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Nested sequences of index filtrations and continuation of the connection matrix

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Abstract

In this paper, we prove the existence of nested sequences of index filtrations for convergent sequences of (admissible) semiflows on a metric space. This result is new even in the context of flows on a locally compact space. The nested index filtration theorem implies the continuation of homology index braids which, in turn, implies the continuation of connection matrices in the infinite-dimensional Conley index theory.

1. Introduction

Morse decompositions (see e.g. [5,24,22,6]) are a useful tool in the analysis of flows or semiflows defined by ordinary, functional and evolutionary partial differential equations. Combined with an appropriate version of the Conley index and a corresponding Morse equation, they often allow us to obtain multiplicity results for solutions of variational problems (see e.g. [1,12]). Through the use of some more refined topological tools like the Conley connection matrix, Morse decompositions can also be used to detect connections, i.e. heteroclinic orbits in dynamical systems.

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The connection matrix theory for flows defined on locally compact spaces was developed by Franzosa in his thesis [6] and in subsequent papers [7–9]. Important contributions to the theory and applications were made in [15,18,19]. (Cf. also the recent volume [16] for various articles on connection and transition matrices and the references contained therein.)

One of the crucial results of Franzosa's theory is the continuation (i.e. homotopy invariance) property for homology index braids and the connection matrices along paths in the parameter space. The importance of this is clear: in order to compute a connection matrix for a complicated flow, one may first try to homotope this flow to a simpler one for which the connection matrix is easier to compute and use the same connection matrix for the original flow. Franzosa's proof of the continuation property heavily relies on continuation results along paths both for the categorial Morse index (established by Conley [5]) and for index triples (established by Kurland [13,14]).

In [10] Franzosa and Mischaikow extended part of the theory of partially ordered Morse decompositions and connection matrices to the setting of Conley index theory (developed in [20,21]) for admissible local semiflows on (not necessarily locally compact) metric spaces. While those authors establish the existence of index filtrations and connection matrices in this general setting, they do not prove any continuation property.

In previous papers [1,3] we proved some continuation results for Morse decompositions under very general assumptions, applicable to local semiflows and even to equations without the uniqueness property of solutions.

It is the purpose of the present paper to prove a homotopy invariance property for homology index braids (and thus for connection matrices) for parameter-dependent admissible local semiflows on general metric spaces. However, instead of simply extending Franzosa's proof method to the noncompact case (which would require imposing an unnecessarily strong admissibility assumption on the parametrized semiflows) we choose a completely different approach, yielding a stronger result under weaker hypotheses. Our approach is similar in spirit to the proof of the continuation property of the homotopy Conley index given in [20,22]. In those works, for sequences of pairs (π_n, S_n) (where π_n is a local semiflow and S_n is an isolated π_n-invariant set) converging to a corresponding limit pair (π_0, S_0) one proves, under the usual admissibility assumptions, the existence of nested sequences of index pairs (for large $n \in \mathbb{N}$)

$$(N_{1,n}, N_{2,n}) \subset (N_{1,0}, N_{2,0}) \subset (\widetilde{N}_{1,n}, \widetilde{N}_{2,n}) \subset (\widetilde{N}_{1,0}, \widetilde{N}_{2,0}).$$

This result immediately implies that the inclusion $N_{1,n}/N_{2,n} \to N_{1,0}/N_{2,0}$ is a homotopy equivalence of pointed spaces and this, in turn, establishes a homotopy invariance property of Conley index not only along paths but also along connected metric parameter spaces. In this paper, we will similarly establish the existence of nested sequences of index filtrations (cf. Theorem 3.4) which immediately imply the homotopy invariance property of homology index braids (cf. Theorem 3.5 and Theorem 3.7) along connected parameter spaces. This gives us continuation results for connection matrices. An advantage of our method is that it can also be adapted to singular perturbation problems in the setting considered in [2]. This is treated in the subsequent publication [4].

The existence of nested sequences of index filtrations, new even in the locally compact case, is a strong statement which is of interest in its own right. The proof of this very technical result combines the ingenious construction of index filtration from [10] together with various ideas employed in the existence proofs, given in [20,21,24], for isolating blocks, index pairs and block pairs.

This paper is organized as follows. In the next section we recall some basic concepts from Conley index and homology index braid theory and establish some preliminary results. In particular, we state (in Theorem 2.10 below) an extension of an important existence result for index filtrations due to Franzosa–Mischaikow. In Section 3 we state the main results of this paper (Theorems 3.4, 3.5 and 3.7 below) and discuss their applicability. In particular, we show that homology index braids for certain types of parabolic equations are isomorphic to the corresponding homology index braids of their (sufficiently high dimensional) Galerkin approximations.

The last three sections of the paper are devoted to the proof of Theorem 3.4. In Section 4, we use Theorem 2.10 to prove an abstract existence result (Theorem 4.1) for a sequence of index filtrations $(\mathcal{N}_n)_{n\in\mathbb{N}_0}$ such that $\mathcal{N}_n$ is, in some sense, 'asymptotically' included in $\mathcal{N}_0$. In Section 5, we construct sequences of index triples having special properties required for the application of Theorem 4.1 (see Theorems 5.5 and 5.9 below). In Section 6, we make two very specific applications of Theorems 5.5, 5.9 and 4.1 to obtain two sequences of index filtrations which are 'almost' nested. By appropriately modifying these index filtrations (using Proposition 2.9 from Section 2) we finally achieve the full nesting property and thus complete the proof of Theorem 3.4.

Notation: In this paper, $\mathbb{N}$, resp. $\mathbb{N}_0$, denotes the set of all positive, resp. nonnegative, integers.

2. Preliminaries

The purpose of this section is to recall a few concepts from Conley index theory and to establish some preliminary results needed later in this paper. We assume the reader's familiarity with the (infinite dimensional) Conley index theory, as expounded in [22], and with the papers [7,10].

In this section, unless otherwise specified, X is a metric space, π is a local semiflow on X and all concepts are defined *relative to* π.

Suppose that Y is a subset of X. By $\mathrm{Inv}_\pi^+(Y)$, resp. $\mathrm{Inv}_\pi^-(Y)$, resp. $\mathrm{Inv}_\pi(Y)$ we denote the largest positively invariant, resp. negatively invariant, resp. invariant subset of Y. Moreover, let the function $\rho_Y = \rho_{Y,\pi}\colon Y \to \mathbb{R}\cup\{\infty\}$ be given by

$$\rho_Y(x) := \sup\{\, t \geqslant 0 \mid x\pi t \text{ is defined and } x\pi[0,t] \subset Y \,\}.$$

Y is called *π-admissible* if Y is closed and whenever $(x_n)_n$ and $(t_n)_n$ are such that $t_n \to \infty$ and $x_n\pi[0,t_n] \subset Y$ for all $n \in \mathbb{N}$, then the sequence $(x_n\pi t_n)_n$ has a convergent subsequence. We say that *π does not explode in Y* if whenever $x \in X$ and $x\pi t \in Y$

as long as $x\pi t$ is defined, then $x\pi t$ is defined for all $t \in [0, \infty[$. Y is called *strongly* π*-admissible* if Y is π-admissible and π does not explode in Y.

Let N and Y be subsets of X. The set Y is called N*-positively invariant* if whenever $x \in Y$, $t \geqslant 0$ are such that $x\pi[0,t] \subset N$, then $x\pi[0,t] \subset Y$.

Let N, Y_1 and Y_2 be subsets of X. The set Y_2 is called an *exit ramp for* N *within* Y_1 if whenever $x \in Y_1$ and $x\pi t' \notin N$ for some $t' \in [0, \infty[$, then there exists a $t_0 \in [0, t']$ such that $x\pi[0, t_0] \subset N$ and $x\pi t_0 \in Y_2$.

If Y_1 and Y_2 are subsets of X then Y_2 is called an *exit ramp for* Y_1 if Y_2 is an exit ramp for N within Y_1, where $N = Y_1$.

Definition 2.1. Let $B \subset X$ be a closed set and $x \in \partial B$. The point x is called a *strict egress* (respectively *strict ingress*, respectively *bounce-off*) *point of* B, if for every solution $\sigma: [-\delta_1, \delta_2] \to X$ through x, with $\delta_1 \geqslant 0$ and $\delta_2 > 0$, the following properties hold:

(1) There exists an $\varepsilon_2 \in]0, \delta_2[$ such that $\sigma(t) \notin B$ (respectively $\sigma(t) \in \operatorname{Int} B$, respectively, $\sigma(t) \notin B$), for $t \in]0, \varepsilon_2]$.
(2) If $\delta_1 > 0$, then there exists an $\varepsilon_1 \in]0, \delta_1[$ such that $\sigma(t) \in \operatorname{Int} B$ (respectively $\sigma(t) \notin B$, respectively, $\sigma(t) \notin B$), for $t \in [-\varepsilon_1, 0[$.

The set of all strict egress (respectively strict ingress, respectively bounce-off) points of B is denoted by B^e (respectively B^i, respectively B^b). Moreover, we call $B^- := B^e \cup B^b$ the *exit set of* B and $B^+ := B^i \cup B^b$ the *entrance set of* B. B is called an *isolating block*, if $\partial B = B^e \cup B^i \cup B^b$ and B^- is closed. If B is also an isolating neighborhood of an invariant set S, then we say that B is an *isolating block for* S.

If B is an isolating block then (B, B^-) is an example of an index pair in B. More generally, we have the following definition.

Definition 2.2. Let N be closed in X. A pair (N_1, N_2) is called an *index pair in* N if:

(1) N_1 and N_2 are closed and N-positively invariant subsets of N;
(2) N_2 is an exit ramp for N within N_1;
(3) $\operatorname{Inv}_\pi(N)$ is closed and $\operatorname{Inv}_\pi(N) \subset \operatorname{Int}(N_1 \setminus N_2)$.

The next definition introduces a more general concept.

Definition 2.3. A pair (N_1, N_2) is called a *Franzosa–Mischaikow-index pair* (*or FM-index pair*) *for* S if:

(1) N_1 and N_2 are closed subsets of X with $N_2 \subset N_1$ and N_2 is N_1-positively invariant;
(2) N_2 is an exit ramp for N_1;
(3) S is closed, $S \subset \operatorname{Int}(N_1 \setminus N_2)$ and S is the largest invariant set in $\operatorname{Cl}(N_1 \setminus N_2)$;

Proposition 2.4 (*cf. Franzosa and Mischaikow* [10]). *Let* (N_1, N_2) *be a pair of closed subsets of* X *with* $N_2 \subset N_1$.

(1) *If* S *is an isolated invariant set,* N_1 *is an isolating neighborhood of* S *and* (N_1, N_2) *is an index pair in* N_1*, then* (N_1, N_2) *is an FM-index pair for* S.

(2) *If* (N_1, N_2) *is an FM-index pair for* S *and* N *is an isolating neighborhood of* S *with* $N_1 \setminus N_2 \subset N$, *then* $N_1 \cap N$ *is an isolating neighborhood of* S *and* $(N_1 \cap N, N_2 \cap N)$ *is an index pair in* $N_1 \cap N$.

Proposition 2.5. *Let* (N_1, N_2) *be an FM-index pair for* S. *Let* Y *be a closed set such that* $S \subset \mathrm{Int}(Y)$ *and such that* N_2 *is an exit ramp for* Y. *Then* $(Y \cap N_1, Y \cap N_2)$ *is an FM-index pair for* S.

Proof. It is clear that $Y \cap N_1$ and $Y \cap N_2$ are closed. If $x \in Y \cap N_2$ and $x\pi[0, t] \subset Y \cap N_1$ for some $t \geqslant 0$, then, since N_2 is N_1-positively invariant, we obtain $x\pi[0, t] \subset N_2$ and so $x\pi[0, t] \subset Y \cap N_2$. This proves that $Y \cap N_2$ is $(Y \cap N_1)$-positively invariant.

There exist open sets V and W such that $S \subset W \subset N_1 \setminus N_2$ and $S \subset V \subset Y$. Thus $S \subset V \cap W \subset Y \cap (N_1 \setminus N_2) \subset (Y \cap N_1) \setminus (Y \cap N_2)$ and so $S \subset \mathrm{Int}((Y \cap N_1) \setminus (Y \cap N_2))$. Thus $S \subset \mathrm{Inv}_\pi \mathrm{Cl}((Y \cap N_1) \setminus (Y \cap N_2)) = \mathrm{Inv}_\pi \mathrm{Cl}(Y \cap (N_1 \setminus N_2)) \subset \mathrm{Inv}_\pi \mathrm{Cl}(N_1 \setminus N_2) = S$.

To complete the proof, let $x \in Y \cap N_1$ and assume that there exists a $t' \geqslant 0$ such that $x\pi t' \notin Y \cap N_1$. Suppose first that $\rho_Y(x) > \rho_{N_1}(x)$. Then $\rho_{N_1}(x) < \infty$ and so $x\pi[0, \rho_{N_1}(x)] \subset Y \cap N_1$ and $x\pi\rho_{N_1}(x) \in N_2$ since (N_1, N_2) is an FM-index pair for S. Thus, $x\pi\rho_{N_1}(x) \in Y \cap N_2$. Now assume that $\rho_Y(x) \leqslant \rho_{N_1}(x)$. Then $\rho_Y(x) < \infty$ and so $x\pi[0, \rho_Y(x)] \subset Y \cap N_1$. Moreover, since N_2 is an exit ramp for Y, we have $x\pi\rho_Y(x) \in N_2$ and so $x\pi\rho_Y(x) \in Y \cap N_2$. The proof is complete. □

Definition 2.6. Let S be an isolated invariant set and (A, A^*) be an attractor–repeller pair in S. A pair (B_1, B_2) is called a *block pair* (*for* (π, S, A, A^*)) if B_1 is an isolating block for A^*, B_2 is an isolating block for A, $B := B_1 \cup B_2$ is an isolating block for S and $B_1 \cap B_2 \subset B_1^- \cap B_2^+$.

If (B_1, B_2) is a block pair then $(B, B_2 \cup B^-, B^-)$ is an example of an FM-index triple:

Definition 2.7. Let S be an isolated invariant set and (A, A^*) be an attractor–repeller pair in S. A triple (N_1, N_2, N_3) with $N_3 \subset N_2 \subset N_1$ is called an *FM-index triple* (*for* (π, S, A, A^*)) if (N_1, N_3) is an FM-index pair for S and (N_2, N_3) is an FM-index pair for A.

Proposition 2.8 (*cf. Franzosa and Mischaikow [10]*). *If* (N_1, N_2, N_3) *is an FM-index triple for* (π, S, A, A^*) *then* (N_1, N_2) *is an FM-index pair for* A^*.

Given an isolated invariant set K having a strongly π-admissible isolating neighborhood we denote by $h(K) = h(\pi, K)$ the Conley-index of K and by $H(K) = H(\pi, K) = H(h(K))$ the homology Conley index, where H is the singular homology functor (with coefficients in some fixed module G over a PID).

If (A, A^*) is an attractor–repeller pair in S and (N_1, N_2, N_3) is an FM-index triple for (π, S, A, A^*) with N_1 strongly π-admissible, then the inclusion induced sequence $N_2/N_3 \xrightarrow{i} N_1/N_3 \xrightarrow{p} N_1/N_2$ induces a long exact homology sequence

$$\longrightarrow H_q(N_2/N_3) \xrightarrow{i} H_q(N_1/N_3) \xrightarrow{p} H_q(N_1/N_2) \xrightarrow{\partial} H_{q-1}(N_2/N_3) \longrightarrow .$$

This sequence is independent of the choice of (N_1, N_2, N_3) and so there is a well-defined long exact sequence

$$\longrightarrow H_q(A) \xrightarrow{i} H_q(S) \xrightarrow{p} H_q(A^*) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow$$

called the *homology index sequence of* (π, S, A, A^*).

Recall that a *strict partial order* on a set P is a relation $\prec\, \subset P \times P$ which is irreflexive and transitive. As usual, we write $x \prec y$ instead of $(x, y) \in\, \prec$. The symbol $<$ will be reserved for the less-than-relation on $\mathbb{R}$.

For the rest of this paper, unless specified otherwise, let P be a fixed finite set and $\prec$ be a fixed strict partial order on P.

A set $I \subset P$ is called *a* $\prec$*-interval* if whenever $i, j, k \in P$, $i, k \in I$ and $i \prec j \prec k$, then $j \in I$. By $\mathcal{I}(\prec)$ we denote the set of all $\prec$-intervals in P. A set I is called *a* $\prec$*-attracting interval* if whenever $i, j \in P$, $j \in I$ and $i \prec j$, then $i \in I$. By $\mathcal{A}(\prec)$ we denote the set of all $\prec$-attracting intervals in P. Of course, $\mathcal{A}(\prec) \subset \mathcal{I}(\prec)$.

An *adjacent n-tuple of* $\prec$*-intervals* is a sequence $(I_j)_{j=1}^n$ of pairwise disjoint $\prec$-intervals whose union is a $\prec$-interval and such that, whenever $j < k$, $p \in I_j$ and $p' \in I_k$, then $p' \not\prec p$ (i.e. $p \prec p'$ or else p and p' are not related by $\prec$). By $\mathcal{I}_n(\prec)$ we denote the set of all adjacent n-tuples of $\prec$-intervals.

Let S be a compact invariant set. A family $(M_i)_{i \in P}$ of subsets of S is called a $\prec$*-ordered Morse decomposition of* S if the following properties hold:

(1) The sets M_i, $i \in P$, are closed, π-invariant and pairwise disjoint.
(2) For every full solution σ of π lying in S either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are $k, l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

Let S be a compact invariant set and $(M_i)_{i \in P}$ be a $\prec$-ordered Morse decomposition of S. If $A, B \subset X$ then the (π, S)*-connection set* $\mathrm{CS}_{\pi,S}(A, B)$ *from A to B* is the set of all points $x \in X$ for which there is a solution $\sigma\colon \mathbb{R} \to S$ of π with $\sigma(0) = x$, $\alpha(\sigma) \subset A$ and $\omega(\sigma) \subset B$.

For an arbitrary $\prec$-interval I set

$$M(I) = M_{\pi,S}(I) = \bigcup_{(i,j) \in I \times I} \mathrm{CS}_{\pi,S}(M_i, M_j).$$

An *index filtration for* $(\pi, S, (M_p)_{p\in P})$ is a family $\mathcal{N} = (N(I))_{I\in\mathcal{A}(\prec)}$ of closed subsets of X such that

(1) for each $I \in \mathcal{A}(\prec)$, the pair $(N(I), N(\emptyset))$ is an FM-index pair for $M(I)$,
(2) for each $I_1, I_2 \in \mathcal{A}(\prec)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$.

$\mathcal{N}$ is called *strongly π-admissible* if $N(P)$ is strongly π-admissible.

The following result is an immediate consequence of Proposition 2.5.

Proposition 2.9. *Let Y be a closed set such that $S \subset \mathrm{Int}(Y)$ and such that $N(\emptyset)$ is an exit ramp for Y. Let $\mathcal{N} = (N(I))_{I\in\mathcal{A}(\prec)}$ be an index filtration for $(\pi, S, (M_p)_{p\in P})$. Then $(Y \cap N(I))_{I\in\mathcal{A}(\prec)}$ is an index filtration for $(\pi, S, (M_p)_{p\in P})$.*

A special version of the following basic result was established in [10].

Theorem 2.10 (*cf. Franzosa and Mischaikow [10, Theorem 3.5]*). *Let N_1^i, N_2^i, N_I^i, $i = 2, 4$, $I \in \mathcal{A}(\prec)$, be sets such that, for each $I \in \mathcal{A}(\prec)$, (N_1^i, N_I^i, N_2^i), $i = 2, 4$, is an FM-index triple for $(\pi, S, M(I), M(P \setminus I))$. Suppose $N_1^2 \subset N_1^4$, $N_2^2 \subset N_2^4$ and $N_I^2 \subset N_I^4$, $I \in \mathcal{A}(\prec)$. For each $p \in P$ define the following sets*:

$$D_p := \Big(\bigcap_{\substack{I\\ p\in I}} \mathrm{Int}(N_I^2 \setminus N_2^4)\Big) \cap \Big(\bigcap_{\substack{I\\ p\notin I}} \mathrm{Int}(N_1^2 \setminus N_I^4)\Big)$$

and

$$E_p := \{\, x \in N_1^2 \mid \textit{there exists a } t \geqslant 0 \textit{ such that } x\pi[0,t] \subset N_1^2 \textit{ and } x\pi t \in D_p \,\}.$$

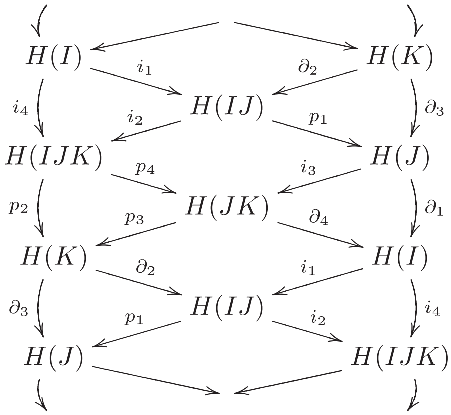
For each $I \in \mathcal{A}(\prec)$, define $N(I) := N_1^2 \setminus \bigcup_{p\in P\setminus I} E_p$. Then $(N(I))_{I\in\mathcal{A}(\prec)}$ is an index filtration for $(\pi, S, (M_p)_{p\in P})$. Moreover, $N_2^2 \subset N(\emptyset)$ and $N_I^2 \subset N(I)$ for all $I \in \mathcal{A}(\prec)$.

Proof. For $N_1^2 = N_1^4$, $N_2^2 = N_2^4$ and $N_I^2 = N_I^4$, $I \in \mathcal{A}(\prec)$, this is just Theorem 3.5 in [10], whose proof carries over almost verbatim to the present, more general case. Details are left to the reader. □

Let $\mathcal{N}$ be a strongly π-admissible index filtration for $(\pi, S, (M_p)_{p\in P})$. For $J \in \mathcal{I}(\prec)$ the set $M(J)$ is an isolated invariant set and we write $H(J) = H(\pi, J) := H(\pi, M(J))$. If $(I, J) \in \mathcal{I}_2(\prec)$, then $(M(I), M(J))$ is an attractor–repeller filtration in $M(IJ)$, where $IJ := I \cup J$. Hence there is the corresponding homology index sequence

$$\longrightarrow H_q(I) \xrightarrow{i_{I,J}} H_q(IJ) \xrightarrow{p_{I,J}} H_q(J) \xrightarrow{\partial_{I,J}} H_{q-1}(I) \longrightarrow$$

of $(\pi, M(IJ), M(I), M(J))$. Using the filtration $\mathcal{N}$ one proves that for every triple $(I, J, K) \in \mathcal{I}_3(\prec)$ the following diagram, made up of the four homology index sequences defined by (I, J, K), commutes:



The collection of all the homology indices $H(\pi, M(J))$, $J \in \mathcal{I}(\prec)$, and all the maps $i_{I,J}$, $p_{I,J}$ and $\partial_{I,J}$, $(I,J) \in \mathcal{I}_2(\prec)$, is called *the homology index braid of* $(\pi, S, (M_p)_{p\in P})$ and is denoted by $\mathcal{H}(\pi, S, (M_p)_{p\in P})$.

For the rest of this section assume that, for $i = 1, 2$, π_i is a local semiflow on the metric space X_i, S_i is an isolated invariant set and $(M_{p,i})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_i, relative to π_i. Write $M_i(I) = M_{\pi_i,S_i}(I)$, $H_i(I) = H(\pi_i, M_i(I))$ and $\mathcal{H}_i := \mathcal{H}(\pi_i, S_i, (M_{p,i})_{p\in P})$, for $i = 1, 2$ and $I \in \mathcal{I}(\prec)$.

Suppose $\theta := (\theta(J))_{J\in\mathcal{I}(\prec)}$ is a family $\theta(J): H_1(J) \to H_2(J)$, $J \in \mathcal{I}(\prec)$, of maps such that, for all $(I, J) \in \mathcal{I}_2(\prec)$, the diagram

$$\begin{array}{ccccccccccc} \longrightarrow & H_{1,q}(I) & \xrightarrow{i_{I,J}} & H_{1,q}(IJ) & \xrightarrow{p_{I,J}} & H_{1,q}(J) & \xrightarrow{\partial_{I,J}} & H_{1,q-1}(I) & \longrightarrow \\ & \downarrow{\scriptstyle \theta(I)} & & \downarrow{\scriptstyle \theta(IJ)} & & \downarrow{\scriptstyle \theta(J)} & & \downarrow{\scriptstyle \theta(I)} & \\ \longrightarrow & H_{2,q}(I) & \xrightarrow[i_{I,J}]{} & H_{2,q}(IJ) & \xrightarrow[p_{I,J}]{} & H_{2,q}(J) & \xrightarrow[\partial_{I,J}]{} & H_{2,q-1}(I) & \longrightarrow \end{array} \tag{2.1}$$

commutes. Then we say that θ is a *morphism from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ and we write $\theta: \mathcal{H}_1 \to \mathcal{H}_2$. If each $\theta(J)$ is an isomorphism, then we say that θ is an *isomorphism* and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* homology index braids.

Remark 2.11. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic homology index braids, then, by Proposition 1.5 in [9], $\mathcal{H}_1$ and $\mathcal{H}_2$ determine the same collection of connection matrices and the same collection of C-connection matrices.

We will now introduce an important class of morphisms between homology index braids. Let $\mathcal{N}_i = (N_i(I))_{I\in\mathcal{A}(\prec)}$ be a strongly π_i-admissible index filtration for $(\pi_i, S_i, (M_{p,i})_{p\in P})$, $i = 1, 2$. Assume the *nesting* property

$$N_1(I) \subset N_2(I), \quad I \in \mathcal{A}(\prec).$$

For $J \in \mathcal{I}(\prec)$ choose I, $K \in \mathcal{A}(\prec)$ with $(I, J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Then, for $i = 1$, 2, $(N_i(K), N_i(I))$ is an FM-index pair for $M_i(J)$, relative to π_i. The inclusion induced map from $N_1(K)/N_1(I)$ to $N_2(K)/N_2(I)$ induces a homomorphism

$$\theta(J) = \theta_{\mathcal{N}_1,\mathcal{N}_2}(J): H(\pi_1, M_1(J)) \to H(\pi_2, M_2(J)).$$

Of course, this homomorphism depends on the choice of $\mathcal{N}_i$, $i = 1, 2$, but we claim that it is independent of the choice of I and K. In fact, if I' and $K' \in \mathcal{A}(\prec)$ are such that $(I', J) \in \mathcal{I}_2(\prec)$ and $K' = I'J$ then property (2) of index filtrations implies that $N_i(K) \setminus N_i(I) = N_i(K') \setminus N_i(I')$, $i = 1, 2$, so there is an inclusion induced, commutative diagram of pointed spaces

$$\begin{array}{ccc} N_1(K)/N_1(I) & \longrightarrow & N_1(K')/N_1(I') \\ \downarrow & & \downarrow \\ N_2(K)/N_2(I) & \longrightarrow & N_2(K')/N_2(I'). \end{array}$$

This proves the claim in view of the identifications made in the definition of the homology Conley index. We write

$$\theta_{\mathcal{N}_1,\mathcal{N}_2} = (\theta_{\mathcal{N}_1,\mathcal{N}_2}(J))_{J\in\mathcal{I}(\prec)}.$$

We also claim that $\theta_{\mathcal{N}_1,\mathcal{N}_2}: \mathcal{H}_1 \to \mathcal{H}_2$. In fact, let $(I, J) \in \mathcal{I}_2(\prec)$ and let B be the set of all $p \in P \setminus (IJ)$ for which there is a $p' \in IJ$ with $p \prec p'$. It follows that B, BI, $BIJ \in \mathcal{A}(\prec)$. Setting, for $i = 1, 2$, $N_{1,i} = N_i(BIJ)$, $N_{2,i} = N_i(BI)$ and $N_{3,i} = N_i(B)$ we see that $(N_{1,i}, N_{2,i}, N_{3,i})$ is an FM-index triple for $(\pi_i, M_i(IJ), M_i(I), M_i(J))$. Thus the inclusion induced commutative diagram

$$\begin{array}{ccccc} N_{2,1}/N_{3,1} & \xrightarrow{i} & N_{1,1}/N_{3,1} & \xrightarrow{p} & N_{1,1}/N_{2,1} \\ \downarrow & & \downarrow & & \downarrow \\ N_{2,2}/N_{3,2} & \xrightarrow{i} & N_{1,2}/N_{3,2} & \xrightarrow{p} & N_{1,2}/N_{2,2} \end{array}$$

implies the commutativity of diagram (2.1). This proves our second claim.

We call $\theta := (\theta(J))_{J\in\mathcal{I}(\prec)}$ the *inclusion induced morphism from $\mathcal{H}_1$ to $\mathcal{H}_2$*.

We now obtain the following result:

Proposition 2.12. *For $i = 1, 2$ let $\mathcal{N}_i = (N_i(I))_{I\in\mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_i = (\widetilde{N}_i(I))_{I\in\mathcal{A}(\prec)}$ be strongly π_i-admissible index filtrations for $(\pi_i, S_i, (M_{p,i})_{p\in P})$. Assume the nesting property*

$$N_1(I) \subset N_2(I) \subset \widetilde{N}_1(I) \subset \widetilde{N}_2(I), \quad I \in \mathcal{A}(\prec). \tag{2.2}$$

Then the inclusion induced morphism $\theta_{\mathcal{N}_1,\mathcal{N}_2}$ is an isomorphism.

Proof. Let $J \in \mathcal{I}(\prec)$ be arbitrary, $a := \theta_{\mathcal{N}_1,\mathcal{N}_2}(J)$, $b := \theta_{\mathcal{N}_2,\widetilde{\mathcal{N}}_1}(J)$ and $c := \theta_{\widetilde{\mathcal{N}}_1,\widetilde{\mathcal{N}}_2}(J)$. Then $b \circ a$ and $c \circ b$ are isomorphisms, being induced by maps lying in the same connected simple system (the categorial Morse index of $(\pi_1, M_1(J))$ and $(\pi_2, M_2(J))$, respectively). It follows that a, b and c are isomorphisms. This proves the proposition. □

3. Continuation of homology index braids

Let π_n, $n \in \mathbb{N}_0$, be local semiflows on the metric space X. We say that the sequence $(\pi_n)_{n\in\mathbb{N}}$ *converges* π_0 and we write $\pi_n \to \pi_0$ if whenever $x_n \to x_0$ in X, $t_n \to t_0$ in $[0,\infty[$ and $x_0\pi_0 t_0$ is defined, then $x_n\pi_n t_n$ is defined for all n large enough and $x_n\pi_n t_n \to x_0\pi_0 t_0$ in X.

Given $Y \subset X$ we say that Y *is* $(\pi_n)_n$*-admissible* if Y is closed and whenever $(x_n)_n$ and $(t_n)_n$ are such that $t_n \to \infty$, $x_n\pi_n t_n$ is defined and $x_n\pi_n[0,t_n] \subset Y$ for all $n \in \mathbb{N}$, then the sequence $(x_n\pi_n t_n)_n$ has a convergent subsequence.

The following continuation result for Morse decompositions was established in [3].

Theorem 3.1 (*cf. Carbinatto and Rybakowski [3, Corollaries 3.5 and 3.6]*). *Let π_n, where $n \in \mathbb{N}_0$, be local semiflows on X and $\widetilde{N}$ be a closed subset of X which is strongly π_n-admissible for every $n \in \mathbb{N}_0$. Moreover, assume that*

(A) *$\pi_n \to \pi_0$ and $\widetilde{N}$ is $(\pi_{n_m})_m$-admissible for every subsequence $(\pi_{n_m})_m$ of $(\pi_n)_n$.*

Suppose that $S_0 := \mathrm{Inv}_{\pi_0}(\widetilde{N}) \subset \mathrm{Int}(\widetilde{N})$ and $(M_{p,0})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_0 relative to π_0. For each $p \in P$, let $\Xi_p \subset \widetilde{N}$ be closed in X and such that $M_{p,0} = \mathrm{Inv}_{\pi_0}(\Xi_p) \subset \mathrm{Int}(\Xi_p)$. (Such sets Ξ_p, $p \in P$, always exist.) For $n \in \mathbb{N}$ and $p \in P$ set $S_n := \mathrm{Inv}_{\pi_n}(\widetilde{N})$ and $M_{p,n} := \mathrm{Inv}_{\pi_n}(\Xi_p)$. Then there is an $\bar{n} \in \mathbb{N}$ such that whenever $n \geqslant \bar{n}$ and $p \in P$ then $S_n \subset \mathrm{Int}(\widetilde{N})$, $M_{p,n} \subset \mathrm{Int}(\Xi_p)$ and the family $(M_{p,n})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_n relative to π_n.

Remark 3.2. It follows from [3, Theorem 3.3 and the proof of Corollary 3.5] that Theorem 3.1 remains valid if we replace assumption (A) by the following weaker assumption:

(B) Whenever $(n_m)_m$ is a sequence in $\mathbb{N}$ with $n_m \to \infty$ and, for every $m \in \mathbb{N}$, u_m is a full solution of π_{n_m} lying in $\widetilde{N}$, then there is a subsequence $(u_{m_k})_k$ of $(u_m)_m$ and a full solution u_0 of π_0 such that $u_{m_k}(t) \to u_0(t)$ as $k \to \infty$, uniformly for t lying in compact subset of $\mathbb{R}$.

However, we require the stronger assumption (A) in Theorem 3.4 below.

The sets S_n and $M_{p,n}$ are asymptotically independent of the choice of $\widetilde{N}$ and Ξ_p, $p \in P$, in the sense that, given other sets $\widetilde{N}'$ and Ξ'_p, $p \in P$ satisfying the same properties as $\widetilde{N}$ and Ξ_p, then, for n large enough, $\mathrm{Inv}_{\pi_n}(\widetilde{N}) = \mathrm{Inv}_{\pi_n}(\widetilde{N}')$ and $\mathrm{Inv}_{\pi_n}(\Xi_p) = \mathrm{Inv}_{\pi_n}(\Xi'_p)$, $p \in P$. This follows from the following result (cf. [2, Proposition 2.17]).

Proposition 3.3. *Suppose $\pi_n \to \pi_0$ and Y_1, Y_2 are two (not necessarily distinct) closed sets which are strongly π_n-admissible for every $n \in \mathbb{N}_0$ and $(\pi_{n_m})_m$-admissible for every subsequence $(\pi_{n_m})_m$ of $(\pi_n)_n$. Suppose that $\mathrm{Inv}_{\pi_0}(Y_1) = \mathrm{Inv}_{\pi_0}(Y_2) \subset \mathrm{Int}(Y_1) \cap \mathrm{Int}(Y_2)$ (resp. suppose that $\mathrm{Inv}_{\pi_0}(Y_1) = \emptyset$). Then there is an $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, $\mathrm{Inv}_{\pi_n}(Y_1) = \mathrm{Inv}_{\pi_n}(Y_2) \subset \mathrm{Int}(Y_1) \cap \mathrm{Int}(Y_2)$ (resp. $\mathrm{Inv}_{\pi_n}(Y_1) = \emptyset$).*

We will tacitly use this result in the sequel.

We can now state the main result of this paper, the *nested index filtration theorem*.

Theorem 3.4. *Assume the hypotheses (and thus also the conclusions) of Theorem 3.1 and let $\overline{n}$ be as in that theorem. Then there is an $n_1 \geqslant \overline{n}$ such that for every $n \in \mathbb{N}_0$ with $n = 0$ or $n \geqslant n_1$ there exist strongly π_n-admissible index filtrations $\mathcal{N}_n = (N_n(I))_{I \in \mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_n = (\widetilde{N}_n(I))_{I \in \mathcal{A}(\prec)}$ for $(\pi_n, S_n, (M_{p,n})_{p \in P})$ such that the following* nesting property *holds*:

$$N_n(I) \subset N_0(I) \subset \widetilde{N}_n(I) \subset \widetilde{N}_0(I) \text{ for all } n \geqslant n_1 \text{ and } I \in \mathcal{A}(\prec). \tag{3.1}$$

Theorem 3.4, Proposition 2.12 and Remark 2.11 immediately imply the following *continuation result for homology index braids and connection matrices*.

Theorem 3.5. *Under the hypotheses of Theorem 3.4 the homology index braids $\mathcal{H}(\pi_0, S_0, (M_{p,0})_{p \in P})$ and $\mathcal{H}(\pi_n, S_n, (M_{p,n})_{p \in P})$, $n \geqslant n_1$, are isomorphic and determine the same collection of connection matrices and the same collection of C-connection matrices.*

Let us make the following definition.

Definition 3.6. Let Λ be a metric space. A family $(\pi_\lambda, S_\lambda, (M_{p,\lambda})_{p \in P})_{\lambda \in \Lambda}$ is called *$\mathcal{S}$-continuous* if for every $\lambda_0 \in \Lambda$ there is a neighborhood W_{λ_0} of λ_0 in Λ and there are closed subsets N_{λ_0}, $\Xi_{p,\lambda_0} \subset N_{\lambda_0}$, $p \in P$, of X such that for every $\lambda \in W_{\lambda_0}$, π_λ is a local semiflow on X, S_λ is a (compact) π_λ-invariant set, $(M_{p,\lambda})_{p \in P}$ is a Morse decomposition of S_λ, relative to π_λ, N_{λ_0} is a strongly π_λ-admissible isolating neighborhood of S_λ and, for $p \in P$, Ξ_{p,λ_0} is an isolating neighborhood of $M_{p,\lambda}$, relative to π_λ. Moreover, whenever $\lambda_n \to \lambda_0$ in W_{λ_0} then $\pi_{\lambda_n} \to \pi_{\lambda_0}$ and N_{λ_0} is $(\pi_{\lambda_n})_n$-admissible.

We can now state our second *continuation result for homology index braids and connection matrices*.

Theorem 3.7. *Let Λ be a metric space and $(\pi_\lambda, S_\lambda, (M_{p,\lambda})_{p \in P})_{\lambda \in \Lambda}$ be an $\mathcal{S}$-continuous family. Then for every $\lambda \in \Lambda$ the homology index braid*

$$\mathcal{H}_\lambda := \mathcal{H}(\pi_\lambda, S_\lambda, (M_{p,\lambda})_{p \in P})$$

is defined and for every $\lambda_0 \in \Lambda$ there is a neighborhood W of λ_0 in Λ such that $\mathcal{H}_\lambda$ is isomorphic to $\mathcal{H}_{\lambda_0}$ for every $\lambda \in W$. In particular, if Λ is connected, then $\mathcal{H}_{\lambda_1}$ and

$\mathcal{H}_{\lambda_2}$ are isomorphic for all λ_1, $\lambda_2 \in \Lambda$ and determine the same collection of connection matrices and the same collection of C-connection matrices.

Proof. A simple application of Theorem 3.4 shows that $\mathcal{H}_{\lambda_0}$ is defined for every $\lambda_0 \in \Lambda$. Thus, if the second assertion is not true then there is a $\lambda_0 \in \Lambda$ and a sequence $(\lambda_n)_n$ with $\lambda_n \to \lambda_0$ and for all $n \in \mathbb{N}$, $\mathcal{H}_{\lambda_n}$ is not isomorphic to $\mathcal{H}_{\lambda_0}$. It is clear that all assumptions of Theorem 3.5 are satisfied with $(\pi_n, S_n, (M_{p,n})_p) := (\pi_{\lambda_n}, S_{\lambda_n}, (M_{p,\lambda_n})_{p\in P})$, $n \in \mathbb{N}_0$, $\widetilde{N} := N_{\lambda_0}$ and $\Xi_p := \Xi_{p,\lambda_0}$, $p \in P$. Thus, by Theorem 3.5, there is an n_1 such that $\mathcal{H}_{\lambda_0}$ is isomorphic to $\mathcal{H}_{\lambda_n}$, for all $n \geqslant n_1$, a contradiction which proves the second assertion of the theorem. The last assertion now follows immediately. □

Remark. Both Definition 3.6 and Theorem 3.7 can be generalized to topological spaces Λ satisfying the first countability axiom, because that is all we use in the proof of Theorem 3.7.

Theorems 3.5 and 3.7 refine the corresponding homotopy invariance results for the (infinite dimensional) Conley index established in [20] (or [22]). The convergence and admissibility assumptions make these results applicable to various classes of parameter dependent evolution equations (e.g. parabolic or damped hyperbolic equations on bounded domains and even some parabolic equations on unbounded domains, see the recent paper [17]).

We will not discuss applications in this paper, reserving them for a subsequent publication. However, we will show that in certain cases homology index braids (and connection matrices) of infinite-dimensional semiflows can be computed by restricting to their finite-dimensional Galerkin approximations.

For the rest of this section let X be a real Hilbert space and $A: D(A) \subset X \to X$ be a positive selfadjoint operator with compact resolvent. Let $(\phi_\nu)_{\nu\in\mathbb{N}}$ be a complete X-orthonormal basis of X consisting of eigenfunctions of A. Let $P_n: X \to X$ be the orthogonal projection of X onto the subspace spanned by the first n eigenfunctions. Moreover, set $Q_n := I - P_n$ where I is the identity map on X. Note that A is sectorial on X and so it generates a family $(X^\alpha)_{\alpha\in[0,\infty[}$ of fractional power spaces. Given $\alpha \in [0, 1[$ and a locally Lipschitzian map $g: X^\alpha \to X$ we denote by π_g the local semiflow on X^α generated by the abstract parabolic equation (see [11])

$$\dot{u} = -Au + g(u), \quad u \in X^\alpha.$$

The following result has been proved in [23] (see [23, Theorem 4.3, Proposition 4.4]).

Proposition 3.8. *Let $f: X^\alpha \to X$ be Lipschitzian on bounded subsets of X^α. For $n \in \mathbb{N}$ and $\tau \in [0, 1]$ let $f_{n,\tau}: X^\alpha \to X$ be defined by*

$$f_{n,\tau}(u) = (1-\tau)f(u) + \tau P_n f(P_n u), \quad u \in X^\alpha.$$

Let $N \subset X^\alpha$ be bounded and closed. Furthermore, let $(n_m)_m$ be a sequence in $\mathbb{N}$ with $n_m \to \infty$ and $(\tau_m)_m$ be an arbitrary sequence in $[0,1]$. For every $m \in \mathbb{N}$ let u_m be a full solution of $\pi_{f_{n_m,\tau_m}}$ lying in N. Then there is a sequence $(m_k)_k$ with $m_k \to \infty$ and there is a full solution u of π_f lying in N such that $u_{m_k}(t) \to u(t)$ in X^α, uniformly for t lying in compact subsets of $\mathbb{R}$.

Corollary 3.9. *Let $f: X^\alpha \to X$ and $f_{n,\tau}$, $n \in \mathbb{N}$, $\tau \in [0,1]$, be as in Proposition 3.8. Let N be bounded and closed in X^α with $S := \mathrm{Inv}_{\pi_f}(N) \subset \mathrm{Int}_{X^\alpha}(N)$. Moreover, let $(M_p)_{p\in P}$ be a $\prec$-ordered Morse decomposition of S, relative to π_f. For each $p \in P$ let $\Xi_p \subset N$ be closed in X^α such that $M_p = \mathrm{Inv}_{\pi_f}(\Xi_p) \subset \mathrm{Int}_{X^\alpha}(\Xi_p)$. For $n \in \mathbb{N}$, $\tau \in [0,1]$ and $p \in P$ define $S_{n,\tau} = \mathrm{Inv}_{\pi_{f_{n,\tau}}}(N)$ and $M_{p,n,\tau} = \mathrm{Inv}_{\pi_{f_{n,\tau}}}(\Xi_p)$. Then there is an $n_0 \in \mathbb{N}$ so that whenever $n \geqslant n_0$ and $\tau \in [0,1]$, then $S_{n,\tau} \subset \mathrm{Int}_{X^\alpha}(N)$, $M_{p,n,\tau} \subset \mathrm{Int}_{X^\alpha}(\Xi_p)$, $p \in P$, and the family $(M_{p,n,\tau})_{p\in P}$ is a Morse decomposition of $S_{n,\tau}$, relative to $\pi_{f_{n,\tau}}$.*

Proof. It is well known (cf. [22]) that N is strongly $\pi_{f_{n,\tau}}$-admissible for all $n \in \mathbb{N}$ and $\tau \in [0,1]$. Now the proof is completed by a contradiction argument, using Proposition 3.8, Remark 3.2 and Theorem 3.1 with assumption (A) replaced by assumption (B). Note that, for sequences $n_m \to \infty$ and τ_m in $[0,1]$ we do *not*, in general, have that $f_{n_m,\tau_m}(u) \to f(u)$ in X locally uniformly in $u \in X^\alpha$, so that we cannot assert that $\pi_{f_{n_m,\tau_m}} \to \pi_f$. This is the reason for having to use the weaker assumption (B). □

Corollary 3.10. *Let n_0 be as in Corollary 3.9. Then for $n \geqslant n_0$ and $\tau \in [0,1]$ the homology index braid of $(\pi_{f_{n,\tau}}, S_{n,\tau}, (M_{p,n,\tau})_{p\in P})$ is isomorphic to the homology index braid of $(\pi_f, S, (M_p)_{p\in P})$.*

Proof. Let $n \geqslant n_0$ be arbitrary. If $\tau_k \to \tau$ in $[0,1]$ then $f_{n,\tau_k}(u) \to f_{n,\tau}(u)$ in X, locally uniformly in $u \in X^\alpha$. This implies, by results in [22], that $\pi_{f_{n,\tau_k}} \to \pi_{f_{n,\tau}}$. Results in [22] also imply that N is $(\pi_{f_{n,\tau_k}})_k$-admissible. Now Corollary 3.9 and Theorem 3.7 complete the proof. □

Given $n \in \mathbb{N}$ and f as in Proposition 3.8 we may consider the local semiflow $\pi'_n = \pi'_{f,n}$ generated on the finite-dimensional space $Y_n := P_n(X^\alpha) = P_n(X)$ by the ordinary differential equation

$$\dot{u} = -Au + P_n f(P_n u), \quad u \in Y_n. \tag{3.2}$$

The local semiflow π'_n is the *n-Galerkin approximation of* π_f.

Moreover, let $\pi''_n = \pi''_{f,n}$ be the semiflow generated on $Z_n := Q_n(X^\alpha)$ by the evolution equation

$$\dot{u} = -Au, \quad u \in Z_n. \tag{3.3}$$

If $f_n := f_{n,1} = P_n \circ f \circ P_n$ then, by Proposition 4.2 in [23] and its proof, the space Y_n is positively invariant relative to the local semiflow π_{f_n} and every bounded π_{f_n}-

invariant set is included in Y_n and is π'_n-invariant. Moreover, every π'_n-invariant set is π_{f_n}-invariant. Setting

$$S_n := S_{n,1} \quad \text{and} \quad M_{p,n} := M_{p,n,1}, \ \ p \in P,$$

we thus see that, whenever $n \geqslant n_0$, then S_n is a compact π'_n-invariant set and $(M_{p,n})_{p\in P}$ is a Morse decomposition of S_n, relative to π'_n. Moreover,

$$M_{\pi_{f_n},S_n}(I) = M_{\pi'_n,S_n}(I) =: M_n(I), \quad I \in \mathcal{I}(\prec).$$

Choose an arbitrary strongly π'_n-admissible index filtration $\mathcal{N}'_n = (N'_n(I))_{I\in\mathcal{A}(\prec)}$ for $(\pi'_n, S_n, (M_{p,n})_{p\in P})$. (Strong π'_n-admissibility means, in this finite-dimensional case, simply that $N'(P)$ is bounded in Y_n.) Let $B = B_n$ be the closed unit ball in Z_n. Since $|u\pi''_n t|_{Z_n} \leqslant e^{-\beta_n t}|u|_{Z_n}$ for some $\beta_n \in]0,\infty[$ and all $u \in Z_n$ and $t \in [0,\infty[$ it follows that, relative to π''_n, B is an isolating block for $\{0\}$ with empty exit set, so in particular, B is positively invariant.

We define $N_n(I) := N'_n(I) + B \cong N'_n(I) \times B$, $I \in \mathcal{A}(\prec)$. It is now a simple exercise to show that $\mathcal{N}_n = (N_n(I))_{I\in\mathcal{A}(\prec)}$ is a strongly π_{f_n}-admissible index filtration for $(\pi_{f_n}, S_n, (M_{p,n})_{p\in P})$. Since $N'_n(I) \subset N_n(I)$ for $I \in \mathcal{A}(\prec)$ there is an inclusion induced morphism $\theta_{\mathcal{N}'_n,\mathcal{N}_n} = (\theta_{\mathcal{N}'_n,\mathcal{N}_n}(J))_{J\in\mathcal{I}(\prec)}$ from the homology index braid $\mathcal{H}'_n$ of $(\pi'_n, S_n, (M_{p,n})_{p\in P})$ to the homology index braid $\mathcal{H}_n$ of $(\pi_{f_n}, S_n, (M_{p,n})_{p\in P})$. We claim that $\theta_{\mathcal{N}'_n,\mathcal{N}_n}$ is an isomorphism. In fact, let $J \in \mathcal{I}(\prec)$ be arbitrary. Choose I, $K \in \mathcal{A}(\prec)$ with $(I, J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Let $\phi: N'_n(K)/N'_n(I) \to N_n(K)/N_n(I)$ be inclusion induced and $\psi: N_n(K)/N_n(I) \to N'_n(K)/N'_n(I)$ be induced by the canonical projection $y + z \mapsto y$ of $X^\alpha = Y_n \oplus Z_n$ onto Y_n. It follows that $\psi \circ \phi$ is the identity on $N'_n(K)/N'_n(I)$ while $\phi \circ \psi$ is homotopic to the identity on $N_n(K)/N_n(I)$ via the homotopy $N_n(K)/N_n(I) \times [0,1] \to N_n(K)/N_n(I)$ induced by the homotopy $X^\alpha \times [0,1] \to X^\alpha$, $(y + z, \tau) \mapsto y + (1-\tau)z$. The homotopy axiom for singular homology now implies that the map

$$\theta_{\mathcal{N}'_n,\mathcal{N}_n}(J): H(\pi'_n, M_n(J)) \to H(\pi_{f_n}, M_n(J))$$

(which is induced by ϕ) is an isomorphism. Using Corollary 3.10 we have now established the following result:

Theorem 3.11. *If n_0 is as in Corollary 3.10, then, for $n \geqslant n_0$, the homology index braids of $(\pi_f, S, (M_p)_{p\in P})$ and $(\pi'_n, S_n, (M_{p,n})_{p\in P})$ are isomorphic so they share the same connection matrices and the same C-connection matrices.*

4. Sequences of index filtrations

The rest of this paper is devoted to the proof of Theorem 3.4. Therefore, for the rest of the paper, assume the hypotheses of Theorem 3.1 and let $\overline{n}$ be as in that theorem. For $I \in \mathcal{I}(\prec)$ and $n \in \mathbb{N}_0$ let $M_n(I) := M_{\pi_n,S_n}(I)$.

If $S_0 = \emptyset$, then, by Proposition 3.3, $S_n = \emptyset$ for all n large enough, so we may choose $N_n(I) := \widetilde{N}_n(I) := \emptyset$, $I \in \mathcal{A}(\prec)$, $n \in \mathbb{N}_0$. Hence Theorem 3.4 holds in this case.

Therefore we may assume that $S_0 \neq \emptyset$. Consequently, using Proposition 3.3 and taking the sets $\widetilde{N}$ and Ξ_p, $p \in P$, smaller and the number $\overline{n}$ larger, if necessary, we may assume from now on that $\widetilde{N}$ is an isolating block relative to π_0. (cf. [20] or [22].) Let $\widetilde{U} = \mathrm{Int}(\widetilde{N})$.

In this section, starting with sequences of FM-index triples satisfying certain inclusion conditions, we will construct index filtrations with an asymptotic nesting property. This will be the crucial abstract step in the proof of Theorem 3.4.

Theorem 4.1. *Let N_1^i, N_2^i, N_I^i, $i = 2, 4$, $I \in \mathcal{A}(\prec)$ be sets such that, for each $I \in \mathcal{A}(\prec)$, (N_1^i, N_I^i, N_2^i) is an FM-index triple for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$, $i = 2, 4$. Moreover let $n_0 \geqslant \overline{n}$ and for each $n \geqslant n_0$, let $N_{1,n}$, $N_{2,n}$, $N_{I,n}$, $I \in \mathcal{A}(\prec)$, be sets such that, for each $I \in \mathcal{A}(\prec)$, $(N_{1,n}, N_{I,n}, N_{2,n})$ is an FM-index triple for $(\pi_n, S_n, M_n(I), M_n(P \setminus I))$. For each $p \in P$ and $n \geqslant n_0$ define the following sets*:

$$D_{p,n} := \Big(\bigcap_{\substack{I \\ p \in I}} \mathrm{Int}(N_{I,n} \setminus N_{2,n})\Big) \cap \Big(\bigcap_{\substack{I \\ p \notin I}} \mathrm{Int}(N_{1,n} \setminus N_{I,n})\Big),$$

$$D_{p,0} := \Big(\bigcap_{\substack{I \\ p \in I}} \mathrm{Int}(N_I^2 \setminus N_2^4)\Big) \cap \Big(\bigcap_{\substack{I \\ p \notin I}} \mathrm{Int}(N_1^2 \setminus N_I^4)\Big),$$

$$E_{p,n} := \{x \in N_{1,n} \mid \text{there is a } t \geqslant 0 \text{ such that } x\pi_n[0,t] \subset N_{1,n} \text{ and } x\pi_n t \in D_{p,n}\}$$

and

$$E_{p,0} := \{x \in N_1^2 \mid \text{there exists a } t \geqslant 0 \text{ such that } x\pi_0[0,t] \subset N_1^2 \text{ and } x\pi_0 t \in D_{p,0}\}.$$

Suppose that there are open sets V_1 and $V_{I,i}$, $i = 3, 4$, $I \in \mathcal{A}(\prec)$, such that for all $n \geqslant n_0$ and $I \in \mathcal{A}(\prec)$ the following inclusions hold:

$$N_1^2 \subset \widetilde{U}, \quad N_1^2 \subset V_1 \subset N_{1,n} \subset N_1^4, \quad N_2^2 \subset N_2^4, \quad N_I^2 \subset N_I^4$$

$$\mathrm{Cl}(N_I^2 \setminus N_2^4) \subset V_{I,3} \subset \mathrm{Int}(N_{I,n} \setminus N_{2,n})$$

$$\mathrm{Cl}(N_1^2 \setminus N_I^4) \subset V_{I,4} \subset \mathrm{Int}(N_{1,n} \setminus N_{I,n}).$$

For each $I \in \mathcal{A}(\prec)$ define

$$N_0(I) := N_1^2 \setminus \bigcup_{p \in P \setminus I} E_{p,0},$$

$$N_n(I) := N_{1,n} \setminus \bigcup_{p \in P \setminus I} E_{p,n}, \quad n \geqslant n_0.$$

Then $(N_n(I))_{I \in \mathcal{A}(\prec)}$ *is an index filtration for* $(\pi_n, S_n, (M_{p,n})_{p \in P})$, *for all* $n \in \mathbb{N}_0$ *with* $n = 0$ *or* $n \geqslant n_0$. *Moreover,*

$$N_2^2 \subset N_0(I), \;\; N_I^2 \subset N_0(\emptyset), \;\; I \in \mathcal{A}(\prec) \tag{4.1}$$

and

$$N_{2,n} \subset N_n(\emptyset) \text{ and } N_{I,n} \subset N_n(I), \;\; n \geqslant n_0, \;\; I \in \mathcal{A}(\prec). \tag{4.2}$$

Furthermore, whenever $I \in \mathcal{A}(\prec)$, $(n_m)_m$ *is a sequence such that* $n_m \to \infty$ *as* $m \to \infty$ *and* $(x_m)_m$ *is a sequence in* N_1^2 *such that* $x_m \in N_{n_m}(I)$ *for all* $m \in \mathbb{N}$ *and* $(x_m)_m$ *is convergent* (*in* X), *then there exists an* $m_0 \in \mathbb{N}$ *such that* $x_m \in N_0(I)$ *for all* $m \geqslant m_0$.

Proof. Theorem 2.10 immediately implies that $(N_0(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_0, S_0, (M_{p,0})_{p \in P})$ satisfying (4.1). Moreover, the same theorem with $N_j^i = N_{j,n}$ and $N_I^i = N_{I,n}$ for $i = 2, 4$ and $j = 1, 2$ implies that for all $n \geqslant n_0$, $(N_n(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_n, S_n, (M_{p,n})_{p \in P})$ satisfying (4.2).

Suppose the second part of the theorem does not hold. Then there exist a $J \in \mathcal{A}(\prec)$, a sequence $(n_m)_m$ with $n_m \to \infty$ as $m \to \infty$ and a sequence $(x_m)_m$ in N_1^2 such that $x_m \to x$ as $m \to \infty$, for some $x \in X$, and $x_m \in N_{n_m}(J) \setminus N_0(J)$ for all $m \in \mathbb{N}$. The definition of the sets $N_n(J)$ and $N_0(J)$ imply that for all $m \in \mathbb{N}$,

$$x_m \in N_{1,n_m} \setminus \bigcup_{p \in P \setminus J} E_{p,n_m} \text{ and } x_m \notin N_1^2 \setminus \bigcup_{p \in P \setminus J} E_{p,0}.$$

Since $x_m \in N_1^2$ for all $m \in \mathbb{N}$, it follows that for all $m \in \mathbb{N}$, $x_m \in \bigcup_{p \in P \setminus J} E_{p,0}$. Thus, taking further subsequences if necessary, we may assume that there exists a $q \in P \setminus J$ such that for all $m \in \mathbb{N}$, $x_m \in E_{q,0}$. So, for each $m \in \mathbb{N}$, there exists a $t_m \geqslant 0$ such that $x_m \pi_0 [0, t_m] \subset N_1^2$ and $x_m \pi_0 t_m \in D_{q,0}$.

We claim that for all $p \in P$, there exists an open set $\widetilde{V}_p$ such that for all $n \geqslant n_0$, $\mathrm{Cl}(D_{p,0}) \subset \widetilde{V}_p \subset D_{p,n}$. In fact, fix $p \in P$ and $n \geqslant n_0$. It follows that

$$\begin{aligned}
\mathrm{Cl}(D_{p,0}) &\subset \Bigg(\bigcap_{\substack{I \\ p \in I}} \mathrm{Cl}(N_I^2 \setminus N_2^4) \Bigg) \cap \Bigg(\bigcap_{\substack{I \\ p \notin I}} \mathrm{Cl}(N_1^2 \setminus N_I^4) \Bigg) \\
&\subset \widetilde{V}_p := \Bigg(\bigcap_{\substack{I \\ p \in I}} V_{I,3} \Bigg) \cap \Bigg(\bigcap_{\substack{I \\ p \notin I}} V_{I,4} \Bigg) \\
&\subset \Bigg(\bigcap_{\substack{I \\ p \in I}} \mathrm{Int}(N_{I,n} \setminus N_{2,n}) \Bigg) \cap \Bigg(\bigcap_{\substack{I \\ p \notin I}} \mathrm{Int}(N_{1,n} \setminus N_{I,n}) \Bigg) = D_{p,n}.
\end{aligned}$$

To complete the proof we will consider two cases.

Case 1: Suppose that $(t_m)_m$ is a bounded sequence. We can assume, taking subsequences if necessary, that there exists a $t \in [0, \infty[$ such that $t_m \to t$ as $m \to \infty$. Since $N_1^2 \subset \widetilde{U} \subset \widetilde{N}$, it follows that π_0 does not explode in N_1^2 and so $x\pi_0[0,t] \subset N_1^2 \subset V_1$. Recall that $\pi_n \to \pi_0$ as $n \to \infty$ and $n_m \to \infty$ as $m \to \infty$. Hence, there exists an $m_0 \in \mathbb{N}$ such that $x_m\pi_{n_m}[0, t_m] \subset V_1 \subset N_{1,n_m}$ for all $m \geqslant m_0$.

Moreover, we have $x\pi_0 t \in \mathrm{Cl}(D_{q,0}) \subset \widetilde{V}_q$ and so there exists an $m_1 \geqslant m_0$ such that $x_m\pi_{n_m}t_m \in \widetilde{V}_q \subset D_{q,n_m}$ for all $m \geqslant m_1$. Therefore, $x_m \in E_{q,n_m}$ for all $m \geqslant m_1$. Since $q \in P \setminus J$, we have a contradiction to our choice of the sequence $(x_m)_m$.

Case 2: Suppose that $(t_m)_m$ is an unbounded sequence. We can assume, taking subsequences if necessary, that $t_m \to \infty$ as $m \to \infty$. Hence $x\pi_0\,[0,\infty[\subset N_1^2$.

Note that if $m \in \mathbb{N}$ and $0 \leqslant t \leqslant t_m$, since $x_m\pi_0[0,t_m] \subset N_1^2$ and $x_m\pi_0 t_m \in D_{q,0}$, it follows that $(x_m\pi_0 t)\pi_0[0, t_m - t] \subset N_1^2$ and $(x_m\pi_0 t)\pi_0(t_m - t) \in D_{q,0}$. In other words, $x_m\pi_0 t \in E_{q,0}$. Thus, for every $t \in [0,\infty[$ and for all $m \geqslant m_t$, for some $m_t \in \mathbb{N}$, we have $x_m\pi_0 t \in E_{q,0}$. Since $E_{q,0} \subset X \setminus N_0(J)$, we conclude that

$$x\pi_0 t \in \mathrm{Cl}(E_{q,0}) \subset X \setminus \mathrm{Int}(N_0(J)) \quad \text{for all } t \geqslant 0. \tag{4.3}$$

Let $(s_k)_k$ be a sequence of positive numbers such that $s_k \to \infty$ as $k \to \infty$. By admissibility, there exist a subsequence of $(x\pi_0 s_k)_k$ which will be denoted again by $(x\pi_0 s_k)_k$ and a $y \in S_0$ such that $x\pi_0 s_k \to y$ as $k \to \infty$. Formula (4.3) implies that $y\pi_0\,[0,\infty[\subset X \setminus \mathrm{Int}(N_0(J))$. The properties of Morse decompositions imply that the ω-limit set $\omega(y)$ of y relative to π_0 is included in $M_{r,0}$, for some $r \in P$. Since $M_{r,0} \subset M_0(J) \subset \mathrm{Int}(N_0(J))$ for all $r \in J$, it follows that $r \in P \setminus J$. Since $M_{r,0} \subset D_{r,0}$ and $D_{r,0}$ is an open set, we have that there exists a $t \geqslant 0$ such that $y\pi_0 t \in D_{r,0}$ and so, for some $k \in \mathbb{N}$, $x\pi_0(s_k + t) \in D_{r,0}$. Hence

$$x_m\pi_{n_m}s \in D_{r,0} \subset D_{r,n_m} \text{ for all } m \text{ large enough,} \tag{4.4}$$

where $s := s_k + t$. Moreover, $x\pi_0[0,s] \subset N_1^2 \subset V_1$ and so $x_m\pi_{n_m}[0,s] \subset V_1 \subset N_{1,n_m}$ for all m large enough. This, together with (4.4), shows that $x_m \in E_{r,n_m}$ for all m large enough, a contradiction as $r \in P \setminus J$. The theorem is proved. □

5. Index triple constructions

In this section we will prove the existence of FM-index triples (relative to the approximating semiflows π_n) with special properties. We use some arguments from the proof of existence of isolating blocks from [20] (or [22]). Define the function $F: X \to [0,1]$ by $F(x) := \min\{1, \mathrm{d}(x, \mathrm{Inv}^-_{\pi_0}(\widetilde{N}))\}$. Furthermore, let

$$s_n^+ := \rho_{\widetilde{N},\pi_n}, \quad t_n^+ := \rho_{\widetilde{U},\pi_n}, \quad n \in \mathbb{N}_0$$

and define the function $g^-: \widetilde{N} \to \mathbb{R}$ by

$$g^-(x) := \sup\{\, \alpha(t)F(x\pi_0 t) \mid \quad t \in [0, s_0^+(x)], \text{ if } s_0^+(x) < \infty \text{ and } t \in [0,\infty[, \text{ if } s_0^+(x) = \infty \,\},$$

where $\alpha\colon [0,\infty[\ \to [1,2[$ is a monotone increasing C^∞-diffeomorphism. Given $\delta > 0$ and $b > 0$, define

$$B_{\delta,b} := \mathrm{Cl}\{\, x \in \widetilde{U} \mid g^-(x) < \delta \text{ and } t_0^+(x) > b \,\}.$$

The sets $B_{\delta,b}$ enjoy the following property.

Lemma 5.1. *There exist δ_0, $\bar{b} \in\ ]0,\infty[$ such that, for all $\delta \in\ ]0,\delta_0]$ and $b \in [\bar{b},\infty[$, $B_{\delta,b} \subset \widetilde{U}$ and $B_{\delta,b}$ is an isolating block for S_0 relative to π_0 with exit set $B^-_{\delta,b} = \{\, x \in \partial B_{\delta,b} \mid g^-(x) \leqslant \delta$ and $t_0^+(x) = b \,\}$.*

Proof. Suppose there exist sequences $(\delta_n)_n$ and $(b_n)_n$ of positive numbers and $(x_n)_n$ such that $\delta_n \to 0$, $b_n \to \infty$ and $x_n \in B_{\delta_n,b_n} \cap (X \setminus \widetilde{U})$ for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists a $y_n \in \widetilde{U}$ with $g^-(y_n) < \delta_n$ and $t_0^+(y_n) > b_n$ such that $\mathrm{d}(x_n, y_n) < 2^{-n}$. By admissibility, we may assume, taking subsequences if necessary, that there exists an $x_0 \in S_0$ such that $y_n \to x_0$ and so $x_n \to x_0$. Therefore, $x_0 \in S_0 \cap (X \setminus \widetilde{U})$ which is a contradiction to the definition of the open set $\widetilde{U}$. Thus, there exist a $\delta_0 \in\ ]0,\infty[$ and a $\bar{b} \in\ ]0,\infty[$ such that $B_{\delta,b} \subset \widetilde{U}$ for all $\delta \in\ ]0,\delta_0]$ and $b \in [\bar{b},\infty[$. Moreover, $\mathrm{Inv}_{\pi_0}(B_{\delta,b}) \subset \mathrm{Inv}_{\pi_0}(\mathrm{Cl}(\widetilde{U})) \subset \mathrm{Inv}_{\pi_0}(\widetilde{N}) = S_0$. On the other hand, if $x \in S_0$, then $x \in \widetilde{U}$, $g^-(x) = 0$ and $t_0^+(x) = \infty$ so $x \in B_{\delta,b}$. Thus $S_0 \subset \mathrm{Inv}_{\pi_0}(B_{\delta,b})$. Therefore, $S_0 = \mathrm{Inv}_{\pi_0}(B_{\delta,b}) \subset \{\, x \in \widetilde{U} \mid g^-(x) < \delta \text{ and } t_0^+(x) > b \,\} \subset \mathrm{Int}(B_{\delta,b})$. The remaining assertions follow immediately since the functions g^- and t_0^+ decrease along solutions of π_0 and g^- is upper-semicontinuous while t_0^+ is lower-semicontinuous. □

Given δ, $b \in\ ]0,\infty]$ and $G \subset X$ define the sets

$$B_{1,\delta,b,G} := B_{\delta,b} \cap G, \quad B_{2,\delta,b,G} := \mathrm{Cl}(B_{\delta,b} \setminus G). \tag{5.1}$$

Given any $I \in \mathcal{A}(\prec)$ with $M_0(P \setminus I) \neq \emptyset$ let $U_{P\setminus I} \subset \widetilde{N}$ be an arbitrary open neighborhood of $M_0(P \setminus I)$ with $\mathrm{Cl}(U_{P\setminus I}) \cap M_0(I) = \emptyset$. Let $g^+_{P\setminus I}\colon U_{P\setminus I} \to [0,\infty[$ be the map given by

$$g^+_{P\setminus I}(x) := \inf\{\, (1+t)^{-1} G(x\pi_0 t) \mid 0 \leqslant t < \rho_{U_{P\setminus I},\pi_0}(x) \,\},$$

where $G(x) := \mathrm{d}(x, M_0(P \setminus I))/(\mathrm{d}(x, M_0(P \setminus I)) + \mathrm{d}(x, X \setminus \mathrm{Cl}(U_{P\setminus I})))$, $x \in U_{P\setminus I}$.

Choose open sets $V_{P\setminus I}$ and $W_{P\setminus I}$ such that $M_0(P\setminus I) \subset V_{P\setminus I} \subset \mathrm{Cl}(V_{P\setminus I}) \subset W_{P\setminus I} \subset \mathrm{Cl}(W_{P\setminus I}) \subset U_{P\setminus I}$ and $g^+_{P\setminus I}|_{\mathrm{Cl}(W_{P\setminus I})}$ is continuous. This is possible by Proposition I.5.2 in [22].

Now, for arbitrary $I \in \mathcal{A}(\prec)$ and $\varepsilon \in\ ]0,\infty[$ define

$$G_{P\setminus I,\varepsilon} := \begin{cases} \mathrm{Cl}\{\, y \in V_{P\setminus I} \mid g^+_{P\setminus I}(y) < \varepsilon \,\} & \text{if } M_0(P \setminus I) \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

and set

$$B_{P\setminus I,\delta,b,\varepsilon} := B_{1,\delta,b,G_{P\setminus I,\varepsilon}},\ B_{I,\delta,b,\varepsilon} := B_{2,\delta,b,G_{P\setminus I,\varepsilon}},\ I\in\mathcal{A}(\prec),\ \varepsilon\in\,]0,\infty[\,.$$

The next lemma is fundamental for what follows.

Lemma 5.2. *Let $\delta_0 \in\,]0,\infty[$ and $\bar{b}\in\,]0,\infty[$ as in Lemma 5.1. Then there exist a $\bar{\delta}\in\,]0,\delta_0]$ and an $\bar{\varepsilon}\in\,]0,\infty[$ such that for all $\delta\in\,]0,\bar{\delta}]$, $\varepsilon\in\,]0,\bar{\varepsilon}]$, $b\in[\bar{b},\infty[$ and for all $I\in\mathcal{A}(\prec)$ with $M_0(P\setminus I)\neq\emptyset$,*

$$B_{P\setminus I,\delta,b,\varepsilon} = B_{\delta,b}\cap G_{P\setminus I,\varepsilon}\subset V_{P\setminus I}.$$

Proof. Otherwise, there exist an $I\in\mathcal{A}(\prec)$ with $M_0(P\setminus I)\neq\emptyset$, sequences $(\delta_n)_n$ and $(\varepsilon_n)_n$ converging to zero, a sequence $(b_n)_n$ in $[\bar{b},\infty[$ and $(x_n)_n$ such that

$$x_n\in B_{\delta_n,b_n}\cap G_{P\setminus I,\varepsilon_n}\cap(X\setminus V_{P\setminus I})\quad\text{for all } n\in\mathbb{N}.$$

Since $x_n\in B_{\delta_n,b_n}$, it follows that $g^-(x_n)\leqslant\delta_n\to 0$ as $n\to\infty$. By admissibility, there exist a subsequence of $(x_n)_n$, denoted again by $(x_n)_n$, and an $x_0\in\mathrm{Inv}^-_{\pi_0}(\widetilde{N})$ such that $x_n\to x_0$ as $n\to\infty$. The definition of the set $G_{P\setminus I,\varepsilon_n}$ implies that $x_0\in\mathrm{Cl}(V_{P\setminus I})\subset\mathrm{Cl}(W_{P\setminus I})$. The continuity of $g^+_{P\setminus I}|_{\mathrm{Cl}(W_{P\setminus I})}$ implies that $g^+_{P\setminus I}|_{\mathrm{Cl}(W_{P\setminus I})}(x_0)=0$. It follows from Proposition I.5.2 in [22] that $x_0\in\mathrm{Inv}^+_{\pi_0}(\mathrm{Cl}(U_{P\setminus I}))\subset\mathrm{Inv}^+_{\pi_0}(\widetilde{N})$ and so $x_0\in S_0$. Since $\mathrm{Cl}(U_{P\setminus I})\cap M_0(I)=\emptyset$ and $(M_0(I),M_0(P\setminus I))$ is an attractor–repeller pair in S_0, relative to π_0, we see that the ω-limit set $\omega(x_0)$ of x_0 relative to π_0 is a subset of $M_0(P\setminus I)$. Theorem III.1.4 in [22] implies that $x_0\in M_0(P\setminus I)$ and so $x_0\in M_0(P\setminus I)\cap(X\setminus V_{P\setminus I})$ which contradicts our choice of the open set $V_{P\setminus I}$. The lemma is proved. □

Let $\bar{b}>0$ be as in Lemma 5.1 and $\bar{\delta}>0$ and $\bar{\varepsilon}>0$ be as in Lemma 5.2.

Corollary 5.3. *For all $\delta\in\,]0,\bar{\delta}]$, $\varepsilon\in\,]0,\bar{\varepsilon}]$, $b\in[\bar{b},\infty[$ and for all $I\in\mathcal{A}(\prec)$, the pair $(B_{P\setminus I,\delta,b,\varepsilon},B_{I,\delta,b,\varepsilon})$ is a block pair for $(\pi_0,S_0,M_0(I),M_0(P\setminus I))$.*

Proof. This follows from Lemma 5.2 and the proof of Theorem III.2.4 in [22]. □

We also have the following result.

Lemma 5.4. *Let $I\in\mathcal{A}(\prec)$ and b_2, b_3, ε_2, ε_3, δ and δ_2 be positive numbers such that $\bar{b}\leqslant b_2<b_3$, $\varepsilon_2<\varepsilon_3\leqslant\bar{\varepsilon}$ and $\delta<\delta_2\leqslant\bar{\delta}$. Then $B_{I,\delta,b_3,\varepsilon_3}\subset\mathrm{Int}(B_{I,\delta_2,b_2,\varepsilon_2})$.*

Proof. Let $I\in\mathcal{A}(\prec)$ be arbitrary. If $M_0(P\setminus I)=\emptyset$ then the result is clear. Therefore, let $M_0(P\setminus I)\neq\emptyset$. We claim that $B_{I,\delta,b_3,\varepsilon_3}\subset\mathrm{Int}(B_{\delta_2,b_2})\setminus G_{P\setminus I,\varepsilon_2}$. Let $y\in B_{I,\delta,b_3,\varepsilon_3}$ be arbitrary. Then there exists a sequence $(x_n)_n$ such that $x_n\in B_{\delta,b_3}\setminus G_{P\setminus I,\varepsilon_3}$ for all n and $x_n \to y$. In particular, $y \in \widetilde{U}$ and $g^-(y) \leqslant \delta < \delta_2$ and $t_0^+(y) \geqslant b_3 > b_2$. Thus $y \in \{ x \in \widetilde{U} \mid g^-(y) < \delta_2 \text{ and } t_0^+(y) > b_2 \} \subset \operatorname{Int}(B_{\delta_2,b_2})$. Suppose $y \in G_{P\setminus I,\varepsilon_2}$. It follows from Lemma 5.2 that $y \in B_{\delta_2,b_2} \cap G_{P\setminus I,\varepsilon_2} \subset V_{P\setminus I}$. So $y \in V_{P\setminus I}$ and $g^+_{P\setminus I}(y) \leqslant \varepsilon_2 < \varepsilon_3$. Then for all n large enough, $x_n \in V_{P\setminus I}$ and $g^+_{P\setminus I}(x_n) < \varepsilon_3$. Thus $x_n \in G_{P\setminus I,\varepsilon_3}$ for all n large enough which is a contradiction. This proves the claim and completes the proof. □

We can now state the main result of this section.

Theorem 5.5. *Let $\overline{b}$ be as in Lemma 5.1 and $\overline{\delta}$ and $\overline{\varepsilon}$ be as in Lemma 5.2. Fix positive numbers b, b_2, b_3, ε_2, ε_3 and δ_2 with $\overline{b} < b_2 < b_3 < b$, $\varepsilon_2 < \varepsilon_3 < \overline{\varepsilon}$ and $\delta_2 < \overline{\delta}$. For all $I \in \mathcal{A}(\prec)$ and all $\delta \in \,]0, \delta_2]$, let $\Lambda_{I,\delta}$ be a closed subset of X with*

$$M_0(I) \subset \operatorname{Int}(\Lambda_{I,\delta}) \subset \Lambda_{I,\delta} \subset B_{I,\delta,b_3,\varepsilon_3}$$

and such that whenever $\delta < \delta'$ then $\Lambda_{I,\delta} \subset \Lambda_{I,\delta'}$.

Assume also that whenever $I \in \mathcal{A}(\prec)$ and $(\delta_n)_n$ is a decreasing sequence converging to zero and $x_n \in \Lambda_{I,\delta_n}$ for all $n \in \mathbb{N}$, then the sequence $(x_n)_n$ has a convergent subsequence.

For $\delta \in \,]0, \delta_2]$, $n \in \mathbb{N}$ and $I \in \mathcal{A}(\prec)$ define the sets

$$N_{1,n}(\delta) := B_{\delta_2,b_2} \cap \operatorname{Cl}\{ y \mid \text{there exist an } x \in B_{\delta,b_3} \text{ and a } t \geqslant 0 \text{ such that } x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \},$$

$$N_{2,n}(\delta) := N_{1,n}(\delta) \cap \{ y \in \widetilde{U} \mid t_n^+(y) \leqslant b \}$$

and

$$N_{I,n}(\delta) := \left[B_{I,\delta_2,b_2,\varepsilon_2} \cap \operatorname{Cl}\{ y \mid \text{there exist an } x \in \Lambda_{I,\delta} \text{ and a } t \geqslant 0 \text{ such that } x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \}\right] \cup N_{2,n}(\delta).$$

Under these assumptions there exists a $\delta_3 \in \,]0, \delta_2[$ such that for all $\delta \in \,]0, \delta_3]$, there exists an $n_0(\delta) \in \mathbb{N}$ such that for all $n \geqslant n_0(\delta)$ and for all $I \in \mathcal{A}(\prec)$ the triple $(N_{1,n}(\delta), N_{I,n}(\delta), N_{2,n}(\delta))$ is an FM-index triple for $(\pi_n, S_n, M_n(I), M_n(P \setminus I))$.

Proof. It is clear that the sets $N_{1,n}(\delta)$, $N_{2,n}(\delta)$ and $N_{I,n}(\delta)$ are closed and $N_{2,n}(\delta) \subset N_{I,n}(\delta) \subset N_{1,n}(\delta)$. Moreover, using arguments completely analogous to those contained in the proofs of Lemmas I.12.5 and I.12.6 in [22], we can establish the validity of the following results.

Lemma 5.6. *There exists a* $\delta' \in \,]0, \delta_2]$ *such that for all* $\delta \in \,]0, \delta']$*, there exists an* $n'(\delta) \in \mathbb{N}$ *such that for all* $n \geqslant n'(\delta)$ *the pair* $(N_{1,n}(\delta), N_{2,n}(\delta))$ *is an FM-index pair for* S_n*, relative to* π_n*.*

Lemma 5.7. *If* $u_n \to u_0$ *in* $\widetilde{U}$ *then* $t_n^+(u_n) \to t_0^+(u)$ *as* $n \to \infty$*.*

Now Lemma 5.8 below completes the proof of Theorem 5.5. □

Lemma 5.8. *There exists a* $\delta'' \in \,]0, \delta']$ *such that for all* $\delta \in \,]0, \delta'']$*, there exists an* $n''(\delta) \in \mathbb{N}$*,* $n''(\delta) \geqslant n'(\delta)$*, such that for all* $n \geqslant n''(\delta)$ *and all* $I \in \mathcal{A}(\prec)$ *the pair* $(N_{I,n}(\delta), N_{2,n}(\delta))$ *is an FM-index pair for* $M_n(I)$*, relative to* π_n*.*

Proof. In this proof we will write $B_I := B_{I,\delta_2,b_2,\varepsilon_2}$ for short.

Using standard arguments together with Proposition 3.3 we see that there is a $\delta'' \in \,]0, \delta']$ such that for all $\delta \in \,]0, \delta'']$, there exists an $n''(\delta) \in \mathbb{N}$, $n''(\delta) \geqslant n'(\delta)$, so that for all $n \geqslant n''(\delta)$ and all $I \in \mathcal{A}(\prec)$, $N_{2,n}(\delta)$ is $N_{I,n}(\delta)$-positively invariant, $M_n(I) \subset \operatorname{Int}(N_{I,n}(\delta) \setminus N_{2,n}(\delta))$ and $M_n(I)$ is the largest π_n-invariant set in $\operatorname{Cl}(N_{I,n}(\delta) \setminus N_{2,n}(\delta))$.

Now notice that the set

$$N_{I,n}^c(\delta) := B_I \cap \operatorname{Cl}\{\, y \mid \text{there exist an } x \in \Lambda_{I,\delta} \text{ and a } t \geqslant 0 \text{ such that } x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \,\}$$

is B_I-positively invariant relative to π_n. Thus, if $N_{2,n}(\delta)$ is not an exit ramp for $N_{I,n}(\delta)$ (relative to π_n) then $\partial B_I \cap N_{I,n}(\delta) \not\subset N_{2,n}(\delta)$. Therefore, the proof of the lemma will be complete if we show that there exists a $\delta''' \in \,]0, \delta'']$, such that for all $\delta \in \,]0, \delta''']$, there exists an $n'''(\delta) \geqslant n''(\delta)$ such that for all $n \geqslant n'''(\delta)$ and all $I \in \mathcal{A}(\prec)$, $\partial B_I \cap N_{I,n}(\delta) \subset N_{2,n}(\delta)$. Suppose this is not true. Then there exist an $I \in \mathcal{A}(\prec)$ and a decreasing sequence $(\gamma_m)_m$, with $\gamma_m < \delta_2$ and sequences $(n_m)_m$ and $(y_m)_m$ such that $\gamma_m \to 0$, $n_m \to \infty$ as $m \to \infty$ and

$$y_m \in (\partial B_I \cap N_{I,n_m}(\gamma_m)) \setminus N_{2,n_m}(\gamma_m), \quad m \in \mathbb{N}. \tag{5.2}$$

Hence, $y_m \in N_{1,n_m}(\gamma_m)$ and so for each $m \in \mathbb{N}$, $t_{n_m}^+(y_m) > b$ and there exists a $x_m \in \Lambda_{I,\gamma_m}$ and a $t_m \geqslant 0$ such that $x_m\pi_{n_m}[0,t_m] \subset \widetilde{U}$ and $\mathrm{d}(y_m, x_m\pi_{n_m}t_m) < 1/2^m$. Admissibility and the properties of the set Λ_{I,γ_m} imply that there exist subsequences of $(x_m)_m$ and $(t_m)_m$, denoted again by $(x_m)_m$ and $(t_m)_m$, and x_0, $y_0 \in X$ such that $x_m \to x_0$ and $x_m\pi_{n_m}t_m \to y_0$ as $m \to \infty$. Since $x_m \in \Lambda_{I,\gamma_m} \subset B_{I,\gamma_m,b_3,\varepsilon_3} \subset B_{I,\gamma_1,b_3,\varepsilon_3}$, with $\gamma_1 < \delta_2$, Lemma 5.4 implies

$$x_0 \in B_{I,\gamma_1,b_3,\varepsilon_3} \subset \operatorname{Int}(B_I). \tag{5.3}$$

We claim that $x_m\pi_{n_m}[0,t_m] \subset B_I$ for all $m \in \mathbb{N}$ large enough. If our claim is not true, then, using (5.3) and taking subsequences if necessary, we may assume that for each

$m \in \mathbb{N}$, there exists a $\tau_m \leqslant t_m$ such that

$$x_m \pi_{n_m}[0, \tau_m] \subset B_I \text{ and } x_m \pi_{n_m} \tau_m \in \partial B_I. \tag{5.4}$$

Let $b' \in]b_2, b[$. Since $y_m \to y_0$ as $m \to \infty$ Lemma 5.7 implies $t_0^+(y_0) \geqslant b > b'$. Hence, again by Lemma 5.7, $t_{n_m}^+(x_m \pi_{n_m} t_m) > b'$ for m large enough.

Suppose that $\tau_m \to 0$ as $m \to \infty$. Since $\pi_n \to \pi_0$, it follows that $x_m \pi_{n_m} \tau_m \to x_0$ as $m \to \infty$. Formulas (5.3) and (5.4) imply that $x_0 \in \text{Int}(B_I) \cap \partial B_I$ which is a contradiction. Hence, we may assume, taking subsequence if necessary, that there exists an $r > 0$ such that $\tau_m > r$ for all m large enough.

Define $z_m := x_m \pi_{n_m}(\tau_m - r)$, $m \in \mathbb{N}$. Without loss of generality, we may assume that there exists a $z_0 \in X$ such that $z_m \to z_0$ as $m \to \infty$. Note that, for all $m \in \mathbb{N}$, $z_m \pi_{n_m}[0, r] = x_m \pi_{n_m}[\tau_m - r, \tau_m] \subset B_I$ and so $z_0 \pi_0[0, r] \subset B_I$. Since, $z_m \pi_{n_m} r = x_m \pi_{n_m} \tau_m$ for all $m \in \mathbb{N}$ and $\pi_n \to \pi_0$, it follows from (5.4) that $z_0 \pi_0 r \in \partial B_I$. Since $z_0 \pi_0[0, r] \subset B_I$ and $z_0 \pi_0 r \in \partial B_I$, it follows that $z_0 \pi_0 r \notin B_I^+$, where, as usual, B_I^+ is the entrance set of the isolating block B_I, relative to π_0. Since $(B_I, B_{P \setminus I, \delta_2, b_2, \varepsilon_2})$ is a block pair for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$ and so $B_I \cap B_{P \setminus I, \delta_2, b_2, \varepsilon_2} \subset B_I^+ \cap B^-_{P \setminus I, \delta_2, b_2, \varepsilon_2}$, it follows that $z_0 \pi_0 r \notin B_{P \setminus I, \delta_2, b_2, \varepsilon_2}$. Hence, $z_0 \pi_0 r \in \partial B_I \setminus B_{P \setminus I, \delta_2, b_2, \varepsilon_2} \subset \partial B_{\delta_2, b_2}$. Since $z_0 \pi_0[0, r] \subset B_I \subset B_{\delta_2, b_2}$, it follows that $z_0 \pi_0 r \in B^-_{\delta_2, b_2}$. Lemma 5.1 implies that $t_0^+(z_0 \pi_0 r) = b_2$. On the other hand, since $\tau_m \leqslant t_m$, it follows that $t_{n_m}^+(x_m \pi_{n_m} \tau_m) \geqslant t_{n_m}^+(x_m \pi_{n_m} t_m) > b'$ for all m large enough and so $b_2 = t_0^+(z_0 \pi_0 r) \geqslant b' > b_2$ which is a contradiction. Therefore, our claim holds.

Suppose now that $t_m \to 0$ as $m \to \infty$. Since $\pi_n \to \pi_0$, it follows that $x_m \pi_{n_m} t_m \to x_0$ as $m \to \infty$ and so $y_m \to x_0$ as $m \to \infty$. Formulas (5.2) and (5.3) imply that $x_0 \in \text{Int}(B_I) \cap \partial B_I$ which is a contradiction. Hence, we may assume that, taking subsequence if necessary, there exists an $s > 0$ such that $t_m > s$ for all m large enough.

Define $w_m := x_m \pi_{n_m}(t_m - s)$, $m \in \mathbb{N}$. Taking subsequence if necessary, we may assume that there exists a $w_0 \in X$ such that $w_m \to w_0$ as $m \to \infty$. Note that, for all $m \in \mathbb{N}$, $w_m \pi_{n_m}[0, s] = x_m \pi_{n_m}[t_m - s, t_m] \subset B_I$ and so $w_0 \pi_0[0, s] \subset B_I$. Since $w_m \pi_{n_m} s \equiv x_m \pi_{n_m} t_m$ it follows that $w_0 \pi_0 s \in \partial B_I$. Thus $w_0 \pi_0 s \notin B_I^+$ and so $w_0 \pi_0 s \in \partial B_I \setminus B_{P \setminus I, \delta_2, b_2, \varepsilon_2} \subset \partial B_{\delta_2, b_2}$. Since $w_0 \pi_0[0, s] \subset B_I \subset B_{\delta_2, b_2}$, it follows that $w_0 \pi_0 s \in B^-_{\delta_2, b_2}$. Lemma 5.1 implies that $t_0^+(w_0 \pi_0 s) = b_2$. On the other hand, since $t_{n_m}^+(x_m \pi_{n_m} t_m) > b'$ for all m large enough, we have $b_2 = t_0^+(w_0 \pi_0 s) \geqslant b' > b_2$ which is a contradiction. The proof of the lemma is complete. □

We conclude this section with the following result.

Theorem 5.9. *Let the numbers $\bar{b}$, b, b_2, b_3, $\bar{\varepsilon}$, ε_2, ε_3, $\bar{\delta}$, δ_2 and the sets $\Lambda_{I,\delta}$ be as in the hypotheses of Theorem 5.5. Let $\delta_3 \in]0, \delta_2]$ be as in the conclusion of that theorem. Let the numbers b_1', b_2' and b_1'' be such that $b_3 < b_1' < b_2' < b < b_1''$. For each $\delta \in]0, \delta_3]$, let $n_0(\delta) \in \mathbb{N}$ such that the conclusions of Theorem 5.5 hold for all $n \geqslant n_0(\delta)$. Then there exists a $\delta_4 \in]0, \delta_3]$ such that for all $\delta \in]0, \delta_4]$, there exists an $n_1(\delta) \geqslant n_0(\delta)$ such that for all $n \geqslant n_1(\delta)$, the following* *inclusions hold*:

$$B_{\delta,b_3}\cap\{\,y\in\widetilde{U}\mid t_0^+(y)\leqslant b_2'\,\}\subset B_{\delta,b_3}\cap\{\,y\in\widetilde{U}\mid t_n^+(y)\leqslant b\,\}$$

and

$$N_{2,n}(\delta)\subset B_{\delta_2,b_2}\cap\{\,y\in\widetilde{U}\mid t_0^+(y)\leqslant b_1''\,\}.$$

Proof. Suppose the conclusions of the theorem do not hold. Then there exist a decreasing sequence $(\gamma_m)_m$ converging to zero, a sequence $(n_m)_m$ in $\mathbb{N}$ with $n_m\to\infty$ and a sequence $(y_m)_m$ such that

$$y_m\in B_{\gamma_m,b_3},\ \ t_0^+(y_m)\leqslant b_2' \text{ and } t_{n_m}^+(y_m)>b \quad \text{for all } m\in\mathbb{N} \tag{5.5}$$

or

$$y_m\in N_{2,n_m}(\gamma_m) \text{ and } t_0^+(y_m)>b_1'' \quad \text{for all } m\in\mathbb{N}. \tag{5.6}$$

By admissibility there exists a subsequence of $(y_m)_m$ which we denote again by $(y_m)_m$ and there is a $y_0\in\widetilde{N}$ such that $y_m\to y_0$ as $m\to\infty$. If (5.5) holds, then, for all $m\in\mathbb{N}$ large enough, $y_m\in B_{\gamma_m,b_3}\subset B_{\delta_3,b_3}\subset B_{\delta_2,b_2}$ and so $y_0\in B_{\delta_2,b_2}\subset\widetilde{U}$. If (5.6) holds, then, for all $m\in\mathbb{N}$, $y_m\in B_{\delta_2,b_2}$ and so again $y_0\in\widetilde{U}$. In both cases Lemma 5.7 and the continuity of t_0^+ imply that $t_{n_m}^+(y_m)\to t_0^+(y_0)$ and $t_0^+(y_m)\to t_0^+(y_0)$ as $m\to\infty$.

Suppose (5.5) holds. Then we get $t_0^+(y_0)\leqslant b_2'<b\leqslant t_0^+(y_0)$ which is a contradiction. If (5.6) holds, the definition of the $N_{2,n_m}(\gamma_m)$ implies that $t_{n_m}^+(y_m)\leqslant b$ for all $m\in\mathbb{N}$ and so $t_0^+(y_0)\leqslant b<b_1''\leqslant t_0^+(y_0)$ which is a contradiction again. The theorem is proved. □

6. The proof of the nested index filtration theorem

Let $\overline{b}$ be as in Lemma 5.1 and $\overline{\delta}$ and $\overline{\varepsilon}$ be as in Lemma 5.2. Fix real numbers ε_i, b_i, $i=1,\dots,5$, δ_i, $i=1,2$, b, b_i', b_i'', $i=1,2$ such that

$$0<\varepsilon_1<\varepsilon_2<\varepsilon_3<\varepsilon_4<\varepsilon_5<\overline{\varepsilon},\ 0<\delta_2<\delta_1<\overline{\delta} \tag{6.1}$$

and

$$\overline{b}<b_1<b_2<b_3<b_4<b_5<b_1'<b_2'<b<b_1''<b_2''. \tag{6.2}$$

For all $\delta\in\,]0,\delta_2]$ and $I\in\mathcal{A}(\prec)$ define $\Lambda_{I,\delta}:=B_{I,\delta,b_3,\varepsilon_3}$.

It is immediately checked that the family $\Lambda_{I,\delta}$ of sets satisfies the hypotheses of Theorem 5.5.

For $\delta \in\,]0, \delta_2]$, $n \in \mathbb{N}$ and $I \in \mathcal{A}(\prec)$ define the sets $N_{1,n}(\delta)$, $N_{I,n}(\delta)$ and $N_{2,n}(\delta)$ as in Theorem 5.5 with respect to this choice of the set $\Lambda_{I,\delta}$. More explicitly,

$$\begin{aligned} N_{1,n}(\delta) &:= B_{\delta_2,b_2} \cap \mathrm{Cl}\{\, y \mid \text{ there exist an } x \in B_{\delta,b_3} \text{ and a } t \geqslant 0 \text{ such that} \\ &\qquad\qquad x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \,\}, \\ N_{2,n}(\delta) &:= N_{1,n}(\delta) \cap \{\, y \in \widetilde{U} \mid t_n^+(y) \leqslant b \,\}, \\ N_{I,n}(\delta) &:= \big[B_{I,\delta_2,b_2,\varepsilon_2} \cap \mathrm{Cl}\{\, y \mid \text{ there exist an } x \in B_{I,\delta,b_3,\varepsilon_3} \text{ and a } t \geqslant 0 \text{ such that} \\ &\qquad\qquad x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \,\}\big] \cup N_{2,n}(\delta). \end{aligned}$$

An application of Theorem 5.5 shows that there is a $\delta_3 \in\,]0, \delta_2[$ such that for every $\delta \in\,]0, \delta_3]$, there exists an $n_0(\delta) \in \mathbb{N}$ such that for all $n \geqslant n_0(\delta)$ and for all $I \in \mathcal{A}(\prec)$ the triple $(N_{1,n}(\delta), N_{I,n}(\delta), N_{2,n}(\delta))$ is an FM-index triple for $(\pi_n, S_n, M_n(I), M_n(P \setminus I))$. Let $\delta_4 \in \big[0, \delta_3\big[$ be as in Theorem 5.9 and $n_2 := n_1(\delta_4) \in \mathbb{N}$ as in the conclusions of that theorem (again for the above choice of $\Lambda_{I,\delta}$). Choose δ_5 with

$$\delta_5 \in\,]0, \delta_4[\,. \tag{6.3}$$

For $n \geqslant n_2$ and $I \in \mathcal{A}(\prec)$ define the following sets:

$$\begin{aligned} &N_1^2 := B_{\delta_5,b_4}, \quad N_2^2 := B_{\delta_5,b_4} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) \leqslant b_1' \,\}, \\ &N_1^4 := B_{\delta_1,b_1}, \quad N_2^4 := B_{\delta_1,b_1} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) \leqslant b_2'' \,\}, \\ &N_{1,n} := N_{1,n}(\delta_4), \quad N_{2,n} := N_{2,n}(\delta_4), \\ &N_I^2 := B_{I,\delta_5,b_4,\varepsilon_4} \cup N_2^2, \quad N_I^4 := B_{I,\delta_1,b_1,\varepsilon_1} \cup N_2^4, \quad N_{I,n} := N_{I,n}(\delta_4), \\ &V_1 := \mathrm{Int}(B_{\delta_4,b_3}), \;\; V_{I,3} := \mathrm{Int}(B_{I,\delta_4,b_3,\varepsilon_3}) \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\}, \\ &V_{I,4} := \mathrm{Int}(B_{\delta_4,b_3}) \cap (X \setminus B_{I,\delta_2,b_2,\varepsilon_2}) \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\}. \end{aligned}$$

It is clear that for each $I \in \mathcal{A}(\prec)$ the triple (N_1^i, N_I^i, N_2^i) is an FM-index triple for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$, $i = 2, 4$. Moreover, a lengthy but straightforward check using Theorem 5.9 confirms that for all $n \geqslant n_2$ and $I \in \mathcal{A}(\prec)$ the following inclusions hold:

$$\begin{aligned} &N_1^2 \subset \widetilde{U}, \quad N_1^2 \subset V_1 \subset N_{1,n} \subset N_1^4, \quad N_2^2 \subset N_2^4, \quad N_I^2 \subset N_I^4, \\ &\mathrm{Cl}(N_I^2 \setminus N_2^4) \subset V_{I,3} \subset \mathrm{Int}(N_{I,n} \setminus N_{2,n}), \\ &\mathrm{Cl}(N_1^2 \setminus N_I^4) \subset V_{I,4} \subset \mathrm{Int}(N_{1,n} \setminus N_{I,n}). \end{aligned}$$

This means that all assumptions of Theorem 4.1 are satisfied. That theorem therefore implies the following result.

Theorem 6.1. *With the notation introduced above, there exists an* $n_2 \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}_0$ *with* $n = 0$ *or* $n \geqslant n_2$ *there exists an index filtration* $(N_n(I))_{I \in \mathcal{A}(\prec)}$ *for* $(\pi_n, S_n, (M_{p,n})_{p \in P})$ *such that for all* $I \in \mathcal{A}(\prec)$ *the following inclusions hold*:

(1) $B_{I,\delta_5,b_4,\varepsilon_4} \cup N_2^2 = N_I^2 \subset N_0(I) \subset N_1^2 = B_{\delta_5,b_4}$,
(2) $N_2^2 \subset N_0(\emptyset)$,
(3) $N_{I,n} \subset N_n(I) \subset N_{1,n}$,
(4) $N_{2,n} \subset N_n(\emptyset)$.

Moreover, whenever $I \in \mathcal{A}(\prec)$, $(n_m)_m$ *is a sequence such that* $n_m \to \infty$ *as* $m \to \infty$ *and* $(x_m)_m$ *is a sequence in* N_1^2 *such that* $x_m \in N_{n_m}(I)$ *for all* $m \in \mathbb{N}$ *and* $(x_m)_m$ *is convergent* (*in X*), *then there exists an* $m_0 \in \mathbb{N}$ *such that* $x_m \in N_0(I)$ *for all* $m \geqslant m_0$.

Let n_2 and $(N_0(I))_{I \in \mathcal{A}(\prec)}$ be as in Theorem 6.1. For $I \in \mathcal{A}(\prec)$ we will now define a new family $\widetilde{G}_{P\setminus I,\varepsilon}$, $\varepsilon \in]0,\infty[$, of closed neighborhoods of $M_0(P \setminus I)$ to which the results of the preceding section, in particular Theorems 5.5 and 5.9, can be applied.

Given $I \in \mathcal{A}(\prec)$ with $M_0(P \setminus I) \neq \emptyset$ let

$$\widetilde{U}_{P\setminus I} := \operatorname{Int}(N_0(P) \setminus N_0(I)). \tag{6.4}$$

$\widetilde{U}_{P\setminus I}$ is an open neighborhood of $M_0(P \setminus I)$ with $\operatorname{Cl}(\widetilde{U}_{P\setminus I}) \cap M_0(I) = \emptyset$. Let $\widetilde{g}^+_{P\setminus I}\colon \widetilde{U}_{P\setminus I} \to [0,\infty[$ be the map given by

$$\widetilde{g}^+_{P\setminus I}(x) := \inf\{\, (1+t)^{-1}\widetilde{G}(x\pi_0 t) \mid 0 \leqslant t < \rho_{\widetilde{U}_{P\setminus I},\pi_0}(x)\,\},$$

where $\widetilde{G}(x) := \mathrm{d}(x, M_0(P \setminus I))/(\mathrm{d}(x, M_0(P \setminus I)) + \mathrm{d}(x, X \setminus \operatorname{Cl}(\widetilde{U}_{P\setminus I})))$, $x \in \widetilde{U}_{P\setminus I}$.

Choose open sets $\widetilde{V}_{P\setminus I}$ and $\widetilde{W}_{P\setminus I}$ such that $M_0(P\setminus I) \subset \widetilde{V}_{P\setminus I} \subset \operatorname{Cl}(\widetilde{V}_{P\setminus I}) \subset \widetilde{W}_{P\setminus I} \subset \operatorname{Cl}(\widetilde{W}_{P\setminus I}) \subset \widetilde{U}_{P\setminus I}$ and $g^+_{P\setminus I}|_{\operatorname{Cl}(\widetilde{W}_{P\setminus I})}$ is continuous. This is possible by Proposition I.5.2 in [22].

Now, for arbitrary $I \in \mathcal{A}(\prec)$ and $\varepsilon \in]0,\infty[$ define

$$\widetilde{G}_{P\setminus I,\varepsilon} := \begin{cases} \operatorname{Cl}\{\, y \in \widetilde{V}_{P\setminus I} \mid \widetilde{g}^+_{P\setminus I}(y) < \varepsilon \,\} & \text{if } M_0(P \setminus I) \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

and set

$$\widetilde{B}_{P\setminus I,\delta,b,\varepsilon} := B_{1,\delta,b,\widetilde{G}_{P\setminus I,\varepsilon}}, \; \widetilde{B}_{I,\delta,b,\varepsilon} := B_{2,\delta,b,\widetilde{G}_{P\setminus I,\varepsilon}}, \; I \in \mathcal{A}(\prec), \; \varepsilon \in]0,\infty[\,.$$

(cf. (5.1)).

Lemma 5.2, Corollary 5.3 and Lemma 5.4 imply that there exist a $\widetilde{\delta}_0 \in]0,\delta_5[$ (with δ_5 as in (6.3)) and an $\widetilde{\varepsilon}_0 \in]0,\overline{\varepsilon}[$ such that for all $\widetilde{\delta} \in]0,\widetilde{\delta}_0]$, $\widetilde{\varepsilon} \in]0,\widetilde{\varepsilon}_0]$,

$b \in [\overline{b}, \infty[$ and for all $I \in \mathcal{A}(\prec)$, the pair $(\widetilde{B}_{P\setminus I,\delta,b,\varepsilon}, \widetilde{B}_{I,\delta,b,\varepsilon})$ is a block pair for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$. Moreover, if $0 < \widetilde{\varepsilon}_3 < \widetilde{\varepsilon}_2 \leqslant \widetilde{\varepsilon}_0$, $0 < \widetilde{\delta}_3 < \widetilde{\delta}_2 \leqslant \widetilde{\delta}_0$ and $\widetilde{b}_2 < \widetilde{b}_3 \leqslant \overline{b}$ then $\widetilde{B}_{I,\widetilde{\delta}_3,\widetilde{b}_3,\widetilde{\varepsilon}_3} \subset \mathrm{Int}(\widetilde{B}_{I,\widetilde{\delta}_2,\widetilde{b}_2,\widetilde{\varepsilon}_2})$. Formula (6.4) implies that

$$N_0(I) \cap B_{\widetilde{\delta},b} \subset \widetilde{B}_{I,\widetilde{\delta},b,\widetilde{\varepsilon}}, \quad \widetilde{\delta} \in \left]0, \widetilde{\delta}_0\right], \, \widetilde{\varepsilon} \in \left]0, \widetilde{\varepsilon}_0\right], \, b \in \left[\overline{b}, \infty\right[, \, I \in \mathcal{A}(\prec). \tag{6.5}$$

Therefore, the following lemma is obvious:

Lemma 6.2. *Let $\widetilde{\varepsilon}_3 \in]0, \widetilde{\varepsilon}_0[$. Then for every $\widetilde{\delta} \in]0, \widetilde{\delta}_0]$ and $I \in \mathcal{A}(\prec)$ the set $\Lambda_{I,\widetilde{\delta}} := N_0(I) \cap B_{\widetilde{\delta},b_3}$ is a closed subset of X and*

$$M_0(I) \subset \mathrm{Int}(\Lambda_{I,\widetilde{\delta}}) \subset \Lambda_{I,\widetilde{\delta}} \subset \widetilde{B}_{I,\widetilde{\delta},b_3,\widetilde{\varepsilon}_3}.$$

Whenever $\widetilde{\delta} < \widetilde{\delta}'$, then $\Lambda_{I,\widetilde{\delta}} \subset \Lambda_{I,\widetilde{\delta}'}$. Furthermore, whenever $(\widetilde{\delta}_n)_n$ is a decreasing sequence converging to zero and $x_n \in \Lambda_{I,\widetilde{\delta}_n}$ for all $n \in \mathbb{N}$, then the sequence $(x_n)_n$ has a convergent subsequence.

For $\widetilde{\delta} \in]0, \widetilde{\delta}_2]$, $n \in \mathbb{N}$ and $I \in \mathcal{A}(\prec)$ define

$$\widetilde{N}_{1,n}(\widetilde{\delta}) := B_{\widetilde{\delta}_2,b_2} \cap \mathrm{Cl}\{\, y \mid \text{ there exist an } x \in B_{\widetilde{\delta},b_3} \text{ and a } t \geqslant 0 \text{ such that } x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \,\},$$

$$\widetilde{N}_{2,n}(\widetilde{\delta}) := \widetilde{N}_{1,n}(\widetilde{\delta}) \cap \{\, y \in \widetilde{U} \mid t_n^+(y) \leqslant b \,\},$$

$$\widetilde{N}_{I,n}(\widetilde{\delta}) := \Big[\widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\varepsilon}_2} \cap \mathrm{Cl}\{\, y \mid \text{ there exist an } x \in N_0(I) \cap B_{\widetilde{\delta},b_3} \text{ and a } t \geqslant 0 \text{ such that } x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \,\}\Big] \cup \widetilde{N}_{2,n}(\widetilde{\delta}).$$

An application of Theorem 5.5 shows that there is a $\widetilde{\delta}_3 \in]0, \widetilde{\delta}_2[$ such that for every $\widetilde{\delta} \in]0, \widetilde{\delta}_3]$, there exists an $\widetilde{n}_0(\widetilde{\delta}) \in \mathbb{N}$ such that for all $n \geqslant \widetilde{n}_0(\widetilde{\delta})$ and for all $I \in \mathcal{A}(\prec)$ the triple $(\widetilde{N}_{1,n}(\widetilde{\delta}), \widetilde{N}_{I,n}(\widetilde{\delta}), \widetilde{N}_{2,n}(\widetilde{\delta}))$ is an FM-index triple for $(\pi_n, S_n, M_n(I), M_n(P \setminus I))$. By Theorem 5.9 there exists a $\widetilde{\delta}_4 \in]0, \widetilde{\delta}_3]$ such that for all $\widetilde{\delta} \in]0, \widetilde{\delta}_4]$, there exists an $\widetilde{n}_1(\widetilde{\delta}) \geqslant \widetilde{n}_0(\widetilde{\delta})$ such that for all $n \geqslant \widetilde{n}_1(\widetilde{\delta})$, the following inclusions hold:

$$B_{\widetilde{\delta},b_3} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) \leqslant b_2' \,\} \subset B_{\widetilde{\delta},b_3} \cap \{\, y \in \widetilde{U} \mid t_n^+(y) \leqslant b \,\} \tag{6.6}$$

and

$$\widetilde{N}_{2,n}(\widetilde{\delta}) \subset B_{\widetilde{\delta}_2,b_2} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) \leqslant b_1'' \,\}. \tag{6.7}$$

Fix positive numbers $\widetilde{\varepsilon}_i$, $i = 1,\dots,4$, $\widetilde{\delta}_1$ and $\widetilde{\delta}_5$ such that

$$0 < \widetilde{\varepsilon}_1 < \widetilde{\varepsilon}_2 < \widetilde{\varepsilon}_3 < \widetilde{\varepsilon}_4 < \widetilde{\varepsilon}_0 \text{ and } 0 < \widetilde{\delta}_5 < \widetilde{\delta}_4 < \widetilde{\delta}_3 < \widetilde{\delta}_2 < \widetilde{\delta}_1 < \widetilde{\delta}_0. \tag{6.8}$$

For $n \geqslant n_3 := \max\{n_2, \widetilde{n}_1(\widetilde{\delta}_4)\}$ Define the following sets:

$$\widetilde{N}_1^2 := B_{\widetilde{\delta}_5, b_5}, \quad \widetilde{N}_2^2 := B_{\widetilde{\delta}_5, b_5} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) \leqslant b_1' \,\},$$
$$\widetilde{N}_1^4 := B_{\widetilde{\delta}_1, b_1}, \quad \widetilde{N}_2^4 := B_{\widetilde{\delta}_1, b_1} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) \leqslant b_2'' \,\},$$
$$\widetilde{N}_{1,n} := \widetilde{N}_{1,n}(\widetilde{\delta}_4), \quad \widetilde{N}_{2,n} := \widetilde{N}_{2,n}(\widetilde{\delta}_4),$$
$$\widetilde{N}_I^2 := B_{I,\widetilde{\delta}_5,b_5,\varepsilon_5} \cup \widetilde{N}_2^2, \quad \widetilde{N}_I^4 := \widetilde{B}_{I,\widetilde{\delta}_1,b_1,\widetilde{\varepsilon}_1} \cup \widetilde{N}_2^4, \quad \widetilde{N}_{I,n} := \widetilde{N}_{I,n}(\widetilde{\delta}_4),$$
$$\widetilde{V}_1 := \operatorname{Int}(B_{\widetilde{\delta}_4, b_3}), \;\; \widetilde{V}_{I,3} := \operatorname{Int}(B_{I,\widetilde{\delta}_4,b_4,\varepsilon_4}) \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\},$$
$$\widetilde{V}_{I,4} := \operatorname{Int}(B_{\widetilde{\delta}_4, b_3}) \cap (X \setminus \widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\varepsilon}_2}) \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\}.$$

Note that there is no tilde $\widetilde{\ }$ over the letter 'B' in the definitions of $\widetilde{N}_I^2$ and $\widetilde{V}_{I,3}$.

It is clear that for each $I \in \mathcal{A}(\prec)$ the triple $(\widetilde{N}_1^i, \widetilde{N}_I^i, \widetilde{N}_2^i)$ is an FM-index triple for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$, $i = 2, 4$. We claim that for all $n \geqslant n_3$ and $I \in \mathcal{A}(\prec)$ the following inclusions hold:

$$\widetilde{N}_1^2 \subset \widetilde{U}, \quad \widetilde{N}_1^2 \subset \widetilde{V}_1 \subset \widetilde{N}_{1,n} \subset \widetilde{N}_1^4, \quad \widetilde{N}_2^2 \subset \widetilde{N}_2^4, \quad \widetilde{N}_I^2 \subset \widetilde{N}_I^4, \tag{6.9}$$
$$\operatorname{Cl}(\widetilde{N}_I^2 \setminus \widetilde{N}_2^4) \subset \widetilde{V}_{I,3} \subset \operatorname{Int}(\widetilde{N}_{I,n} \setminus \widetilde{N}_{2,n}), \tag{6.10}$$
$$\operatorname{Cl}(\widetilde{N}_1^2 \setminus \widetilde{N}_I^4) \subset \widetilde{V}_{I,4} \subset \operatorname{Int}(\widetilde{N}_{1,n} \setminus \widetilde{N}_{I,n}). \tag{6.11}$$

We prove inclusion (6.10) and leave the other inclusions to the reader. Let $n \geqslant n_3$ and $I \in \mathcal{A}(\prec)$ be arbitrary. Notice that $\widetilde{N}_I^2 \setminus \widetilde{N}_2^4 = (B_{I,\widetilde{\delta}_5,b_5,\varepsilon_5} \cup \widetilde{N}_2^2) \setminus \widetilde{N}_2^4$. Clearly, $\widetilde{N}_2^2 \subset \widetilde{N}_2^4$, and so $\widetilde{N}_I^2 \setminus \widetilde{N}_2^4 \subset B_{I,\widetilde{\delta}_5,b_5,\varepsilon_5} \setminus \widetilde{N}_2^4$ and thus $\widetilde{N}_I^2 \setminus \widetilde{N}_2^4 \subset B_{I,\widetilde{\delta}_5,b_5,\varepsilon_5} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) \geqslant b_2'' \,\} =: T$. By Lemma 5.4, $B_{I,\widetilde{\delta}_5,b_5,\varepsilon_5} \subset \operatorname{Int}(B_{I,\widetilde{\delta}_4,b_4,\varepsilon_4})$ so we obtain that $T \subset \widetilde{V}_{I,3}$. Since T is closed we only need to show that $\widetilde{V}_{I,3} \subset \widetilde{N}_{I,n} \setminus \widetilde{N}_{2,n}$. The definition of the set $\widetilde{N}_{I,n}$ implies that $N_0(I) \cap B_{\widetilde{\delta}_4,b_3} \subset \widetilde{N}_{I,n} \subset B_{\widetilde{\delta}_2,b_2}$. Furthermore, since $\widetilde{\delta}_4 < \delta_5$ it follows that $\operatorname{Int}(B_{I,\widetilde{\delta}_4,b_4,\varepsilon_4}) \subset B_{I,\delta_5,b_4,\varepsilon_4} \subset B_{I,\delta_5,b_4,\varepsilon_4} \cup N_2^2 = N_I^2$ and $\operatorname{Int}(B_{I,\widetilde{\delta}_4,b_4,\varepsilon_4}) \subset B_{\widetilde{\delta}_4,b_3}$. Therefore, the previous inclusions and Theorem 6.1 imply that $\widetilde{V}_{I,3} \subset (N_I^2 \cap B_{\widetilde{\delta}_4,b_3}) \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\} \subset (N_0(I) \cap B_{\widetilde{\delta}_4,b_3}) \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\} \subset \widetilde{N}_{I,n} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\} \subset B_{\widetilde{\delta}_2,b_2} \cap \{\, y \in \widetilde{U} \mid t_0^+(y) > b_1'' \,\} \subset X \setminus \widetilde{N}_{2,n}$. (The last inclusion follows from (6.7).) This proves (6.10).

Inclusions (6.9)–(6.11) imply that the assumptions of Theorem 4.1 are satisfied. That theorem therefore implies the following result.

Theorem 6.3. *With the notation introduced above, there exists an $n_3 \geqslant n_2$ such that for every $n \in \mathbb{N}_0$ with $n = 0$ or $n \geqslant n_3$ there exists an index filtration $(\widetilde{N}_n(I))_{I \in \mathcal{A}(\prec)}$ for $(\pi_n, S_n, (M_{p,n})_{p \in P})$ such that for all $I \in \mathcal{A}(\prec)$ the following inclusions hold*:

(1) $\widetilde{N}_I^2 \subset \widetilde{N}_0(I) \subset \widetilde{N}_1^2 = B_{\widetilde{\delta}_5,b_5}$,

(2) $\widetilde{N}_2^2 \subset \widetilde{N}_0(\emptyset)$,

(3) $N_0(I) \cap B_{\widetilde{\delta}_4,b_3} \subset \widetilde{N}_{I,n} \subset \widetilde{N}_n(I) \subset \widetilde{N}_{1,n}$,
(4) $\widetilde{N}_{2,n} \subset \widetilde{N}_n(\emptyset)$.

Moreover, whenever $I \in \mathcal{A}(\prec)$, $(n_m)_m$ is a sequence such that $n_m \to \infty$ as $m \to \infty$ and $(x_m)_m$ is a sequence in $\widetilde{N}_1^2$ such that $x_m \in \widetilde{N}_{n_m}(I)$ for all $m \in \mathbb{N}$ and $(x_m)_m$ is convergent (in X), then there exists an $m_0 \in \mathbb{N}$ such that $x_m \in \widetilde{N}_0(I)$ for all $m \geqslant m_0$.

The index filtrations that we have obtained from Theorems 6.1 and 6.3 do not as yet form a nested sequence as described in Theorem 3.4. However, after intersecting these index filtrations with appropriate sets and using Proposition 2.9 we will now obtain new index filtrations which do satisfy the nesting property. This will complete the proof of Theorem 3.4.

For $\gamma \in]0, \infty[$ define the sets

$$\widetilde{Y}_{1,n}(\gamma) := B_{\widetilde{\delta}_5,b_5} \cap \mathrm{Cl}\{\, y \mid \text{ there exist an } x \in B_{\gamma,b_1'} \text{ and a } t \geqslant 0 \text{ such that } x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t\,\},$$
$$\widetilde{Y}_{2,n}(\gamma) := \widetilde{Y}_{1,n}(\gamma) \cap \{\, y \in \widetilde{U} \mid t_n^+(y) \leqslant b_1' \,\}.$$

Then there exists a $\widetilde{\gamma}_0 > 0$ such that for all $\gamma \in]0, \widetilde{\gamma}_0]$, there exists an $\widetilde{n}_0(\gamma) \in \mathbb{N}$ such that for all $n \geqslant \widetilde{n}_0(\gamma)$, $S_n \subset \mathrm{Int}(\widetilde{Y}_{1,n}(\gamma))$ and the pair $(\widetilde{Y}_{1,n}(\gamma), \widetilde{Y}_{2,n}(\gamma))$ is an FM-index pair for S_n, relative to π_n. (cf. Lemma 5.6).

Given such an n, we see, using (6.2), (6.8) and Theorem 6.3 that $\widetilde{Y}_{2,n}(\gamma) \subset \widetilde{N}_{2,n} \subset \widetilde{N}_n(\emptyset)$. Since $\widetilde{Y}_{2,n}(\gamma)$ is an exit ramp for $\widetilde{Y}_{1,n}(\gamma)$, relative to π_n, it thus follows that $\widetilde{N}_n(\emptyset)$ is an exit ramp for $\widetilde{Y}_{1,n}(\gamma)$, relative to π_n.

An application of Proposition 2.9 now implies the following result.

Lemma 6.4. *For every $\gamma \in]0, \widetilde{\gamma}_0]$, and all $n \geqslant \widetilde{n}_0(\gamma)$, $(\widetilde{Y}_{1,n}(\gamma) \cap \widetilde{N}_n(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_n, S_n, (M_{p,n})_{p \in P})$.*

We also have the following

Lemma 6.5. *There exists a $\widetilde{\gamma}_1 \in]0, \widetilde{\gamma}_0]$ such that for all $\gamma \in]0, \widetilde{\gamma}_1]$, there exists an $\widetilde{n}_1(\gamma) \geqslant \widetilde{n}_0(\gamma)$ such that $\widetilde{Y}_{1,n}(\gamma) \cap \widetilde{N}_n(I) \subset \widetilde{N}_0(I)$ for all $n \geqslant \widetilde{n}_1(\gamma)$ and all $I \in \mathcal{A}(\prec)$.*

Proof. Suppose the conclusion of the lemma is not true. Then for some $I \in \mathcal{A}(\prec)$ there exist sequences $(\gamma_m)_m$, $(n_m)_m$ and $(x_m)_m$ such that $\gamma_m \to 0$, $n_m \to \infty$ and

$$x_m \in (\widetilde{Y}_{1,n_m}(\gamma_m) \cap \widetilde{N}_{n_m}(I)) \setminus \widetilde{N}_0(I) \quad \text{for all } m.$$

Notice that $x_m \in \widetilde{Y}_{1,n_m}(\gamma_m) \subset B_{\widetilde{\delta}_5,b_5} = \widetilde{N}_1^2$. By admissibility, $(x_m)_m$ has a convergent subsequence, denoted again by $(x_m)_m$. Theorem 6.3 implies that there exists an $m_0 \in \mathbb{N}$ such that $x_m \in \widetilde{N}_0(I)$ for all $m \geqslant m_0$, which is a contradiction. □

Define $\widetilde{\gamma} := \min\{\widetilde{\gamma}_1, \widetilde{\delta}_5\}$. Hence, setting $\widetilde{n}_1 := \widetilde{n}_1(\widetilde{\gamma})$ we have the following corollary.

Corollary 6.6. *There is a* $\widetilde{\gamma} \in]0, \widetilde{\delta}_5]$ *and an* $\widetilde{n}_1 \in \mathbb{N}$ *such that, for all* $n \geqslant \widetilde{n}_1$, $(\widetilde{Y}_{1,n}(\widetilde{\gamma}) \cap \widetilde{N}_n(I))_{I \in \mathcal{A}(\prec)}$ *is an index filtration for* $(\pi_n, S_n, (M_{p,n})_{p \in P})$ *and* $\widetilde{Y}_{1,n}(\widetilde{\gamma}) \cap \widetilde{N}_n(I) \subset \widetilde{N}_0(I)$ *for* $I \in \mathcal{A}(\prec)$.

We claim that

$$B^-_{\widetilde{\gamma},b'_1} \subset N^2_2 \subset N_0(\emptyset). \tag{6.12}$$

Here, of course, $B^-_{\widetilde{\gamma},b'_1}$ is the exit set of the isolating block $B_{\widetilde{\gamma},b'_1}$ relative to π_0.

In fact, the inequalities $\widetilde{\gamma} \leqslant \widetilde{\delta}_5 < \delta_5$ and $b_5 > b_4$ and the definition of the set $\widetilde{Y}_{1,n}(\widetilde{\gamma})$ imply that $B_{\widetilde{\gamma},b'_1} \subset \widetilde{Y}_{1,n}(\widetilde{\gamma}) \subset B_{\widetilde{\delta}_5,b_5} \subset B_{\delta_5,b_4}$. Moreover, if $x \in B^-_{\widetilde{\gamma},b'_1}$, then $x \in \widetilde{U}$ and $t^+_0(x) = b'_1$. It follows that $B^-_{\widetilde{\gamma},b'_1} \subset B_{\delta_5,b_4} \cap \{\, y \in \widetilde{U} \mid t^+_0(x) \leqslant b'_1 \,\} = N^2_2$. The last inclusion in (6.12) follows from Theorem 6.1. This proves (6.12).

Since $B^-_{\widetilde{\gamma},b'_1}$ is an exit ramp for $B_{\widetilde{\gamma},b'_1}$, relative to π_0, it follows from (6.12) that $N_0(\emptyset)$ is an exit set for $B_{\widetilde{\gamma},b'_1}$, relative to π_0, and since $S_0 \subset \mathrm{Int}(B_{\widetilde{\gamma},b'_1})$, Proposition 2.9 implies the following result.

Lemma 6.7. $(N_0(I) \cap B_{\widetilde{\gamma},b'_1})_{I \in \mathcal{A}(\prec)}$ *is an index filtration for* $(\pi_0, S_0, (M_{p,0})_{p \in P})$.

Since $\widetilde{\gamma} \leqslant \widetilde{\delta}_5 < \widetilde{\delta}_4$ and $b_3 < b'_1$, it follows that $B_{\widetilde{\gamma},b'_1} \subset B_{\widetilde{\delta}_4,b_3}$ so $B_{\widetilde{\gamma},b'_1} = B_{\widetilde{\gamma},b'_1} \cap B_{\widetilde{\delta}_4,b_3}$ and so $N_0(I) \cap B_{\widetilde{\gamma},b'_1} = N_0(I) \cap (B_{\widetilde{\gamma},b'_1} \cap B_{\widetilde{\delta}_4,b_3}) \subset (N_0(I) \cap B_{\widetilde{\delta}_4,b_3}) \cap \widetilde{Y}_{1,n}(\widetilde{\gamma})$. This fact, together with Corollary 6.6 and Theorem 6.3, implies that, setting $n'_0 := \max\{n_3, \widetilde{n}_1\}$,

$$N_0(I) \cap B_{\widetilde{\gamma},b'_1} \subset \widetilde{Y}_{1,n}(\widetilde{\gamma}) \cap \widetilde{N}_n(I) \subset \widetilde{N}_0(I), \quad n \geqslant n'_0,\ I \in \mathcal{A}(\prec). \tag{6.13}$$

Let $\delta' := \min\{\widetilde{\gamma}, \delta_5\}$. For $\gamma \in]0, \infty[$ define the sets

$$\begin{aligned} Y_{1,n}(\gamma) &:= B_{\delta',b'_1} \cap \mathrm{Cl}\{\, y \mid \text{ there exist an } x \in B_{\gamma,b} \text{ and a } t \geqslant 0 \text{ such that} \\ &\qquad\qquad x\pi_n[0,t] \subset \widetilde{U} \text{ and } y = x\pi_n t \,\}, \\ Y_{2,n}(\gamma) &:= Y_{1,n}(\gamma) \cap \{\, y \in \widetilde{U} \mid t^+_n(y) \leqslant b \,\}. \end{aligned}$$

Using arguments which are completely analogous to those leading to Corollary 6.6 we obtain the following:

Corollary 6.8. *There is a* $\gamma_1 \in]0, \infty[$ *and an* $n_1 \in \mathbb{N}$ *such that, for all* $n \geqslant n_1$, $(Y_{1,n}(\gamma_1) \cap N_n(I))_{I \in \mathcal{A}(\prec)}$ *is an index filtration for* $(\pi_n, S_n, (M_{p,n})_{p \in P})$ *and* $Y_{1,n}(\gamma_1) \cap N_n(I) \subset N_0(I) \cap B_{\delta',b'_1}$ *for all* $I \in \mathcal{A}(\prec)$.

Proof of Theorem 3.4. Let $n_0 := \max\{n'_0, n_1\}$ and $n \geqslant n_0$ be arbitrary. Define $\mathcal{N}_0 := (N_0(I) \cap B_{\widetilde{\gamma},b'_1})_{I \in \mathcal{A}(\prec)}$, $\widetilde{\mathcal{N}_0} := (\widetilde{N}_0(I))_{I \in \mathcal{A}(\prec)}$, $\mathcal{N}_n := (Y_{1,n}(\gamma_1) \cap N_n(I))_{I \in \mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_n := (\widetilde{Y}_{1,n}(\widetilde{\gamma}) \cap \widetilde{N}_n(I))_{I \in \mathcal{A}(\prec)}$. Lemma 6.7 and Corollaries 6.6 and 6.8 imply that, for each $n \in \mathbb{N}_0$ with $n = 0$ or $n \geqslant n_0$, $\mathcal{N}_n$ and $\widetilde{\mathcal{N}}_n$ are index filtrations for $(\pi_n, S_n, (M_{p,n})_{p \in P})$. Formula (6.13) and Corollary 6.8 imply that

$$Y_{1,n}(\gamma_1) \cap N_n(I) \subset N_0(I) \cap B_{\widetilde{\gamma},b'_1} \subset \widetilde{Y}_{1,n}(\widetilde{\gamma}) \cap \widetilde{N}_n(I) \subset \widetilde{N}_0(I), \ n \geqslant n_0, \ I \in \mathcal{A}(\prec),$$

i.e. the nesting property (3.1) holds. Since $\widetilde{N}_0(P) \subset \widetilde{U}$ it also follows that for each $n \in \mathbb{N}_0$ with $n = 0$ or $n \geqslant n_0$, $\mathcal{N}_n$ and $\widetilde{\mathcal{N}}_n$ are strongly π_n-admissible. This completes the proof of Theorem 3.4. □

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