\gamma} - 1]}, \quad t_c = t_0 \left( \frac{h_0 + \pi_0 a_0}{\pi_0 a_0} \right)^{-\gamma/h_0}, \quad (57)$$

where

$$a_0 = \frac{(\Gamma + 1)\sqrt{2\Gamma(\Gamma - 1)}}{2(3\Gamma - 1)}, \quad h_0 = \frac{6\sqrt{2\Gamma(\Gamma - 1) - 3\Gamma + 5}}{2(3\Gamma - 1)}. \quad (58)$$

In the sequel, we assume that $t_p$, the time when the incident discontinuity meets the shock line, is less than $t_c$.

Let us assume now to have a 2-shock propagating in the $\eta, \tau$ plane with velocity $\Sigma = \Lambda_2 = U - 1$ so that the Lax conditions read

$$U - C < \Sigma + 1 < U^* + C^* \quad (59)$$

and we have one reflected and one transmitted wave.

The situation is similar to that considered in [2] but the propagation occurs in a non-constant state (55) to which corresponds a constant state in the transformed variables. Calculation may be performed in a similar way as in [2] taking into account that

$$[11] = (R - R^*) \begin{bmatrix} 1 & U \\ U & \frac{U^2}{2} \end{bmatrix}, \quad (60)$$
and from (51) we obtain
\[
\sum = \frac{2\Pi C^2(C - C^*)}{C^*[R]},
\]
\[
\alpha_3 = \frac{2\Pi C^4}{C^*(C + C^*)}, \quad \beta_1 = \frac{\Pi(C^* - C)}{C + C^*}. \tag{61}
\]

The same considerations as in [2] concerning the critical times of the transmitted and the incident waves may be performed in a similar way.

Let us consider now the case \( \theta \neq 0 \) and \( m \neq 0 \) and we assume that, at the point \( x_1 \) in the \( x, t \) plane at the time \( t_0 \), a strong discontinuity takes place and the shock curve is a similarity line of the type \( x = \xi t^{1/\gamma} \) such that on its left side the field is characterized by a similarity solution which is a stationary solution (independent of \( \tau \)) of system (43), while on its right side we have \( R^*_0 \neq 0, U^*_0 = 0, P^*_0 = 0 \). At the point \( x_0 \), a weak discontinuity originates propagating, with velocity \( \lambda_3 = u + c \), in a non-constant state that we assume, for simplicity, to be characterized by the non-constant solution,

\[
\rho = R_0, \quad u = 0, \quad p = \frac{x^2}{\gamma^2 t^2} P_0 e^{-2\eta}. \tag{62}
\]

The Lax conditions for a 3-shock are in general
\[
U - C < U \leq \Sigma + 1 \tag{63}
\]
and we have two reflected waves. Requiring the shock to be a similarity line is equivalent to having \( \Sigma = 0 \), so that in the \( \eta, \tau \) plane we have a straight line \( \eta = \eta_0 \).

Now, if we want the result of the interaction of the discontinuity line with the strong shock to be unique [1, 20], as \( \Sigma = 0 \), in order to have \( \Sigma \neq 0 \) we must require that, after the interaction, that is, for \( t > t_{p+} \), the shock curve is not a similarity curve anymore. In other words, the interaction breaks the similarity character of the shock curve if one still wants to determine uniquely the reflected waves.

Taking into account the Rankine–Hugoniot conditions for a strong shock (14), (17), (20), we obtain the expressions
\[
R - R^*_0 = \frac{2[1 - \theta(\Gamma + 1)R^*_0]}{\Gamma - 1 + 2\theta(\Gamma + 1)} R^*_0, \quad \Gamma = \frac{1}{\Gamma + 1} - \theta R^*_0,
\]
\[
U = 2\left(\frac{1}{\Gamma + 1} - \theta R^*_0\right),
\]
\[
P = 2R^*_0\left(\frac{1}{\Gamma + 1} - \theta R^*_0\right),
\]
as \( U^*_0 = P^*_0 = 0 \), allowing us to determine [11].
In this case, as we have only reflected waves, from (52) it follows

\[ \sum_{\sigma} = -\frac{\Pi \Lambda_3^2 \mathbf{r}_3 \cdot \mathbf{l}_3}{[\Pi] \cdot \mathbf{l}_3}, \]

\[ \beta_1 = -\frac{[\Pi] \cdot \mathbf{l}_1}{\Lambda_1^2 \mathbf{r}_1 \cdot \mathbf{l}_1} \sum_{\sigma}, \]

\[ \beta_2 = -\frac{[\Pi] \cdot \mathbf{l}_2}{\Lambda_2^2 \mathbf{r}_2 \cdot \mathbf{l}_2} \sum_{\sigma}, \]

where \( \Pi \) may be computed, in terms of the known solution (62), by similar arguments as in [5] and \( \mathbf{l}_1, \mathbf{l}_2, \) and \( \mathbf{l}_3 \) are the left eigenvectors of \( \nabla_{\Pi\sigma} \) given by

\[ \mathbf{l}_1 = \left[ \frac{I - 1}{1 - \theta R} \left( \frac{U^2}{2} - I \right) + CU + C^2 \quad -C - \frac{I - 1}{1 - \theta R} U \quad \frac{I - 1}{1 - \theta R} \right], \]

\[ \mathbf{l}_2 = \left[ \frac{U^2}{2} - I \quad -U \quad 1 \right], \]

\[ \mathbf{l}_3 = \left[ \frac{I - 1}{1 - \theta R} \left( \frac{U^2}{2} - I \right) - CU + C^2 \quad C - \frac{I - 1}{1 - \theta R} U \quad \frac{I - 1}{1 - \theta R} \right], \]

with \( I = E + \frac{p}{\rho}. \)

REFERENCES


