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Multiple q-Mahler measures and zeta functions

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Abstract

We introduce multiple q-Mahler measures and we calculate some specific examples, where multiple q-analogues of zeta functions appear. We study also limits as the multiple q goes to 1.

1. Introduction

The Mahler measure was introduced by Mahler [M] in studying transcendental numbers and it is defined by

$$m(f) = \mathrm{Re} \int_0^1 \cdots \int_0^1 \log\bigl(f\bigl(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}\bigr)\bigr)\, d\theta_1 \cdots d\theta_n$$

for a rational function $f(x_1, \dots, x_n) \in \mathbb{C}(x_1, \dots, x_n)$. It is known that Mahler measure has interesting connections to special values of zeta functions as explained in Smyth [S], Boyd [B] and Deninger [D]. There are also some results connected with dynamical systems in Lind–Schmidt–Ward [LSW]. We refer to Everest–Ward [EW] for a survey of the Mahler measure.

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The q-analogue $m_q(f)$ of $m(f)$ for $q \in \mathbb{C}$ was introduced in [K1] and some calculations are in [O2] also.

In this paper we introduce multiple $\underline{q}$-Mahler measures, where $\underline{q} = (q_1, \ldots, q_r)$. We deal with the following cases:

(i) $0 < q_1, \ldots, q_r < 1$.
(ii) $\underline{q} = (1, q)$ with $0 < q < 1$.

In case (i), the $\underline{q}$-logarithm function is defined by

$$l_{\underline{q}}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{[n]_{\underline{q}}}$$

in $|x-1| < 1$ originally, where $[n]_{\underline{q}} = [n]_{q_1} \cdots [n]_{q_r} = \frac{1-{q_1}^n}{1-q_1} \cdots \frac{1-q_r^n}{1-q_r}$. We note that

$$\lim_{q_1,\ldots,q_r \uparrow 1} l_{\underline{q}}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n^r} = -\mathrm{Li}_r(1-x) \quad \text{in } |x-1| < 1,$$

where $\mathrm{Li}_r(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^r}$ is the polylogarithm function, since $\lim_{q_j \uparrow 1} [n]_{q_j} = n$. Moreover, $l_{\underline{q}}(x)$ has an analytic continuation to all $x \in \mathbb{C}$ as a meromorphic function via the expression

$$l_{\underline{q}}(x) = (1-q_1)\cdots(1-q_r) \sum_{m_1,\ldots,m_r \geqslant 0} \frac{x-1}{x-1+{q_1}^{-m_1}\cdots {q_r}^{-m_r}},$$

which is easily deduced from the definition.

Definition 1. Let $0 < q_j < 1$. Then the $\underline{q}$-Mahler measure $m_{\underline{q}}(f)$ of a rational function $f(x_1, \ldots, x_n) \in \mathbb{C}(x_1, \ldots, x_n)$ is defined as

$$m_{\underline{q}}(f) = \mathrm{Re} \int_0^1 \cdots \int_0^1 l_{\underline{q}}\big(f\big(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\big)\big)\, d\theta_1 \cdots d\theta_n.$$

Then, we have the following results.

Theorem 1. *For* $0 < a \leqslant 1$,

$$m_{\underline{q}}\left(az\frac{x+1}{y+1} + 1\right) = \frac{4}{\pi^2} \sum_{\substack{n \geqslant 1 \\ \text{odd}}} \frac{a^n}{[n]_{\underline{q}} n^2}.$$

In particular, when $a = 1$,

$$m_{\underline{q}}\left(z\frac{x+1}{y+1} + 1\right) = \frac{4}{\pi^2} \sum_{\substack{n \geqslant 1 \\ \text{odd}}} \frac{1}{[n]_{\underline{q}} n^2}.$$

Theorem 2.

$$\lim_{\underline{q}\uparrow\underline{1}_r} m_{\underline{q}}\left(z\frac{x+1}{y+1}+1\right) = \frac{4}{\pi^2}\sum_{\substack{n\geqslant 1\\ \text{odd}}}\frac{1}{n^{r+2}} = \frac{4}{\pi^2}\left(1-2^{-2-r}\right)\zeta(r+2),$$

where $\underline{1}_r = (1,\dots,1)\in\mathbb{R}^r$.

Theorem 3. *For* $0<a\leqslant 1$,

$$m_{\underline{q}}\left(ay\frac{x+1}{x-1}+1\right) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)[2n+1]_{\underline{q}}}.$$

In particular, when $a=1$,

$$m_{\underline{q}}\left(y\frac{x+1}{x-1}+1\right) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)[2n+1]_{\underline{q}}}.$$

Theorem 4.

$$\lim_{\underline{q}\uparrow\underline{1}_r} m_{\underline{q}}\left(y\frac{x+1}{x-1}+1\right) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{r+1}} = \frac{2}{\pi}L(r+1,\chi_{-4}).$$

Here, $\chi_{-4} = (\frac{-4}{*})$ *is the non-trivial Dirichlet character of modulo* 4.

Secondly we deal with the case $\underline{q}=(1,q)$ with $0<q<1$. In this case, the $(1,q)$-logarithm function is defined by

$$l_{1,q}(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{[n]_{1,q}}$$

in $|x-1|<1$ originally, where $[n]_{1,q} = n\cdot[n]_q = n\cdot\frac{1-q^n}{1-q}$. We note that for $|x-1|<1$

$$\lim_{q\uparrow 1} l_{1,q}(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n^2} = -\mathrm{Li}_2(1-x)$$

and

$$\lim_{q\downarrow 0} l_{1,q}(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n} = \log x,$$

where $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is the dilogarithm function, since $\lim_{q\uparrow 1}[n]_{1,q} = n^2$ and $\lim_{q\downarrow 0}[n]_{1,q} = n$. Moreover, $l_{1,q}(x)$ has an analytic continuation to all $x \in \mathbb{C}$ via the expression

$$l_{1,q}(x) = (1-q)\sum_{m=0}^{\infty} \log\bigl(q^m(x-1)+1\bigr),$$

which is easily deduced from the definition. We remark that $l_{1,q}(x)$ has the following expression:

$$l_{1,q}(x) = (1-q)\log\left(\sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}(x-1)^j}{(1-q)(1-q^2)\cdots(1-q^j)}\right).$$

Definition 2. Let $0 < q < 1$. Then the $(1,q)$-Mahler measure $m_{1,q}(f)$ of a rational function $f(x_1,\ldots,x_n) \in \mathbb{C}(x_1,\ldots,x_n)$ is defined as

$$m_{1,q}(f) = \operatorname{Re}\int_0^1 \cdots \int_0^1 l_{1,q}\bigl(f\bigl(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_n}\bigr)\bigr)\,d\theta_1\cdots d\theta_n.$$

Then, we have the following results.

Theorem 5. *For $a \geqslant 1$,*

$$m_{1,q}(x+a) = l_{1,q}(a).$$

Theorem 6.

$$m_{1,q}\bigl(1+(1+x)y\bigr) = \frac{1-q}{\pi}\sum_{m=0}^{[-\frac{\log 2}{\log q}]}\left\{2\pi\alpha_m\log\bigl(q^m\bigr) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n\pi\alpha_m)}{n^2}\right\},$$

with $\alpha_m = \frac{1}{\pi}\cos^{-1}(\frac{1}{2q^m})$.

Theorem 7. *Let $\chi_{-3} = (\frac{-3}{*})$, $\chi_{-4} = (\frac{-4}{*})$ be the Dirichlet characters of modulo 3 and 4, respectively.*

(i) $$\lim_{q\downarrow 0} m_{1,q}\bigl(1+(1+x)y\bigr) = \frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3}) = L'(-1,\chi_{-3}).$$

(ii) $$m_{1,\frac{1}{2}}\bigl(1+(1+x)y\bigr) = \frac{3\sqrt{3}}{8\pi}L(2,\chi_{-3}) = \frac{1}{2}L'(-1,\chi_{-3}).$$

(iii) $$\begin{aligned} m_{1,\frac{1}{\sqrt{3}}}\bigl(1+(1+x)y\bigr) &= \left(1-\frac{1}{\sqrt{3}}\right)\left\{\frac{5\sqrt{3}}{4\pi}L(2,\chi_{-3}) - \frac{1}{6}\log 3\right\} \\ &= \left(1-\frac{1}{\sqrt{3}}\right)\left\{\frac{5}{3}L'(-1,\chi_{-3}) - \frac{1}{6}\log 3\right\}. \end{aligned}$$

(iv)
$$m_{1,\frac{1}{\sqrt{2}}}\big(1+(1+x)y\big)=\left(1-\frac{1}{\sqrt{2}}\right)\left\{\frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3})+\frac{1}{\pi}L(2,\chi_{-4})-\frac{1}{4}\log 2\right\}$$
$$=\left(1-\frac{1}{\sqrt{2}}\right)\left\{L'(-1,\chi_{-3})+\frac{1}{2}L'(-1,\chi_{-4})-\frac{1}{4}\log 2\right\}.$$

Theorem 8.

$$\lim_{q\uparrow 1}m_{1,q}\big(1+(1+x)y\big)=\frac{5}{81}\pi^2+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{c(n)}{n^2}$$
$$=\frac{5}{162}\pi^2-\frac{1}{\pi}\operatorname{Im}\operatorname{Li}_{1,2}\big(e^{\frac{\pi i}{3}},1\big).$$

Here $c(n)=\sum_{k=1}^{n}\frac{(-1)^k\sin(\frac{2k\pi}{3})}{k}=-\frac{\sqrt{3}}{2}\sum_{k=1}^{n}\frac{(-1)^{k-1}\chi_{-3}(k)}{k}$ *and the doublepolylogarithm function* $\operatorname{Li}_{k,l}(u,v)$ *is defined as* $\operatorname{Li}_{k,l}(u,v)=\sum_{1\leqslant m<n}\frac{u^m v^n}{m^k n^l}$.

Theorem 9.

$$m_{1,q}\left(1+\frac{1+x}{1-x}y\right)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2[2n+1]_q}.$$

Theorem 10.

(i)
$$\lim_{q\downarrow 0}m_{1,q}\left(1+\frac{1+x}{1-x}y\right)=\frac{2}{\pi}L(2,\chi_{-4})=L'(-1,\chi_{-4}).$$

(ii)
$$\lim_{q\uparrow 1}m_{1,q}\left(1+\frac{1+x}{1-x}y\right)=\frac{2}{\pi}L(3,\chi_{-4})=\frac{\pi^2}{16}.$$

This paper belongs to a series of papers making attempts to extend Mahler measures as in [GO,K1,K2,KO,O1,O2] mainly related to multiple sine functions and their generalizations. Our multiple q-deformation examples are mainly suggested by the usual q-deformation examples treated in [K1,K2,KO,O2]. In particular, the fractional form is crucial for q-deformations in comparison with the usual case. We also note that the continuity of $m_{\underline{q}}(f)$ in $\underline{q}$ as $\underline{q}\to\underline{1}$ is a non-trivial problem in general since we must pay attention to complicated analytic continuation of associated $\underline{1}$-logarithm (polylogarithm) and in our examples we calculated the limits only in principle. We hope to treat this problem in a future paper.

In some cases our results give multiple q-deformations of known formulas. For example, the following four results Theorem 2 ($r=1$), Theorem 4 ($r=1$), Theorems 7(i) and 10(i) specialize to formulas of Smyth [S]. We refer to recent interesting papers of Boyd–Rodriguez-Villegas [BRV], Lalin [L1,L2] and Rodriguez-Villegas [RV] for various generalizations different from our way.

2. $(q_1,\ldots,q_r)$-Mahler measures

Proofs of Theorems 1 and 2. From the definition, we have

$$m_{\underline{q}}\left(az\frac{x+1}{y+1}+1\right)$$
$$=\operatorname{Re}\frac{(1-q_1)\cdots(1-q_r)}{(2\pi i)^3}\int\limits_{|x|,|y|,|z|=1}\sum_{m_1,\ldots,m_r\geqslant 0}^{\infty}\frac{az\frac{x+1}{y+1}}{az\frac{x+1}{y+1}+q_1{}^{-m_1}\cdots q_r{}^{-m_r}}\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}.$$

Here we put $\underline{q}^{\underline{m}}=q_1{}^{m_1}\cdots q_r{}^{m_r}$, then

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\frac{1}{z+\frac{y+1}{a\underline{q}^{\underline{m}}(x+1)}}\,dz=\begin{cases}1 & \text{if } |\frac{y+1}{a\underline{q}^{\underline{m}}(x+1)}|<1,\\ 0 & \text{if } |\frac{y+1}{a\underline{q}^{\underline{m}}(x+1)}|\geqslant 1.\end{cases}$$

Hence, for $x=e^{2\pi i t_1}$, $y=e^{2\pi i t_2}$ with $-\frac{1}{2}<t_1<\frac{1}{2}$, $-\frac{1}{2}<t_2<\frac{1}{2}$, we know that $|\frac{y+1}{a\underline{q}^{\underline{m}}(x+1)}|<1$ if and only if $a\underline{q}^{\underline{m}}\cos\pi t_1>\cos\pi t_2$.

Then, putting $\alpha_{\underline{m}}(t_1)=\frac{1}{\pi}\cos^{-1}(a\underline{q}^{\underline{m}}\cos\pi t_1)$ we have

$$m_{\underline{q}}\left(az\frac{x+1}{y+1}+1\right)$$
$$=(1-q_1)\cdots(1-q_r)\sum_{m_1,\ldots,m_r\geqslant 0}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int\limits_{\alpha_{\underline{m}}(t_1)}^{\frac{1}{2}}dt_2+\int\limits_{-\frac{1}{2}}^{-\alpha_{\underline{m}}(t_1)}dt_2\right)dt_1$$
$$=4(1-q_1)\cdots(1-q_r)\sum_{m_1,\ldots,m_r\geqslant 0}\int\limits_{0}^{\frac{1}{2}}\left(\frac{1}{2}-\alpha_{\underline{m}}(t_1)\right)dt_1.$$

Using

$$\cos^{-1}x=\frac{\pi}{2}-\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}$$

and

$$(1-q_j)\sum_{m_j=0}^{\infty}q_j^{(2n+1)m_j}=\frac{1-q_j}{1-q_j^{2n+1}}=\frac{1}{[2n+1]_{q_j}},$$

we obtain

$$m_{\underline{q}}\left(az\frac{x+1}{y+1}+1\right)$$
$$=4(1-q_1)\cdots(1-q_r)\sum_{m_1,\ldots,m_r\geqslant 0}\int\limits_{0}^{\frac{1}{2}}\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{(a\underline{q}^{\underline{m}}\cos\pi t_1)^{2n+1}}{2n+1}\,dt_1$$

$$
\begin{aligned}
&= \frac{4(1-q_1)\cdots(1-q_r)}{\pi} \sum_{m_1,\dots,m_r \geqslant 0} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{(a\underline{q}^{\underline{m}})^{2n+1}}{2n+1} \frac{1}{\pi} \frac{(2n)!!}{(2n+1)!!} \\
&= \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(2n+1)^2} \prod_{j=1}^{r} \frac{1}{[2n+1]_{q_j}} \\
&= \frac{4}{\pi^2} \sum_{\substack{n \geqslant 1 \\ \text{odd}}} \frac{a^n}{n^2 [n]_{\underline{q}}}.
\end{aligned}
$$

So we have proved Theorem 1.

Theorem 2 is easily proved from Theorem 1. □

Proofs of Theorems 3 and 4. From the definition, we have

$$
\begin{aligned}
&m_{\underline{q}}\left(ay\frac{x+1}{x-1}+1\right) \\
&\quad = \operatorname{Re} \frac{(1-q_1)\cdots(1-q_r)}{(2\pi i)^2} \int_{|x|,|y|=1} \sum_{m_1,\dots,m_r \geqslant 0}^{\infty} \frac{ay\frac{x+1}{x-1}}{ay\frac{x+1}{x-1} + {q_1}^{-m_1}\cdots{q_r}^{-m_r}} \frac{dx}{x}\frac{dy}{y}.
\end{aligned}
$$

Here we put $\underline{q}^{\underline{m}} = q_1^{m_1}\cdots q_r^{m_r}$, then

$$
\frac{1}{2\pi i} \int_{|y|=1} \frac{1}{y + \frac{x-1}{a\underline{q}^{\underline{m}}(x+1)}}\, dy = \begin{cases} 1 & \text{if } \left|\frac{x-1}{a\underline{q}^{\underline{m}}(x+1)}\right| < 1, \\ 0 & \text{if } \left|\frac{x-1}{a\underline{q}^{\underline{m}}(x+1)}\right| \geqslant 1. \end{cases}
$$

Hence, for $x = e^{2\pi i t}$, with $-\frac{1}{2} < t < \frac{1}{2}$, we know that $\left|\frac{x-1}{a\underline{q}^{\underline{m}}(x+1)}\right| < 1$ if and only if $|\tan(\pi t)| < a\underline{q}^{\underline{m}}$.

Then, putting $\beta_{\underline{m},a} = \frac{1}{\pi}\tan^{-1}(a\underline{q}^{\underline{m}})$ we have

$$
\begin{aligned}
&m_{\underline{q}}\left(ay\frac{x+1}{x-1}+1\right) \\
&\quad = (1-q_1)\cdots(1-q_r) \sum_{m_1,\dots,m_r \geqslant 0} \int_{-\beta_{\underline{m},a}}^{\beta_{\underline{m},a}} dt \\
&\quad = \frac{2}{\pi}(1-q_1)\cdots(1-q_r) \sum_{m_1,\dots,m_r \geqslant 0} \tan^{-1}\bigl(a\underline{q}^{\underline{m}}\bigr).
\end{aligned}
$$

Using

$$
\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}
$$

and

$$(1-q_j)\sum_{m_j=0}^{\infty} q_j^{(2n+1)m_j} = \frac{1-q_j}{1-q_j^{2n+1}} = \frac{1}{[2n+1]_{q_j}},$$

we obtain

$$\begin{aligned}
&m_{\underline{q}}\left(ay\frac{x+1}{x-1}+1\right) \\
&= \frac{2}{\pi}(1-q_1)\cdots(1-q_r)\left\{\tan^{-1}(a) + \sum_{\substack{m_1,\ldots,m_r\geqslant 0\\(m_1,\ldots,m_r)\neq(0,\ldots,0)}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(a\underline{q}^{\underline{m}}\right)^{2n+1}\right\} \\
&= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{2n+1}\prod_{j=1}^{r}\frac{1}{[2n+1]_{q_j}} \\
&= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)[2n+1]_{\underline{q}}}.
\end{aligned}$$

So we have proved Theorem 3.

Theorem 4 is easily proved from Theorem 3. □

3. $(1, q)$-Mahler measures

Proof of Theorem 5. By definition and Jensen's formula, we have

$$\begin{aligned}
&m_{1,q}(x+a) \\
&= (1-q)\sum_{m=0}^{\infty}\operatorname{Re}\int_0^1\left\{\log\left(e^{2\pi it}+a-1+q^{-m}\right)-\log\left(q^{-m}\right)\right\}dt \\
&= (1-q)\sum_{m=0}^{\infty}\left\{\log^+\left(a-1+q^{-m}\right)-\log\left(q^{-m}\right)\right\} \\
&= (1-q)\sum_{m=0}^{\infty}\log\left(q^m(a-1)+1\right) \\
&= l_{1,q}(a).
\end{aligned}$$

Here, $\log^+ x = \max\{\log x, 0\}$ for $x > 0$. □

Proof of Theorem 6. By Jensen's formula,

$$\begin{aligned} m_{1,q}\big(1+(1+x)y\big) &= (1-q)\sum_{m=0}^{\infty}\int_{0}^{1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log\big|q^m\big(e^{2\pi i t_2}\big(1+e^{2\pi i t_1}\big)\big)+1\big|\,dt_1\,dt_2 \\ &= (1-q)\sum_{m=0}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log^+\big|q^m\big(1+e^{2\pi i t_1}\big)\big|\,dt_1. \end{aligned}$$

Let α_m be a real number such that $0 \leqslant \alpha_m < 1/2$ and $2q^m\cos(\pi\alpha_m) = 1$ for $m \leqslant -\frac{\log 2}{\log q}$. For an integer $m \leqslant -\frac{\log 2}{\log q}$, put

$$I_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log^+\big|q^m\big(1+e^{2\pi i t}\big)\big|\,dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log^+\big|2q^m(\cos\pi t)\big|\,dt.$$

Integrating by parts, we obtain

$$\begin{aligned} I_m &= 2\int_{0}^{\alpha_m}\big\{\log\big(2q^m\big)+\log(\cos\pi t)\big\}\,dt \\ &= 2\left\{\alpha_m\log\big(2q^m\big) + \big[t\log(\cos\pi t)\big]_0^{\alpha_m} + \int_0^{\alpha_m}\pi t\tan\pi t\,dt\right\} \\ &= 2\int_0^{\alpha_m}\pi t\tan\pi t\,dt. \end{aligned}$$

We evaluate I_m by using the double sine function $\mathcal{S}_2(z)$ in [KK]. Firstly, by using the formula (Theorem 2.5 in [KK, p.847])

$$\log\mathcal{S}_2(z) = \int_0^z \pi t\cot\pi t\,dt$$

and

$$\log\mathcal{S}_2\left(z+\frac{1}{2}\right)+\log\mathcal{S}_2\left(z-\frac{1}{2}\right) = -2\int_0^z \pi t\tan\pi t\,dt$$

we find that

$$I_m = -\log\left(\mathcal{S}_2\left(\alpha_m+\frac{1}{2}\right)\mathcal{S}_2\left(\alpha_m-\frac{1}{2}\right)\right).$$

Secondly we use the functional equation of the double sine function (Theorem 2.10 and Examples 2.12 in [KK, pp. 852–854]):

$$\mathcal{S}_2(x+1) = -\mathcal{S}_2(x) \cdot 2\sin(\pi x).$$

This formula gives that

$$\begin{aligned}\mathcal{S}_2\left(\alpha_m + \frac{1}{2}\right)^2 &= \mathcal{S}_2\left(\alpha_m + \frac{1}{2}\right)\mathcal{S}_2\left(\alpha_m - \frac{1}{2}\right) \cdot 2\cos(\pi\alpha_m)\\ &= \mathcal{S}_2\left(\alpha_m + \frac{1}{2}\right)\mathcal{S}_2\left(\alpha_m - \frac{1}{2}\right) \cdot q^{-m}.\end{aligned}$$

Hence we obtain

$$I_m = -\log\left(q^m\right) - 2\log\mathcal{S}_2\left(\alpha_m + \frac{1}{2}\right).$$

Thirdly we use the formula (Theorem 2.8 and Examples 2.9 in [KK, pp. 849–852]):

$$\mathcal{S}_2(x) = (2\sin\pi x)^x \exp\left(\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n x)}{n^2}\right)$$

for $0 < x < 1$. Note that $0 < \alpha_m + 1/2 < 1$. This gives

$$\begin{aligned}\log\mathcal{S}_2\left(\alpha_m + \frac{1}{2}\right) &= \left(\alpha_m + \frac{1}{2}\right)\log(2\cos\pi\alpha_m) + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n\alpha_m + \pi n)}{n^2}\\ &= -\left(\alpha_m + \frac{1}{2}\right)\log\left(q^m\right) + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(2\pi n\alpha_m)}{n^2}.\end{aligned}$$

At last we have

$$I_m = 2\alpha_m\log\left(q^m\right) + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2\pi n\alpha_m)}{n^2}.$$

Since $m_{1,q}(1+(1+x)y) = (1-q)\sum_{m=0}^{[-\frac{\log 2}{\log q}]} I_m$, the proof is completed. $\square$

Proof of Theorem 7. Let α_m and I_m be as in the proof of Theorem 6.

(i) We note that $[-\frac{\log 2}{\log q}] = 0$ for $0 < q < 1/2$. Therefore we see that

$$m_{1,q}\left(1+(1+x)y\right) = (1-q)I_0 \quad (0 < q < 1/2).$$

The value $\pi\alpha_0 = \cos^{-1}(1/2q^0) = \pi/3$ implies that

$$\begin{aligned}
I_0 &= 2\alpha_0 \log\bigl(q^0\bigr) + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(\frac{2\pi n}{3})}{n^2} \\
&= \frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^2} - 2\sum_{n=1}^{\infty}\frac{\chi_{-3}(2n)}{(2n)^2}\right) \\
&= \frac{\sqrt{3}}{2\pi}\left(1+\frac{2}{4}\right)L(2,\chi_{-3}) = \frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3}).
\end{aligned}$$

Thus we obtain

$$\lim_{q\downarrow 0} m_{1,q}\bigl(1+(1+x)y\bigr) = I_0 = \frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3}).$$

(ii) If $q = 1/2$ then $[-\frac{\log 2}{\log q}] = 1$. In this case, $\pi\alpha_1 = \cos^{-1}(1/2q) = 0$. So we have $I_1 = 0$ and

$$m_{1,\frac{1}{2}}\bigl(1+(1+x)y\bigr) = \left(1-\frac{1}{2}\right)(I_0+I_1) = \frac{3\sqrt{3}}{8\pi}L(2,\chi_{-3}).$$

(iii) If $q = 1/\sqrt{3}$ then $[-\frac{\log 2}{\log q}] = 1$. In this case, $\pi\alpha_1 = \cos^{-1}(1/2q) = \pi/6$ and

$$\begin{aligned}
I_1 &= 2\alpha_1 \log(q) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(\frac{\pi n}{3})}{n^2} \\
&= -\frac{1}{6}\log 3 + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\frac{2\pi n}{3})}{n^2} \\
&= -\frac{1}{6}\log 3 + \frac{\sqrt{3}}{2\pi}L(2,\chi_{-3}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
m_{1,\frac{1}{\sqrt{3}}}\bigl(1+(1+x)y\bigr) &= \left(1-\frac{1}{\sqrt{3}}\right)(I_0+I_1) \\
&= \left(1-\frac{1}{\sqrt{3}}\right)\left\{\frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3}) - \frac{1}{6}\log 3 + \frac{\sqrt{3}}{2\pi}L(2,\chi_{-3})\right\} \\
&= \left(1-\frac{1}{\sqrt{3}}\right)\left\{\frac{5\sqrt{3}}{4\pi}L(2,\chi_{-3}) - \frac{1}{6}\log 3\right\}.
\end{aligned}$$

(iv) If $q = 1/\sqrt{2}$ then $[-\frac{\log 2}{\log q}] = 2$. In this case, $\pi\alpha_1 = \cos^{-1}(1/2q) = \pi/4$ and $\pi\alpha_2 = \cos^{-1}(1/2q^2) = 0$. Thus $I_2 = 0$ and we can easily check that

$$I_1 = -\frac{1}{4}\log 2 + \frac{1}{\pi}L(2,\chi_{-4})$$

in the same way. The proof is finished. □

Proof of Theorem 8. At first we show that

$$\lim_{q\uparrow 1} m_{1,q}\big(1+(1+x)y\big) = \frac{2}{\pi}J_1 + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a(n)}{n^2}$$

with

$$J_1 = \int_{\frac{1}{2}}^{1} \frac{\cos^{-1}(\frac{1}{2t})\log t}{t}\,dt$$

and

$$a(n) = \int_{\frac{1}{2}}^{1} \frac{\sin(2n\cos^{-1}(\frac{1}{2t}))}{t}\,dt.$$

This is proved by the Jackson integral. Since $\lim_{q\uparrow 1} q^{[-\frac{\log 2}{\log q}]} = 1/2$, we find that

$$\lim_{q\uparrow 1}(1-q)\sum_{m=0}^{[-\frac{\log 2}{\log q}]} q^m \frac{\cos^{-1}(\frac{1}{2q^m})\log(q^m)}{q^m} = J_1$$

and

$$\lim_{q\uparrow 1}(1-q)\sum_{m=0}^{[-\frac{\log 2}{\log q}]} q^m \frac{\sin(2n\cos^{-1}(\frac{1}{2q^m}))}{q^m} = a(n).$$

Secondly we evaluate the integral J_1. We make following changes of variables. Let $u = \cos^{-1}((2t)^{-1})$, then $\cos u = (2t)^{-1}$, $dt = 2t^2 \sin u\,du$ and $dt/t = \tan u\,du$. We find that

$$\begin{aligned}
J_1 &= \int_0^{\frac{\pi}{3}} u\log\big((2\cos u)^{-1}\big)\tan u\,du \\
&= -\int_0^{\frac{\pi}{3}} u\log(2\cos u)\frac{-(2\cos u)'}{2\cos u}\,du \\
&= \frac{1}{2}\int_0^{\frac{\pi}{3}} \big\{\log^2(2\cos u)\big\}' u\,du
\end{aligned}$$

$$= \frac{1}{2}\big[u \log^2(2\cos u)\big]_0^{\frac{\pi}{3}} - \frac{1}{2}\int_0^{\frac{\pi}{3}} \log^2(2\cos u)\,du$$
$$= -\frac{1}{2}\int_0^{\frac{\pi}{3}} \log^2(2\cos u)\,du.$$

After the change of variable $t = \pi/2 - u$, we see that

$$\begin{aligned}
J_1 &= -\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \log^2(2\sin t)\,dt \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \log^2(2\sin t)\,dt + \frac{1}{2}\int_0^{\frac{\pi}{6}} \log^2(2\sin t)\,dt \\
&= -\frac{1}{2}\left(\frac{\pi^3}{24} - \frac{7\pi^3}{216}\right) \\
&= -\frac{\pi^3}{216}.
\end{aligned}$$

In the above calculation, we used the fact that $\int_0^{\pi/2} \log^2(2\sin t)\,dt = \frac{\pi^3}{24}$ (see 4.224 no. 7 in [GR, p. 559]) and $\int_0^{\pi/6} \log^2(2\sin t)\,dt = \frac{7\pi^3}{216}$ (see (1.3) in [ZW, p. 272]).

Thirdly we calculate $a(n)$. Changing the variable $u = \cos^{-1}((2t)^{-1})$, we obtain

$$\begin{aligned}
a(n) &= \int_0^{\frac{\pi}{3}} \sin(2nu)\frac{\sin u}{\cos u}\,du \\
&= \frac{1}{2}\int_0^{\frac{\pi}{3}} \left\{\frac{\cos(2n-1)u}{\cos u} - \frac{\cos(2n+1)u}{\cos u}\right\} du \\
&= \frac{1}{2}\left[\sum_{k=1}^{n-1}(-1)^{n-1-k}\frac{\sin 2ku}{k} + (-1)^{n-1}u - \sum_{k=1}^{n}(-1)^{n-k}\frac{\sin 2ku}{k} - (-1)^n u\right]_0^{\frac{\pi}{3}} \\
&= \frac{1}{2}(-1)^{n-1}\left[2\sum_{k=1}^{n}(-1)^k\frac{\sin 2ku}{k} + 2u - \frac{(-1)^n\sin(2nu)}{n}\right]_0^{\frac{\pi}{3}} \\
&= (-1)^{n-1}c(n) + \frac{\pi}{3}(-1)^{n-1} + \frac{1}{2}\frac{\sin\frac{2\pi n}{3}}{n}
\end{aligned}$$

with $c(n) = \sum_{k=1}^{n} \frac{(-1)^k \sin\frac{2k\pi}{3}}{k}$.

Therefore, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a(n)}{n^2} &= \sum_{n=1}^{\infty}\frac{c(n)}{n^2}+\frac{\pi}{3}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\frac{2\pi n}{3}}{n^3}\\
&= \sum_{n=1}^{\infty}\frac{c(n)}{n^2}+\frac{\pi}{3}\zeta(2)+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\left(\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^3}-2\sum_{n=1}^{\infty}\frac{\chi_{-3}(2n)}{(2n)^3}\right)\\
&= \sum_{n=1}^{\infty}\frac{c(n)}{n^2}+\frac{\pi}{3}\zeta(2)+\frac{5\sqrt{3}}{16}L(3,\chi_{-3}).
\end{aligned}$$

Since $\zeta(2)=\frac{\pi^2}{6}$ and $L(3,\chi_{-3})=\frac{4\pi^3}{81\sqrt{3}}$, we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a(n)}{n^2} &= \sum_{n=1}^{\infty}\frac{c(n)}{n^2}+\frac{\pi}{3}\cdot\frac{\pi^2}{6}+\frac{5\sqrt{3}}{16}\cdot\frac{4\pi^3}{81\sqrt{3}}\\
&= \sum_{n=1}^{\infty}\frac{c(n)}{n^2}+\frac{23}{324}\pi^3.
\end{aligned}$$

At last we have

$$\begin{aligned}
\lim_{q\uparrow 1} m_{1,q}\big(1+(1+x)y\big) &= \frac{2}{\pi}J_1+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a(n)}{n^2}\\
&= -\frac{2}{\pi}\cdot\frac{\pi^3}{216}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{c(n)}{n^2}+\frac{1}{\pi}\frac{23}{324}\pi^3\\
&= \frac{5}{81}\pi^2+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{c(n)}{n^2}=\frac{5}{162}\pi^2-\frac{1}{\pi}\operatorname{Im}\operatorname{Li}_{1,2}\big(e^{\frac{\pi i}{3}},1\big).
\end{aligned}$$

The proof is finished. □

Proof of Theorem 9. By Jensen's formula,

$$\begin{aligned}
m_{1,q}\left(1+\frac{1+x}{1-x}y\right) &= (1-q)\sum_{m=0}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\log\left|q^m\left(\frac{1+e^{2\pi i t_1}}{1-e^{2\pi i t_1}}e^{2\pi i t_2}\right)+1\right|dt_1\,dt_2\\
&= (1-q)\sum_{m=0}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\log^{+}\left|q^m\cot\pi t_1\right|dt_1.
\end{aligned}$$

For a non-negative integer m, we put

$$I_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log^+ \left| q^m \cot \pi t \right| dt$$

and we evaluate the integral I_m. We know that $q^m |\cot \pi t| > 1$ if and only if $|\tan \pi t| < q^m$. Let $\beta_m = \frac{1}{\pi} \tan^{-1}(q^m)$. Note that $0 \leqslant \beta_m \leqslant 1/4$ for $m \geqslant 0$. We find that

$$\begin{aligned}
I_m &= \int_{-\beta_m}^{\beta_m} \left\{ \log\left(q^m\right) + \log(\cos \pi t) - \log(\sin \pi t) \right\} dt \\
&= 2\beta_m \log\left(q^m\right) + 2 \int_0^{\beta_m} \left\{ \log(\cos \pi t) - \log(\sin \pi t) \right\} dt.
\end{aligned}$$

Integration by parts gives:

$$\begin{aligned}
\int_0^t \log(\cos \pi t)\, dt &= t \log(\cos \pi t) + \int_0^t \pi t \tan(\pi t)\, dt, \\
\int_0^t \log(\sin \pi t)\, dt &= t \log(\sin \pi t) - \int_0^t \pi t \cot(\pi t)\, dt.
\end{aligned}$$

Therefore, we see that

$$\begin{aligned}
I_m &= 2\beta_m \log\left(q^m\right) + 2 \left[t \log(\cot \pi t) + \int_0^t \{ \pi t \cot \pi t + \pi t \tan \pi t \}\, dt \right]_0^{\beta_m} \\
&= 2\beta_m \log\left(q^m\right) + 2\beta_m \log\left(1/q^m\right) + 2 \int_0^{\beta_m} \{ \pi t \cot \pi t + \pi t \tan \pi t \}\, dt \\
&= 2 \int_0^{\beta_m} \{ \pi t \cot \pi t + \pi t \tan \pi t \}\, dt.
\end{aligned}$$

We evaluate I_m by using the various formulas of $\mathcal{S}_2(z)$ similar to the proof of Theorem 6. Firstly, noting that

$$\int_0^{\beta_m} \pi t \cot \pi t\, dt = \log \mathcal{S}_2(\beta_m),$$

$$\int_0^{\beta_m} \pi t \tan \pi t \, dt = -\frac{1}{2} \log\left(\mathcal{S}_2\left(\beta_m + \frac{1}{2}\right)\mathcal{S}_2\left(\beta_m - \frac{1}{2}\right)\right)$$

and

$$\mathcal{S}_2\left(\beta_m + \frac{1}{2}\right) = \mathcal{S}_2\left(\beta_m - \frac{1}{2}\right) \cdot 2\cos(\pi\beta_m),$$

we obtain

$$\begin{aligned}
\frac{1}{2} I_m &= \int_0^{\beta_m} \pi t \cot \pi t \, dt + \int_0^{\beta_m} \pi t \tan \pi t \, dt \\
&= \log \mathcal{S}_2(\beta_m) - \log\left\{\frac{1}{2\cos\pi\beta_m} \mathcal{S}_2\left(\beta_m + \frac{1}{2}\right)^2\right\} \\
&= \frac{1}{2}\log(2\cos\pi\beta_m) + \log\left\{\mathcal{S}_2(\beta_m)\mathcal{S}_2\left(\beta_m + \frac{1}{2}\right)^{-1}\right\}.
\end{aligned}$$

Secondly, utilizing the expansion formula of the double sine function (Theorem 2.8 and Examples 2.9 in [KK, pp. 849–852])

$$\log \mathcal{S}_2(x) = x \log(2\sin\pi x) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^2} \quad (0 < x < 1),$$

we see that

$$\begin{aligned}
&\log \mathcal{S}_2(x) - \log \mathcal{S}_2\left(x + \frac{1}{2}\right) \\
&\quad = x \log(2\sin\pi x) - \left(x + \frac{1}{2}\right)\log(2\cos\pi x) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x) - \sin(2\pi n x + \pi n)}{n^2} \\
&\quad = x \log(\tan\pi x) - \frac{1}{2}\log(2\cos\pi x) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x) - (-1)^n \sin(2\pi n x)}{n^2}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{2} I_m &= \frac{1}{2}\log(2\cos\pi\beta_m) + \beta_m \log(\tan\pi\beta_m) - \frac{1}{2}\log(2\cos\pi\beta_m) \\
&\quad + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\{1 - (-1)^n\}\sin(2\pi n\beta_m)}{n^2} \\
&= \beta_m \log\left(q^m\right) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2\pi(2n+1)\beta_m)}{(2n+1)^2}.
\end{aligned}$$

Therefore we obtain

$$
\begin{aligned}
m_{1,q}\left(1+\frac{1+x}{1-x}y\right) &= (1-q)\sum_{m=0}^{\infty} I_m \\
&= \frac{2(1-q)}{\pi}\sum_{m=0}^{\infty}\left\{\tan^{-1}\bigl(q^m\bigr)\log\bigl(q^m\bigr)+\sum_{n=0}^{\infty}\frac{\sin(2\pi(2n+1)\beta_m)}{(2n+1)^2}\right\}.
\end{aligned}
$$

Finally we use the formula of Ramanujan (Entry 17 in [BJ, p.41]):

$$
\sum_{n=0}^{\infty}\frac{(-1)^n\tan^{2n+1}x}{(2n+1)^2} = x\log|\tan x| + \sum_{n=0}^{\infty}\frac{\sin(4n+2)x}{(2n+1)^2}
$$

with $|x|<\pi/2$. We make the substitution $x=\pi\beta_m=\tan^{-1}(q^m)$ in the above formula, and we have

$$
\tan^{-1}\bigl(q^m\bigr)\log\bigl(q^m\bigr)+\sum_{n=0}^{\infty}\frac{\sin(4n+2)\pi\beta_m}{(2n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(q^m)^{2n+1}}{(2n+1)^2}.
$$

Hence, we get

$$
\begin{aligned}
m_{1,q}\left(1+\frac{1+x}{1-x}y\right) &= \frac{2(1-q)}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(q^{2n+1})^m}{(2n+1)^2} \\
&= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\cdot\frac{1-q}{1-q^{2n+1}} \\
&= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2[2n+1]_q}.
\end{aligned}
$$

The proof is finished. □

Theorem 10 is easily proved from Theorem 9.

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References

[BJ] B.C. Berndt, P.T. Joshi, Ramanujan's Second Notebook, Infinite Series Identities, Transformations, and Evaluations, Contemp. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1983, vii+83 pp. (Chapter 9).

[B] D.W. Boyd, Mahler's measure and special values of L-functions, Experiment. Math. 7 (1998) 37–82.

[BRV] D.W. Boyd, F. Rodriguez-Villegas, Mahler's measures and the dilogarithm (I), Canad. J. Math. 54 (2002) 468–492.

[D] C. Deninger, Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions, J. Amer. Math. Soc. 10 (2) (1997) 259–281.
[EW] G. Everest, T. Ward, Heights of Polynomials and Entropy in Algebraic Dynamics, Springer, 1999.
[GO] Y. Gon, H. Oyanagi, Generalized Mahler measures and multiple sine functions, Internat. J. Math. 15 (5) (2004) 425–442.
[GR] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, Boston, 1994.
[K1] N. Kurokawa, A q-Mahler measure, Proc. Japan Acad. Ser. A 80 (2004) 70–73.
[K2] N. Kurokawa, Quantum deformations of Catalan's constant, Mahler's measures, and Hölder–Shintani's double sine functions, Proc. Edinb. Math. Soc. 49 (2006) 667–681.
[KK] N. Kurokawa, S. Koyama, Multiple sine functions, Forum Math. 15 (2003) 839–876.
[KO] N. Kurokawa, H. Ochiai, Mahler measures via the crystallization, Comment. Math. Univ. St. Pauli 54 (2) (2005) 121–137.
[L1] M. Lalin, Some examples of Mahler measures as multiple polylogarithms, J. Number Theory 103 (1) (2003) 85–108.
[L2] M. Lalin, Mahler measure of some n-variable polynomial families, J. Number Theory 116 (1) (2006) 102–139.
[LSW] D. Lind, K. Schmidt, T. Ward, Mahler measure and entropy for commuting automorphisms of compact groups, Invent. Math. 101 (1990) 593–629.
[M] K. Mahler, An application of Jensen's formula to polynomials, Mathematika 7 (1960) 98–100.
[O1] H. Oyanagi, Differential equations for Mahler measures, J. Ramanujan Math. Soc. 18 (2) (2003) 181–194.
[O2] H. Oyanagi, Q-analogues of Mahler measures, J. Ramanujan Math. Soc. 19 (3) (2004) 203–212.
[RV] F. Rodriguez-Villegas, Modular Mahler measures I, in: Topics in Number Theory, University Park, PA, 1997, in: Math. Appl., vol. 467, Kluwer Acad. Publ., Dordrecht, 1999, pp. 17–48.
[S] C.J. Smyth, On measures of polynomials in several variables, Bull. Austral. Math. Soc. 23 (1981) 49–63.
[ZW] N.Y. Zhang, K.S. Williams, Values of the Riemann zeta function and integrals involving $\log(2\sinh\frac{\theta}{2})$ and $\log(2\sin\frac{\theta}{2})$, Pacific J. Math. 168 (2) (1995) 271–289.