Multiple $q$-Mahler measures and zeta functions

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Abstract

We introduce multiple $q$-Mahler measures and we calculate some specific examples, where multiple $q$-analogues of zeta functions appear. We study also limits as the multiple $q$ goes to 1.

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1. Introduction

The Mahler measure was introduced by Mahler [M] in studying transcendental numbers and it is defined by

$$m(f) = \text{Re} \int_0^1 \cdots \int_0^1 \log(f(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})) \, d\theta_1 \cdots d\theta_n$$

for a rational function $f(x_1, \ldots, x_n) \in \mathbb{C}(x_1, \ldots, x_n)$. It is known that Mahler measure has interesting connections to special values of zeta functions as explained in Smyth [S], Boyd [B] and Deninger [D]. There are also some results connected with dynamical systems in Lind–Schmidt–Ward [LSW]. We refer to Everest–Ward [EW] for a survey of the Mahler measure.
The \( q \)-analogue \( m_q(f) \) of \( m(f) \) for \( q \in \mathbb{C} \) was introduced in [K1] and some calculations are in [O2] also.

In this paper we introduce multiple \( q \)-Mahler measures, where \( q = (q_1, \ldots, q_r) \). We deal with the following cases:

(i) \( 0 < q_1, \ldots, q_r < 1 \).
(ii) \( q = (1, q) \) with \( 0 < q < 1 \).

In case (i), the \( q \)-logarithm function is defined by

\[
\ell_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x - 1)^n}{[n]_q}
\]

for \( |x - 1| < 1 \) originally, where \([n]_q = [n]_{q_1} \cdots [n]_{q_r} = \frac{1 - q^n}{1 - q} \cdots \frac{1 - q^n}{1 - q^n} \). We note that

\[
\lim_{q_1, \ldots, q_r \uparrow 1} \ell_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x - 1)^n}{n^r} = -\text{Li}_r(1 - x) \quad \text{in} \quad |x - 1| < 1,
\]

where \( \text{Li}_r(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^r} \) is the polylogarithm function, since \( \lim_{q_j \uparrow 1} [n]_{q_j} = n \). Moreover, \( \ell_q(x) \) has an analytic continuation to all \( x \in \mathbb{C} \) as a meromorphic function via the expression

\[
\ell_q(x) = (1 - q_1) \cdots (1 - q_r) \sum_{m_1, \ldots, m_r \geq 0} \frac{x - 1}{x - 1 + q_1^{-m_1} \cdots q_r^{-m_r}}
\]

which is easily deduced from the definition.

**Definition 1.** Let \( 0 < q_j < 1 \). Then the \( q \)-Mahler measure \( m_q(f) \) of a rational function \( f(x_1, \ldots, x_n) \in \mathbb{C}(x_1, \ldots, x_n) \) is defined as

\[
m_q(f) = \text{Re} \int_0^1 \cdots \int_0^1 \ell_q(f(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})) \, d\theta_1 \cdots d\theta_n.
\]

Then, we have the following results.

**Theorem 1.** For \( 0 < a \leq 1 \),

\[
m_q(az^x + 1) = \frac{4}{\pi^2} \sum_{n \geq 1} \frac{a^n}{[n]_q n^2}.
\]

In particular, when \( a = 1 \),

\[
m_q(z^x + 1) = \frac{4}{\pi^2} \sum_{n \geq 1} \frac{1}{[n]_q n^2}.
\]
Theorem 2.
\[
\lim_{q \uparrow 1} m_q \left( \frac{x+1}{y+1} + 1 \right) = \frac{4}{\pi^2} \sum_{n \geq 1 \text{ odd}} \frac{1}{n^{r+2}} = \frac{4}{\pi^2} (1 - 2^{-2r}) \zeta(r + 2),
\]
where \( \mathbf{1}_r = (1, \ldots, 1) \in \mathbb{R}^r \).

Theorem 3. For \( 0 < a \leq 1 \),
\[
m_q \left( ay \frac{x+1}{x-1} + 1 \right) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)[2n+1]_q}.
\]
In particular, when \( a = 1 \),
\[
m_q \left( \frac{y}{x-1} + 1 \right) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[2n+1]_q}.
\]

Theorem 4.
\[
\lim_{q \uparrow 1} m_q \left( \frac{y}{x-1} + 1 \right) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[2n+1]_q} = \frac{2}{\pi} L(r+1, \chi_{-4}).
\]
Here, \( \chi_{-4} = (\frac{-4}{x}) \) is the non-trivial Dirichlet character of modulo 4.

Secondly we deal with the case \( q = (1, q) \) with \( 0 < q < 1 \). In this case, the \( (1, q) \)-logarithm function is defined by
\[
l_{1,q}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{[n]_{1,q}}
\]
in \( |x-1| < 1 \) originally, where \([n]_{1,q} = n \cdot [n]_q = n \cdot \frac{1-q^n}{1-q}\). We note that for \( |x-1| < 1 \)
\[
\lim_{q \uparrow 1} l_{1,q}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n^2} = -\text{Li}_2(1-x)
\]
and
\[
\lim_{q \downarrow 0} l_{1,q}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} = \log x.
\]
where \( Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \) is the dilogarithm function, since \( \lim_{q \uparrow 1} [n]_1,q = n^2 \) and \( \lim_{q \downarrow 0} [n]_{1,q} = n \). Moreover, \( l_{1,q}(x) \) has an analytic continuation to all \( x \in \mathbb{C} \) via the expression

\[
l_{1,q}(x) = (1 - q) \sum_{m=0}^{\infty} \log(q^n(x - 1) + 1),
\]

which is easily deduced from the definition. We remark that \( l_{1,q}(x) \) has the following expression:

\[
l_{1,q}(x) = (1 - q) \log \left( \sum_{j=0}^{\infty} \frac{q^j (x-1)^j}{(1-q)(1-q^2) \cdots (1-q^j)} \right).
\]

**Definition 2.** Let \( 0 < q < 1 \). Then the \( (1,q) \)-Mahler measure \( m_{1,q}(f) \) of a rational function \( f(x_1, \ldots, x_n) \in \mathbb{C}(x_1, \ldots, x_n) \) is defined as

\[
m_{1,q}(f) = \text{Re} \int_0^1 \cdots \int_0^1 l_{1,q}(f(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})) \, d\theta_1 \cdots d\theta_n.
\]

Then, we have the following results.

**Theorem 5.** For \( a \geq 1 \),

\[
m_{1,q}(x + a) = l_{1,q}(a).
\]

**Theorem 6.**

\[
m_{1,q}(1 + (1 + x)y) = \frac{1 - q}{\pi} \sum_{m=0}^{\left\lfloor \frac{\log_2\left(\frac{1}{\log_2 q}\right)}{\log_2(1)} \right\rfloor} 2\pi \alpha_m \log(q^m) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(2n\pi \alpha_m)}{n^2},
\]

with \( \alpha_m = \frac{1}{\pi} \cos^{-1}\left(\frac{1}{2q^m}\right) \).

**Theorem 7.** Let \( \chi_{-3} = (\frac{-3}{*}) \), \( \chi_{-4} = (\frac{-4}{*}) \) be the Dirichlet characters of modulo 3 and 4, respectively.

(i) \[
\lim_{q \downarrow 0} m_{1,q}(1 + (1 + x)y) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}) = L'(-1, \chi_{-3}).
\]

(ii) \[
m_{1,2}(1 + (1 + x)y) = \frac{3\sqrt{3}}{8\pi} L(2, \chi_{-3}) = \frac{1}{2} L'(-1, \chi_{-3}).
\]

(iii) \[
m_{1,\frac{1}{\sqrt{3}}}(1 + (1 + x)y) = \left(1 - \frac{1}{\sqrt{3}}\right) \left\{ \frac{5\sqrt{3}}{4\pi} L(2, \chi_{-3}) - \frac{1}{6} \log 3 \right\}
\]

\[
= \left(1 - \frac{1}{\sqrt{3}}\right) \left\{ \frac{5}{3} L'(-1, \chi_{-3}) - \frac{1}{6} \log 3 \right\}.
\]
\( m_{1, \frac{1}{\sqrt{2}}} (1 + (1 + x)y) = \left( 1 - \frac{1}{\sqrt{2}} \right) \left\{ \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}) + \frac{1}{\pi} L(2, \chi_{-4}) - \frac{1}{4} \log 2 \right\} = \left( 1 - \frac{1}{\sqrt{2}} \right) \left\{ L'(-1, \chi_{-3}) + \frac{1}{2} L'(-1, \chi_{-4}) - \frac{1}{4} \log 2 \right\}. \)

Theorem 8.

\[
\lim_{q \uparrow 1} m_{1,q} (1 + (1 + x)y) = \frac{5}{81} \pi^2 + \frac{1}{\pi} \sum_{n=1}^\infty \frac{c(n)}{n^2} = \frac{5}{162} \pi^2 - \frac{1}{\pi} \text{Im Li}_1(e^{\pi i/3}, 1).
\]

Here \( c(n) = \sum_{k=1}^n \frac{(-1)^k \sin \left( \frac{2k\pi}{3} \right)}{k} \) and the doublepolylogarithm function \( \text{Li}_{k,l}(u, v) \) is defined as \( \text{Li}_{k,l}(u, v) = \sum_{1 \leq m < n} \frac{u^m v^n}{m^k n^l}. \)

Theorem 9.

\[
m_{1,q} \left( 1 + \frac{1 + x}{1 - x} y \right) = \frac{2}{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{(2n + 1)^2 [2n + 1]_q}.
\]

Theorem 10.

(i) \( \lim_{q \downarrow 0} m_{1,q} \left( 1 + \frac{1 + x}{1 - x} y \right) = \frac{2}{\pi} L(2, \chi_{-4}) = L'(-1, \chi_{-4}). \)

(ii) \( \lim_{q \uparrow 1} m_{1,q} \left( 1 + \frac{1 + x}{1 - x} y \right) = \frac{2}{\pi} L(3, \chi_{-4}) = \frac{\pi^2}{16}. \)

This paper belongs to a series of papers making attempts to extend Mahler measures as in [GO,K1,K2,KO,O1,O2] mainly related to multiple sine functions and their generalizations. Our multiple \( q \)-deformation examples are mainly suggested by the usual \( q \)-deformation examples treated in [K1,K2,KO,O2]. In particular, the fractional form is crucial for \( q \)-deformations in comparison with the usual case. We also note that the continuity of \( m_q(f) \) in \( q \) as \( q \to 1 \) is a non-trivial problem in general since we must pay attention to complicated analytic continuation of associated \( 1 \)-logarithm (polylogarithm) and in our examples we calculated the limits only in principle. We hope to treat this problem in a future paper.

In some cases our results give multiple \( q \)-deformations of known formulas. For example, the following four results Theorem 2 \( (r = 1) \), Theorem 4 \( (r = 1) \), Theorems 7(i) and 10(i) specialize to formulas of Smyth [S]. We refer to recent interesting papers of Boyd–Rodriguez-Villegas [BRV], Lalin [L1,L2] and Rodriguez-Villegas [RV] for various generalizations different from our way.

2. \((q_1, \ldots, q_r)\)-Mahler measures

Proofs of Theorems 1 and 2. From the definition, we have
Here we put $q = q_m^1 \cdots q_m^r$, then

$$
\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z + \frac{y+1}{aq_m(x+1)}} \, dz = \begin{cases} 
1 & \text{if } |\frac{y+1}{aq_m(x+1)}| < 1, \\
0 & \text{if } |\frac{y+1}{aq_m(x+1)}| \geq 1.
\end{cases}
$$

Hence, for $x = e^{2\pi i t_1}$, $y = e^{2\pi i t_2}$ with $-\frac{1}{2} < t_1 < \frac{1}{2}$, $-\frac{1}{2} < t_2 < \frac{1}{2}$, we know that $|\frac{y+1}{aq_m(x+1)}| < 1$ if and only if $aq_m \cos \pi t_1 > \cos \pi t_2$.

Then, putting $\alpha_m(t_1) = \frac{1}{\pi} \cos^{-1}(aq_m \cos \pi t_1)$ we have

$$
m_q \left( \frac{ax + 1}{y + 1} + 1 \right) \\
= (1 - q_1) \cdots (1 - q_r) \sum_{m_1, \ldots, m_r \geq 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_m(t_1) \, dt_2 \right) dt_1 \\
= 4(1 - q_1) \cdots (1 - q_r) \sum_{m_1, \ldots, m_r \geq 0} \int_{0}^{\frac{1}{2}} \left( \frac{1}{2} - \alpha_m(t_1) \right) dt_1.
$$

Using

$$
\cos^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{(2n)!!} \frac{x^{2n+1}}{2n + 1}
$$

and

$$
(1 - q_j) \sum_{m_j=0}^{\infty} q_j^{(2n+1)m_j} = \frac{1 - q_j}{1 - q_j^{2n+1}} = \frac{1}{[2n+1] q_j},
$$

we obtain

$$
m_q \left( \frac{ax + 1}{y + 1} + 1 \right) \\
= 4(1 - q_1) \cdots (1 - q_r) \sum_{m_1, \ldots, m_r \geq 0} \int_{0}^{\frac{1}{2}} \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{(2n)!!} \frac{(aq_m \cos \pi t_1)^{2n+1}}{2n + 1} dt_1.
$$
\[
\frac{4(1 - q_1) \cdots (1 - q_r)}{\pi} \sum_{m_1, \ldots, m_r \geq 0} \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{(2n)!!} \frac{\pi (2n)!!}{2n + 1} \frac{1}{\pi (2n + 1)!!}
\]
\[
= \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(2n+1)^2} \prod_{j=1}^{r} \frac{1}{[2n+1]q_j}
\]
\[
= \frac{4}{\pi^2} \sum_{n \geq 1} \frac{a^n}{n^2[n]_q}.
\]

So we have proved Theorem 1.
Theorem 2 is easily proved from Theorem 1. \qed

**Proofs of Theorems 3 and 4.** From the definition, we have

\[
m_q \left( ay \frac{x + 1}{x - 1} + 1 \right) = \text{Re} \left( \frac{(1 - q_1) \cdots (1 - q_r)}{(2\pi i)^2} \int_{y_ \leq 1, |y| = 1} \sum_{m_1, \ldots, m_r \geq 0} \frac{a^{y+1}}{x - 1} + q_1^{-m_1} \cdots q_r^{-m_r} \frac{dx \, dy}{x \, y} \right).
\]

Here we put \( q_m = q_1^{m_1} \cdots q_r^{m_r} \), then

\[
\frac{1}{2\pi i} \int_{y \leq 1} \frac{1}{y + aq^{m}(x+1)} \, dy = \begin{cases} 
1 & \text{if } |\frac{x-1}{aq^{m}(x+1)}| < 1, \\
0 & \text{if } |\frac{x-1}{aq^{m}(x+1)}| \geq 1.
\end{cases}
\]

Hence, for \( x = e^{2\pi it} \), with \(-\frac{1}{2} < t < \frac{1}{2}\), we know that \( |\frac{x-1}{aq^{m}(x+1)}| < 1 \) if and only if \( |\tan(\pi t)| < aq^{m} \).

Then, putting \( \beta_{m,a} = \frac{1}{\pi} \tan^{-1}(aq^{m}) \) we have

\[
m_q \left( ay \frac{x + 1}{x - 1} + 1 \right) = (1 - q_1) \cdots (1 - q_r) \sum_{m_1, \ldots, m_r \geq 0} \int_{-\beta_{m,a}}^{\beta_{m,a}} dt
\]
\[
= \frac{2}{\pi} (1 - q_1) \cdots (1 - q_r) \sum_{m_1, \ldots, m_r \geq 0} \tan^{-1}(aq^{m}).
\]

Using

\[
\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1}
\]
and

\[(1 - q_j) \sum_{m_j=0}^{\infty} q_j^{(2n+1)m_j} = \frac{1 - q_j}{1 - q_j^{2n+1}} = \frac{1}{[2n+1]q_j},\]

we obtain

\[
m_q\left(ay \frac{x + 1}{x - 1} + 1\right) = \frac{2}{\pi} (1 - q_1) \cdots (1 - q_r) \left\{ \tan^{-1}(a) + \sum_{m_1, \ldots, m_r \geq 0}^{\infty} \sum_{(m_1, \ldots, m_r) \neq (0, \ldots, 0)} \frac{(-1)^n}{2n + 1} (aq_m^{2n+1}) \right\}
\]

\[
= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} a^{2n+1} \prod_{j=1}^{r} \frac{1}{[2n+1]q_j}
\]

\[
= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)[2n+1]q_j}.
\]

So we have proved Theorem 3.

Theorem 4 is easily proved from Theorem 3. \(\square\)

3. \((1, q)\)-Mahler measures

**Proof of Theorem 5.** By definition and Jensen’s formula, we have

\[
m_{1, q}(x + a)
\]

\[
= (1 - q) \sum_{m=0}^{\infty} \text{Re} \int_{0}^{1} \{ \log(e^{2\pi i t} + a - 1 + q^{-m}) - \log(q^{-m}) \} \, dt
\]

\[
= (1 - q) \sum_{m=0}^{\infty} \{ \log^+(a - 1 + q^{-m}) - \log(q^{-m}) \}
\]

\[
= (1 - q) \sum_{m=0}^{\infty} \log(q^n(a - 1) + 1)
\]

\[
= l_{1, q}(a).
\]

Here, \(\log^+ x = \max[\log x, 0]\) for \(x > 0.\) \(\square\)
Proof of Theorem 6. By Jensen’s formula,

\[ m_{1,q}(1 + (1 + x)y) = (1 - q) \sum_{m=0}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log|q^m(e^{2\pi it}(1 + e^{2\pi it})) + 1| dt_1 dt_2 \]

\[ = (1 - q) \sum_{m=0}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log^+|q^m(1 + e^{2\pi it})| dt_1. \]

Let \( \alpha_m \) be a real number such that \( 0 \leq \alpha_m < 1/2 \) and \( 2q^m \cos(\pi \alpha_m) = 1 \) for \( m \leq -\log \frac{2}{\log q}. \) For an integer \( m \leq -\log \frac{2}{\log q} \), put

\[ I_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log^+|q^m(1 + e^{2\pi it})| dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log^+|2q^m(\cos \pi t)| dt. \]

Integrating by parts, we obtain

\[ I_m = 2 \int_{0}^{\alpha_m} \{ \log(2q^m) + \log(\cos \pi t) \} dt \]

\[ = 2 \left\{ \alpha_m \log(2q^m) + [t \log(\cos \pi t)]_0^{\alpha_m} + \int_{0}^{\alpha_m} \pi t \tan \pi t dt \right\} \]

\[ = 2 \int_{0}^{\alpha_m} \pi t \tan \pi t dt. \]

We evaluate \( I_m \) by using the double sine function \( S_2(z) \) in [KK]. Firstly, by using the formula (Theorem 2.5 in [KK, p.847])

\[ \log S_2(z) = \int_{0}^{z} \pi t \cot \pi t dt \]

and

\[ \log S_2\left(z + \frac{1}{2}\right) + \log S_2\left(z - \frac{1}{2}\right) = -2 \int_{0}^{z} \pi t \tan \pi t dt \]

we find that

\[ I_m = - \log \left( S_2\left(\alpha_m + \frac{1}{2}\right) S_2\left(\alpha_m - \frac{1}{2}\right) \right). \]
Secondly we use the functional equation of the double sine function (Theorem 2.10 and Examples 2.12 in [KK, pp. 852–854]):

\[ S_2(x + 1) = -S_2(x) \cdot 2 \sin(\pi x). \]

This formula gives that

\[
S_2 \left( \alpha_m + \frac{1}{2} \right)^2 = S_2 \left( \alpha_m + \frac{1}{2} \right) S_2 \left( \alpha_m - \frac{1}{2} \right) \cdot 2 \cos(\pi \alpha_m)
\]
\[ = S_2 \left( \alpha_m + \frac{1}{2} \right) S_2 \left( \alpha_m - \frac{1}{2} \right) \cdot q^{-m}. \]

Hence we obtain

\[ I_m = - \log(q^m) - 2 \log S_2 \left( \alpha_m + \frac{1}{2} \right). \]

Thirdly we use the formula (Theorem 2.8 and Examples 2.9 in [KK, pp. 849–852]):

\[ S_2(x) = (2 \sin \pi x)^x \exp \left( \frac{1}{2\pi} \sum_{n=1}^{\infty} \sin(2\pi nx) \frac{n^2}{n^2} \right) \]
for \(0 < x < 1\). Note that \(0 < \alpha_m + 1/2 < 1\). This gives

\[ \log S_2 \left( \alpha_m + \frac{1}{2} \right) = \left( \alpha_m + \frac{1}{2} \right) \log(2 \cos \pi \alpha_m) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \sin(2\pi n\alpha_m + \pi n) \frac{n^2}{n^2} \]
\[ = - \left( \alpha_m + \frac{1}{2} \right) \log(q^m) + \frac{1}{2\pi} \sum_{n=1}^{\infty} (-1)^n \sin(2\pi n\alpha_m) \frac{n^2}{n^2}. \]

At last we have

\[ I_m = 2\alpha_m \log(q^m) + \frac{1}{2\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \sin(2\pi n\alpha_m) \frac{n^2}{n^2}. \]

Since \(m_{1,q}(1 + (1 + x)y) = (1 - q) \sum_{m=0}^{\left\lfloor \frac{\log 2}{\log q} \right\rfloor} I_m\), the proof is completed. \(\square\)

**Proof of Theorem 7.** Let \(\alpha_m\) and \(I_m\) be as in the proof of Theorem 6.

(i) We note that \(\lfloor -\log 2/\log q \rfloor = 0\) for \(0 < q < 1/2\). Therefore we see that

\[ m_{1,q}(1 + (1 + x)y) = (1 - q)I_0 \quad (0 < q < 1/2). \]

The value \(\pi \alpha_0 = \cos^{-1}(1/2q^0) = \pi/3\) implies that
\[ I_0 = 2\alpha_0 \log(q^0) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin\left(\frac{2\pi n}{3}\right)}{n^2} \]
\[ = \frac{\sqrt{3}}{2\pi} \left( \sum_{n=1}^{\infty} \frac{\chi_3(n)}{n^2} - 2 \sum_{n=1}^{\infty} \frac{\chi_3(2n)}{(2n)^2} \right) \]
\[ = \frac{\sqrt{3}}{2\pi} \left( 1 + \frac{2}{4} \right)L(2, \chi_3) = \frac{3\sqrt{3}}{4\pi}L(2, \chi_3). \]

Thus we obtain

\[ \lim_{q \downarrow 0} m_{1,q} \left( 1 + (1 + x)y \right) = I_0 = \frac{3\sqrt{3}}{4\pi}L(2, \chi_3). \]

(ii) If \( q = 1/2 \) then \([-\log \frac{2}{\log q}] = 1\). In this case, \( \pi \alpha_1 = \cos^{-1}(1/2q) = 0 \). So we have \( I_1 = 0 \) and

\[ m_{1,\frac{1}{2}} \left( 1 + (1 + x)y \right) = \left( 1 - \frac{1}{2} \right)(I_0 + I_1) = \frac{3\sqrt{3}}{8\pi}L(2, \chi_3). \]

(iii) If \( q = 1/\sqrt{3} \) then \([-\log \frac{2}{\log q}] = 1\). In this case, \( \pi \alpha_1 = \cos^{-1}(1/2q) = \pi/6 \) and

\[ I_1 = 2\alpha_1 \log(q) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin\left(\frac{\pi n}{3}\right)}{n^2} \]
\[ = -\frac{1}{6} \log 3 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2\pi n}{3}\right)}{n^2} \]
\[ = -\frac{1}{6} \log 3 + \frac{\sqrt{3}}{2\pi}L(2, \chi_3). \]

Thus we have

\[ m_{1,\frac{1}{\sqrt{3}}} \left( 1 + (1 + x)y \right) = \left( 1 - \frac{1}{\sqrt{3}} \right)(I_0 + I_1) \]
\[ = \left( 1 - \frac{1}{\sqrt{3}} \right) \left\{ \frac{3\sqrt{3}}{4\pi}L(2, \chi_3) - \frac{1}{6} \log 3 + \frac{\sqrt{3}}{2\pi}L(2, \chi_3) \right\} \]
\[ = \left( 1 - \frac{1}{\sqrt{3}} \right) \left\{ \frac{5\sqrt{3}}{4\pi}L(2, \chi_3) - \frac{1}{6} \log 3 \right\}. \]

(iv) If \( q = 1/\sqrt{2} \) then \([-\log \frac{2}{\log q}] = 2\). In this case, \( \pi \alpha_1 = \cos^{-1}(1/2q) = \pi/4 \) and \( \pi \alpha_2 = \cos^{-1}(1/2q^2) = 0 \). Thus \( I_2 = 0 \) and we can easily check that

\[ I_1 = -\frac{1}{4} \log 2 + \frac{1}{\pi}L(2, \chi_{-4}) \]

in the same way. The proof is finished. \[ \Box \]
Proof of Theorem 8. At first we show that

\[ \lim_{q \uparrow 1} m_{1,q} (1 + (1 + x)y) = \frac{2}{\pi} J_1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - a(n)}{n^2} \]

with

\[ J_1 = \int_0^{1/2} \cos^{-1}\left( \frac{1}{2t} \right) \frac{\log t}{t} \ dt \]

and

\[ a(n) = \int_0^{1/2} \frac{\sin(2n \cos^{-1}\left( \frac{1}{2t} \right))}{t} \ dt. \]

This is proved by the Jackson integral. Since \( \lim_{q \uparrow 1} q \left[ \log \frac{2}{\log q} \right] = 1/2 \), we find that

\[ \lim_{q \uparrow 1} (1 - q) \sum_{m=0}^{\left[ \log \frac{2}{\log q} \right]} q^m \frac{\cos^{-1}\left( \frac{1}{2q^m} \right) \log(q^m)}{q^m} = J_1 \]

and

\[ \lim_{q \uparrow 1} (1 - q) \sum_{m=0}^{\left[ \log \frac{2}{\log q} \right]} q^m \frac{\sin(2n \cos^{-1}\left( \frac{1}{2q^m} \right))}{q^m} = a(n). \]

Secondly we evaluate the integral \( J_1 \). We make following changes of variables. Let \( u = \cos^{-1}\left( (2t)^{-1} \right) \), then \( \cos u = (2t)^{-1} \), \( dt = 2t \sin u \ du \) and \( dt/t = \tan u \ du \). We find that

\[ J_1 = \int_0^{\frac{\pi}{2}} u \log((2 \cos u)^{-1}) \tan u \ du \]

\[ = - \int_0^{\frac{\pi}{2}} u \log(2 \cos u) - (2 \cos u)' \log(2 \cos u) \ du \]

\[ = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left\{ \log^2(2 \cos u) \right\}' u \ du \]
\[\begin{align*}
= \frac{1}{2} \left[ u \log^2(2 \cos u) \right]_0^{\frac{\pi}{3}} - \frac{1}{2} \int_0^{\frac{\pi}{3}} \log^2(2 \cos u) \, du \\
= -\frac{1}{2} \int_0^{\frac{\pi}{3}} \log^2(2 \cos u) \, du.
\end{align*}\]

After the change of variable \( t = \frac{\pi}{2} - u \), we see that
\[J_1 = -\frac{1}{2} \int_0^{\frac{\pi}{3}} \log^2(2 \sin t) \, dt\]
\[= -\frac{1}{2} \int_0^{\frac{\pi}{3}} \log^2(2 \sin t) \, dt + \frac{1}{2} \int_0^{\frac{\pi}{3}} \log^2(2 \sin t) \, dt\]
\[= -\frac{1}{2} \left( \frac{\pi^3}{24} - \frac{7\pi^3}{216} \right)\]
\[= -\frac{\pi^3}{216}.\]

In the above calculation, we used the fact that \( \int_0^{\pi/2} \log^2(2 \sin t) \, dt = \frac{\pi^3}{24} \) (see 4.224 no. 7 in [GR, p. 559]) and \( \int_0^{\pi/6} \log^2(2 \sin t) \, dt = \frac{7\pi^3}{216} \) (see (1.3) in [ZW, p. 272]).

Thirdly, we calculate \( a(n) \). Changing the variable \( u = \cos^{-1}((2t)^{-1}) \), we obtain
\[a(n) = \int_0^{\frac{\pi}{3}} \sin(2nu) \frac{\sin u}{\cos u} \, du\]
\[= \frac{1}{2} \int_0^{\frac{\pi}{3}} \left\{ \frac{\cos(2n-1)u}{\cos u} - \frac{\cos(2n+1)u}{\cos u} \right\} \, du\]
\[= \frac{1}{2} \left[ \sum_{k=1}^{n-1} (-1)^{n-1-k} \frac{\sin 2ku}{k} + (-1)^{n-1}u - \sum_{k=1}^{n} (-1)^{n-k} \frac{\sin 2ku}{k} - (-1)^nu \right]_0^{\frac{\pi}{3}}\]
\[= \frac{1}{2} (-1)^{n-1} \left[ 2 \sum_{k=1}^{n} (-1)^k \frac{\sin 2ku}{k} + 2u - \frac{(-1)^n \sin(2nu)}{n} \right]_0^{\frac{\pi}{3}}\]
\[= (-1)^{n-1} c(n) + \frac{\pi}{3} (-1)^{n-1} + \frac{1}{2} \sin \frac{2\pi n}{3}\]
with \( c(n) = \sum_{k=1}^{n} (-1)^k \frac{\sin \frac{2\pi n}{3}}{k} \).
Therefore, we have

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} a(n)}{n^2} = \sum_{n=1}^{\infty} \frac{c(n)}{n^2} + \frac{\pi}{3} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin \frac{2\pi n}{3}}{n^3} \\
= \sum_{n=1}^{\infty} \frac{c(n)}{n^2} + \frac{\pi}{3} \xi(2) + \frac{1}{2} \cdot \sqrt{3} \cdot \left( \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^3} - 2 \sum_{n=1}^{\infty} \frac{\chi_{-3}(2n)}{(2n)^3} \right) \\
= \sum_{n=1}^{\infty} \frac{c(n)}{n^2} + \frac{\pi}{3} \xi(2) + \frac{5 \sqrt{3}}{16} L(3, \chi_{-3}).
\]

Since \( \xi(2) = \frac{\pi^2}{6} \) and \( L(3, \chi_{-3}) = \frac{4\pi^3}{81\sqrt{3}} \), we get

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} a(n)}{n^2} = \sum_{n=1}^{\infty} \frac{c(n)}{n^2} + \frac{\pi}{3} \cdot \frac{\pi^2}{6} + \frac{5 \sqrt{3}}{16} \cdot \frac{4\pi^3}{81\sqrt{3}}
\]

\[
= \sum_{n=1}^{\infty} \frac{c(n)}{n^2} + \frac{23}{324} \pi^3.
\]

At last we have

\[
\lim_{q \uparrow 1} m_{1,q} \left( 1 + (1 + x)y \right) = \frac{2}{\pi} J_1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a(n)}{n^2}
\]

\[
= \frac{2}{\pi} \cdot \frac{\pi^3}{216} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{c(n)}{n^2} + \frac{1}{\pi} \frac{23}{324} \pi^3
\]

\[
= \frac{5}{81} \pi^2 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{c(n)}{n^2} = \frac{5}{162} \pi^2 - \frac{1}{\pi} \text{Im Li}_{1,2} \left( e^{\frac{\pi i}{3}}, 1 \right).
\]

The proof is finished. \( \square \)

**Proof of Theorem 9.** By Jensen’s formula,

\[
m_{1,q} \left( 1 + \frac{1+x}{1-x} y \right) = (1-q) \sum_{m=0}^{\infty} \int_{-\frac{1}{2}}^{1/2} \int_{-\frac{1}{2}}^{1/2} \log \left| q^m \left( \frac{1 + e^{2\pi i t_1}}{1 - e^{2\pi i t_1}} e^{2\pi i t_2} \right) \right| dt_1 dt_2
\]

\[
= (1-q) \sum_{m=0}^{\infty} \int_{-\frac{1}{2}}^{1/2} \log^+ \left| q^m \cot \pi t_1 \right| dt_1.
\]
For a non-negative integer $m$, we put

$$I_m = \int_{\frac{-1}{2}}^{\frac{1}{2}} \log^+ |q^m \cot \pi t| \, dt$$

and we evaluate the integral $I_m$. We know that $q^m |\cot \pi t| > 1$ if and only if $|\tan \pi t| < q^m$. Let $\beta_m = \frac{1}{\pi} \tan^{-1}(q^m)$. Note that $0 \leq \beta_m \leq 1/4$ for $m \geq 0$. We find that

$$I_m = \int_{-\beta_m}^{\beta_m} \{ \log(q^m) + \log(\cos \pi t) - \log(\sin \pi t) \} \, dt$$

$$= 2\beta_m \log(q^m) + 2 \int_{0}^{\beta_m} \{ \log(\cos \pi t) - \log(\sin \pi t) \} \, dt.$$

Integration by parts gives:

$$\int_{0}^{t} \log(\cos \pi t) \, dt = t \log(\cos \pi t) + \int_{0}^{t} \pi t \tan(\pi t) \, dt,$$

$$\int_{0}^{t} \log(\sin \pi t) \, dt = t \log(\sin \pi t) - \int_{0}^{t} \pi t \cot(\pi t) \, dt.$$

Therefore, we see that

$$I_m = 2\beta_m \log(q^m) + 2 \left[ t \log(\cot \pi t) + \int_{0}^{t} \{ \pi t \cot \pi t + \pi t \tan \pi t \} \, dt \right]_{0}^{\beta_m}$$

$$= 2\beta_m \log(q^m) + 2\beta_m \log(1/q^m) + 2 \int_{0}^{\beta_m} \{ \pi t \cot \pi t + \pi t \tan \pi t \} \, dt$$

$$= 2 \int_{0}^{\beta_m} \{ \pi t \cot \pi t + \pi t \tan \pi t \} \, dt.$$

We evaluate $I_m$ by using the various formulas of $S_2(z)$ similar to the proof of Theorem 6. Firstly, noting that

$$\int_{0}^{\beta_m} \pi t \cot \pi t \, dt = \log S_2(\beta_m),$$
\[
\beta_m \int_0^{\pi t} \tan \pi t \, dt = -\frac{1}{2} \log \left( S_2 \left( \beta_m + \frac{1}{2} \right) S_2 \left( \beta_m - \frac{1}{2} \right) \right)
\]

and

\[
S_2 \left( \beta_m + \frac{1}{2} \right) = S_2 \left( \beta_m - \frac{1}{2} \right) \cdot 2 \cos(\pi \beta_m),
\]

we obtain

\[
\frac{1}{2} I_m = \int_0^{\beta_m} \pi t \cot \pi t \, dt + \int_0^{\beta_m} \pi t \tan \pi t \, dt
\]

\[
= \log S_2(\beta_m) - \log \left\{ \frac{1}{2} \cos \pi \beta_m \right\} S_2 \left( \beta_m + \frac{1}{2} \right)^2 \}
\]

\[
= \frac{1}{2} \log(2 \cos \pi \beta_m) + \log \left\{ S_2(\beta_m) S_2 \left( \beta_m + \frac{1}{2} \right)^{-1} \right\}.
\]

Secondly, utilizing the expansion formula of the double sine function (Theorem 2.8 and Examples 2.9 in [KK, pp. 849–852])

\[
\log S_2(x) = x \log(2 \sin \pi x) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^2} \quad (0 < x < 1),
\]

we see that

\[
\log S_2(x) - \log S_2 \left( x + \frac{1}{2} \right)
\]

\[
= x \log(2 \sin \pi x) - \left( x + \frac{1}{2} \right) \log(2 \cos \pi x) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx) - \sin(2\pi nx + \pi n)}{n^2}
\]

\[
= x \log(\tan \pi x) - \frac{1}{2} \log(2 \cos \pi x) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx) - (-1)^n \sin(2\pi nx)}{n^2}.
\]

Thus we have

\[
\frac{1}{2} I_m = \frac{1}{2} \log(2 \cos \pi \beta_m) + \beta_m \log(\tan \pi \beta_m) - \frac{1}{2} \log(2 \cos \pi \beta_m)
\]

\[
+ \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(1 - (-1)^n) \sin(2\pi n \beta_m)}{n^2}
\]

\[
= \beta_m \log(q^m) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2\pi (2n + 1) \beta_m)}{(2n + 1)^2}.
\]
Therefore we obtain
\[
m_{1,q}\left(1 + \frac{1 + x}{1 - x}ight) = (1 - q) \sum_{m=0}^{\infty} I_m
\]
\[
= \frac{2(1 - q)}{\pi} \sum_{m=0}^{\infty} \tan^{-1}(q^m) \log(q^m) + \sum_{n=0}^{\infty} \frac{\sin(2\pi(2n+1)\beta_m)}{(2n+1)^2}.
\]

Finally we use the formula of Ramanujan (Entry 17 in [BJ, p.41]):
\[
\sum_{n=0}^{\infty} \frac{(-1)^n \tan^{2n+1} x}{(2n+1)^2} = x \log |\tan x| + \sum_{n=0}^{\infty} \frac{\sin(4n+2)x}{(2n+1)^2}
\]
with $|x| < \pi/2$. We make the substitution $x = \pi \beta_m = \tan^{-1}(q^m)$ in the above formula, and we have
\[
\tan^{-1}(q^m) \log(q^m) + \sum_{n=0}^{\infty} \frac{\sin(4n+2)\pi \beta_m}{(2n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (q^m)^{2n+1}}{(2n+1)^2}.
\]

Hence, we get
\[
m_{1,q}\left(1 + \frac{1 + x}{1 - x}ight) = \frac{2(1 - q)}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^n (q^{2n+1})^m}{(2n+1)^2}
\]
\[
= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{1 - q}{1 - q^{2n+1}} = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2[2n+1]_q}.
\]

The proof is finished. \qed

Theorem 10 is easily proved from Theorem 9.

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References


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