A 3-local characterization of $U_6(2)$ and $\text{Fi}_{22}$

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Abstract

The unitary group $U_6(2)$, often referred to as $\text{Fi}_{21}$, and the sporadic simple group $\text{Fi}_{22}$, discovered by Fischer [B. Fischer, Finite groups generated by 3-transpositions. I, Invent. Math. 13 (1971) 232–246 [6]], are characterized by specifying partial information about the structure of the normalizer of a non-trivial 3-central cyclic subgroup.

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1. Introduction

There are two objectives of this paper. The first is to recognize the 3 transposition groups $U_6(2)$ and $\text{Fi}_{22}$ from a fragment of their 3-local structure. In our case the fragment will be the normalizer of a cyclic subgroup of order 3 contained in the centre of a Sylow 3-subgroup of $G$. The second objective is to explore the methods for recognizing groups given this kind of partial $p$-local information at an odd prime $p$, without invoking a hypothesis which involves knowing all the finite simple groups. The famous Brauer–Fowler Theorem [3] tells us that there are only a finite number of simple groups possessing a given isomorphism type of the centralizer of some involution in $G$. Using the classification of finite simple groups, Hartley [13] proved a similar result for centralizers of elements of arbitrary orders. A number of theorems recognizing small groups from their odd $p$-local

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centralizers were proved in the mid 1970s and early 1980s. We shall, in particular, exploit a theorem of Hayden [11] which identifies \( \text{PSP}_4(3) \) by the centralizer of its non-trivial 3-central elements and a certain further 3-local condition. We mention that other contributions in this area include work by Higman [8], Hayden [12] and Prince [19–21]. These results form the foundation upon which we build our identifications.

Before we move on to state our theorems we establish some notation and recall some group theoretic terms. Throughout this article all the groups are finite groups. For a group \( G \) and a prime \( p \), we say that \( X \subseteq G^\# \) is \( p \)-central if \( C_G(X) \) contains a Sylow \( p \)-subgroup of \( G \). Suppose that \( A \leq B \leq G \) are groups. Then \( A \) is weakly closed in \( B \) with respect to \( G \), provided whenever \( A^g \leq B \), \( A^g = A \) and \( A \) is strongly closed in \( B \) with respect to \( G \) so long as, for all \( g \in G \), \( A^g \cap B \leq A \). For a \( p \)-group \( P \), \( J(P) \) is the Thompson subgroup of \( P \). That is the subgroup of \( P \) generated by the set \( \mathcal{A}(P) \) of abelian subgroups of \( P \) of maximal order. We normally use ATLAS [4] notation for groups and group extensions. In particular, we follow the ATLAS conventions for describing shapes of groups. For an odd prime \( p \), the extraspecial group of exponent \( p \) and order \( p^{1+2n} \) are denoted \( p_+^{1+2n} \). For \( p = 2 \), the extraspecial group of order \( 2^{1+2n} \) which has maximal order elementary abelian subgroups of order \( 2^{n+1} \) is denoted \( 2_+^{1+2n} \) where \( \epsilon = \pm \). The symmetric and alternating groups of degree \( n \) are suggestively denoted by \( \text{Sym}(n) \) and \( \text{Alt}(n) \), respectively. The dihedral group of order \( n \) is \( \text{Dih}(n) \) and \( Q_8 \) is the quaternion group of order 8. The rest of our group theoretical notation is standard and can be found in [1] and [17], for example.

**Definition 1.** A group \( X \) is similar to a 3-normalizer in \( U_6(2) \) if

1. \( |X| = 2^7 \cdot 3^6 \);
2. \( O_3(X) \) is extraspecial of order \( 3^5 \);
3. \( O_2(X) = 1 \);
4. \( O_2(X/O_3(X)) \cong Q_8 \times Q_8 \);
5. \( |X/O_3(X)| = 2 \); and
6. \( O_3(X)/Z(O_3(X)) \) is an \( X \)-chief factor.

So if \( X \) is similar to a 3-normalizer in \( U_6(2) \), then it has shape \( 3_+^{1+4} \cdot (Q_8 \times Q_8) \cdot 3.2 \). The subgroup \( GU_3(2) \cdot 2 \) of \( GU_6(2) \) projects to the normalizer of a 3-central cyclic subgroup of order 3 in \( PGU_6(2) \) and then this intersects with \( U_6(2) \) in a subgroup of index 3. Thus if \( Z \) is a 3-central subgroup in \( G := U_6(2) \), then \( N_G(Z) \) is similar to a 3-normalizer in \( U_6(2) \).

**Theorem 1.** Suppose that \( G \) is a group, \( S \in \text{Syl}_3(G) \) and \( Z := Z(S) \). If \( N_G(Z) \) is similar to a 3-normalizer in \( U_6(2) \), then either \( Z \) is weakly closed in \( S \) or \( G \cong U_6(2) \).

Let \( E \) be an elementary abelian 2-group of order \( 2^{18} \). Then \( \text{Aut}(E) \cong \text{GL}_{18}(2) \) contains a subgroup \( X \) similar to a 3-normalizer in \( U_6(2) \). Hence the semidirect product \( EX \) satisfies the hypothesis of Theorem 1 and, in this case, \( Z \) is weakly closed in \( S \). In fact, if we were prepared to invoke the classification of finite simple groups, then we could apply [7, Remark 7.8.3] to see that if \( Z \) were weakly closed in \( S \), \( ZO_3'(G)/O_3'(G) \leq Z(G/O_3'(G)) \).

Suppose now that \( G \cong \text{Fi}_{22} \). Let \( S \in \text{Syl}_3(G) \) and \( Z := Z(S) \). Then from the ATLAS [4] we have \( N_G(Z) \sim 3_+^{1+6} \cdot 2^{1+4} \cdot 3^2 \cdot 2 \). Furthermore, \( S \) contains a subgroup \( J \), the Thompson
subgroup of $S$, of order $3^5$. In Section 4 we shall give a precise definition of what we mean when we say that a group $X$ is similar to a 3-normalizer in $\text{Fi}_{22}$ and then we shall prove the following theorem.

**Theorem 2.** Suppose that $G$ is a group, $S \in \text{Syl}_3(G)$ and $Z = Z(S)$. If $N_G(Z)$ is similar to a 3-normalizer in $\text{Fi}_{22}$, then either $Z$ is weakly closed in $J(S)$ or $G \cong \text{Fi}_{22}$.

One application of Theorems 1 and 2 is to the end game of the investigation of groups of local characteristic $p$, which is being led by Meierfrankenfeld, Stellmacher and Stroth [18]. At the close of their work they will provide a collection of $p$-local subgroups forming possible amalgams in groups of local characteristic $p$. For the most part, they will then go on to recognize the possible groups via geometric methods. That is either by constructing a building or some other simply connected object upon which the group to hand acts. For a number of the smaller groups this will not be possible and other methods will be required. It is intended that the results in this paper will be applied at this stage of the programme. Of course in their project, the groups they are investigating are what are called $K$-proper, that is, all of their proper subgroups have composition factors from the list of known simple groups. The proof of Theorem 1 becomes somewhat easier in this context. For example using the $K$-proper hypothesis would close the proof of Theorem 1 once we have proved Lemma 25. However, it is very instructive to produce the extra handful of arguments and do without this sledge hammer. We mention here that there are odd $p$-local characterizations of some of the sporadic simple groups by Parker and Rowley [15] and Parker and Wiedorn [16] which do use the powerful $K$-proper assumption.

We now give an overview of the proof of Theorem 1. We begin by getting a tighter hold on the structure of the group $M := N_G(Z)$. In particular, for $S \in \text{Syl}_3(G)$, we show in Lemma 15 that the Thompson subgroup $J := J(S)$ is abelian of order $3^4$ and then, in Lemma 18, we show that $N_G(J)/J \cong \text{Sym}(6)$. With this information to hand, we use Hayden’s Theorem in the proof of Lemma 22 to show $C_G(X) \cong 3 \times \text{PSp}_4(3)$ for a certain cyclic subgroup $X$ of $J$. From the way that $X$ is defined, it is obviously inverted by a certain specified involution $t_2$ and this involution in turn centralizes the component $K \cong \text{PSp}_4(3)$ of $C_G(X)$. The proof of Theorem 1 requires that we determine $C_G(t_2)$. With this in mind, we analyze the centralizers of subgroups of $J \cap K$, $J \cap K$ being elementary abelian of order 27. This allows us to show that the set of maximal $J_K$-invariant $3'$-subgroup of $C_G(t_2)$, $\mathcal{C}_G(t_2)(J_K, 3')$, consists of a single subgroup $R$ and coincides with $\mathcal{C}_G(t_2)(K, 3')$. Furthermore, we show that $R \cong 2^{1+8}_+$. Putting $L := RK$ we set about showing that $L = C_G(t_2)$. To do this we prove that $L$ is strongly 3-embedded in $C_G(t_2)$. We then reach our contradiction by invoking Theorem 37 which is proved in Appendix A. At this stage we have proved $C_G(t_2) \sim 2^{1+8}_+.U_4(2)$ and we call upon a characterization theorem by Parrott [22] to get that $G \cong U_6(2)$.

The proof of Theorem 2 is a much more straightforward business. Directly from the structure of a group $M$ with 3-normalizer similar to that in $\text{Fi}_{22}$ we see an involution $u_1$ which has $C_M(u_1)/\langle u_1 \rangle$ similar to a 3-normalizer in $U_6(2)$. It is then relatively painless to show that, if $S \in \text{Syl}_3(G)$ and $Z := Z(S)$, then $Z$ is not weakly closed in $C_M(u_1)$ with respect to $C_G(u_1)$. Applying Theorem 1 then delivers $C_G(u_1)/\langle u_1 \rangle \cong U_6(2)$ and
C\textsubscript{G}(u\textsubscript{1}) \cong 2:U\textsubscript{6}(2), the unique double cover of U\textsubscript{6}(2). Finally we apply a theorem of Hunt [9] to get \(G \cong \text{Fi}_{22}\).

2. Preliminary results

The following result is a well-known elementary application of Sylow’s Theorem.

**Lemma 3.** Suppose that \(p\) is a prime, \(X\) is a group and \(P \in \text{Syl}_p(X)\). If \(x, y \in Z(J(P))\) are \(X\)-conjugate, then \(x\) and \(y\) are \(N_X(J(P))\)-conjugate.

**Proof.** Let \(T \in \text{Syl}_p(C_X(x))\) with \(T \geq J(P)\) and \(R \in \text{Syl}_p(C_X(y))\) with \(R \geq J(P)\). Then \(J(P) = J(T) = J(R)\). Suppose that \(g \in X\) and \(x^g = y\). Then \(T^g \leq C_X(y)\). Hence there exists \(k \in C_X(y)\) such that \(T^gk = R\). But \(J(P)gk = J(T)gk = J(R) = J(P)\).

Hence \(gk \in N_X(J(P))\). But then \(x^{gk} = y^k = y\) and we are done. \(\square\)

**Lemma 4.** Suppose that \(p\) is a prime, \(X\) is a \(p\)-constrained group and \(P\) is a \(p\)-subgroup of \(X\). Then \(O_{p'}(N_X(P)) = 1\).

**Proof.** See [17, 8.2.12, p. 169]. \(\square\)

**Theorem 5.** (Hayden [11]) Suppose that \(X\) is isomorphic to the centralizer of a non-trivial 3-central element in \(\text{PSp}_4(3)\) and that \(H\) is a group with an element \(d\) such that \(C_H(d) \cong X\). Let \(P \in \text{Syl}_3(C_H(d))\) and \(E\) be the elementary abelian subgroup of \(P\) of order 27. If \(E\) does not normalize any non-trivial 3’-subgroup of \(H\) and \(d\) is not \(H\)-conjugate to its inverse, then either \(H\) has a normal subgroup of index 3 or \(H \cong \text{PSp}_4(3)\).

The next lemma will be required when we apply Hayden’s Theorem.

**Lemma 6.** Suppose that \(X\) is a group of shape \(3^{1+2}.\text{SL}_2(3)\), \(O_2(X) = 1\) and that a Sylow 3-subgroup of \(X\) contains an elementary abelian subgroup of order 3\(^3\). Then \(X\) is isomorphic to the centralizer of a non-trivial 3-central element in \(\text{PSp}_4(3)\).

**Proof.** Since the centralizer of a non-trivial 3-central element in \(\text{PSp}_4(3)\) has the structure described in the lemma, it suffices to show that a group \(X\) with the above properties is uniquely determined up to isomorphism. So suppose that \(X\) has the described properties. Let \(R := O_3(X)\), \(P \in \text{Syl}_3(X)\), \(Q \in \text{Syl}_2(X)\), \(Z := Z(Q)\) and let \(Y\) be the elementary abelian subgroup of \(P\) of order 3\(^3\). Then \(R\) is extraspecial, \(P = RY\) and \(Y\) does not centralize \(R\) as otherwise \(O_2(X) > 1\). In particular, \(Z(P) = Z(R)\). If \(Y\) is not the unique abelian subgroup of order 3\(^3\) in \(P\), then by intersecting two of them we have \(|Z(P)| \geq 3^2\) which is impossible. Therefore \(Y\) is the unique elementary abelian subgroup of order 3\(^3\) in \(P\). Now \(N_X(P) = PZ\) and so \(Z\) normalizes \(Y\), and, as \(X/R \cong \text{SL}_2(3)\) and \(Z\) centralizes \(Z(R)\),
$C_Y(Z)$ has order 9 and is elementary abelian. Let $H := C_X(Z)$. Then, as $C_Y(Z)$ is elementary abelian, $H$ contains three subgroups isomorphic to $\text{SL}_2(3)$ which complement $R$. Since their action on $R$ is the same (they differ only by the central elements of $R$), we have that $X$ is the semidirect product of $3^{1+2}$ and $\text{SL}_2(3)$ and is uniquely determined up to isomorphism. □

During the course of the proof of Theorem 5, Hayden also proved the following nice result which we shall require in the proof of Lemma 28.

**Lemma 7.** (Hayden [11, 3.3]) Let $X$ be a group which has an elementary abelian Sylow 3-subgroup $T$ of order 9. Suppose that $C_X(T) = T$, $N_X(T)/T$ is a fours group and that $C_X(t) \leq N_X(T)$ for all non-trivial $t \in T$. Then $T$ is normal in $X$.

**Lemma 8.** Suppose that $X \cong \text{Sym}(4)$ and $V$ is a faithful, 3-dimensional, $\text{GF}(3)X$-module. Then

(i) there is a set of 1-dimensional subspaces $\mathcal{B} := \{\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle\}$ such that $X/O_2(X)$ acts as $\text{Sym}(3)$ on $\mathcal{B}$ and each subspace in $\mathcal{B}$ is inverted by $O_2(X)$;
(ii) $X$ has orbits of length 3, 4 and 6 on the 1-dimensional subspaces of $V$ with representatives $\langle v_1 \rangle$, $\langle v_1 + v_2 \rangle$ and $\langle v_1 + v_2 + v_3 \rangle$, respectively; and
(iii) $X$ has orbits of length 3, 4 and 6 on the 2-dimensional subspaces of $V$ with representatives $\langle v_1, v_2 \rangle$, $\langle v_1 + v_2, v_2 + v_3 \rangle$ and $\langle v_1, v_1 + v_2 + v_3 \rangle$, respectively.

**Proof.** Let $Q := O_2(X)$ and $Q^# := \{q_1, q_2, q_3\}$. Then, as $V$ is a faithful, irreducible $\text{GF}(3)X$-module and $X$ acts transitively on $Q^#$ by conjugation, we have that $V = C_V(q_1) \oplus C_V(q_2) \oplus C_V(q_3)$ and that $X$ permutes the subspace $\{C_V(q_i) \mid 1 \leq i \leq 3\}$ transitively. Setting $\langle v_i \rangle := C_V(q_i)$, we have that (i) holds.

Obviously $\{\langle v_i \rangle \mid 1 \leq i \leq 3\}$ is an orbit of length 3 on the 1-dimensional subspace of $V$. The subspaces $\langle v_1 \pm v_2 \pm v_3 \rangle$ form an orbit of length 4 and the subspaces $\langle v_i \pm v_j \rangle$ with $i \neq j$ give an orbit of length 6. This proves part (ii). A similar calculation provides a proof of (iii). □

Let $P$ be the stabilizer of the subspace $\langle v_1 \rangle$ from Lemma 8. Then $P \cong \text{Dih}(8)$ and $\langle v_1 \rangle$ is inverted by $Q$. The isomorphism type of $V$ is determined by $C_P(V)$. This group is either the cyclic subgroup of order 4 or the fours group of $P$ different from $Q$.

**Lemma 9.** Suppose that $X \cong \text{PSp}_4(3)$, let $T \in \text{Syl}_3(X)$ and set $K := J(T)$. Then $K$ is elementary abelian of order $3^3$, $N_X(K) \sim 3^3 \cdot \text{Sym}(4)$ and every cyclic subgroup of order 3 in $X$ is conjugate to a subgroup of $K$.

**Proof.** We know that $T$ has centre of order 3 and $X$ contains subgroups of shape $3^3: \text{Sym}(4)$. Thus the 3-rank of $T$ is 3 and, as $|Z(T)| = 3$, $T$ has a unique abelian subgroup of order $3^3$. By Lemmas 3 and 8(ii), $K$ contains 3-conjugacy classes of non-trivial cyclic subgroups and they are not fused in $X$. On the other hand, [4, p. 27] shows that $G$ has 3 conjugacy classes of non-trivial cyclic subgroup of order 3. This proves the lemma. □

We continue with a well-known lemma about extraspecial groups.
Lemma 10. Assume that $p$ is an odd prime and that $Q$ is an extraspecial $p$-group of order $p^{1+2n}$. Then

(i) if $Q$ has exponent $p$, then $\text{Aut}(Q)/\text{Inn}(Q) \cong \text{GSp}_{2n}(p)$; and
(ii) if $L \leq \text{Aut}(Q)$ and $L$ acts irreducibly on $Q/Z(Q)$, then $Q$ has exponent $p$.

Proof. See [5, Theorems 20.8 and 20.9].

In the next lemma we use ATLAS [4] notation for the conjugacy classes of $U_4(2)$.

Lemma 11. Suppose that $X \cong U_4(2)$ and $V$ is the natural $\text{GF}(4)X$-module.

(i) $X$ has two orbits on the non-trivial vectors of $V$. One orbit consists of isotropic vectors the other of non-isotropic vectors. If $v$ is isotropic, then $C_X(v) \sim 2^{1+4}.\text{Sym}(3)$ and, if $v$ is non-isotropic, $C_X(v) \sim 3^{1+2}.\mathbb{Q}_8$.

(ii) For $x \in X$ in class 2A, $C_X(x) \sim 2^{1+4}.(\text{Sym}(3) \times 3)$ and $|C_V(x)| = 2^6$.

(iii) For $x \in X$ in class 2B, $C_X(x) \sim 2^4.\text{Sym}(3) \sim 2^{1+2+1}.\text{Sym}(3)$ and $|C_V(x)| = 2^4$.

(iv) For $x \in X$ in class 2B and $E \in \text{Syl}_3(C_X(x))$, $C_V(E) = 0$ and $|C_X(E)| = 108$.

(v) For $v$ isotropic and $E \in \text{Syl}_3(C_X(v))$, $|C_V(E)| = 2^4$.

(vi) For $x \in X$ in class 2A, $C_X(x)$ has three orbits on $C_V(x)/[V,x]$. These orbits have lengths 1, 6 and 9.

(vii) Assume that $E \leq X$ has order at least $2^5$ and $x \in Z(E)#$. Then $C_V(E) \subseteq [C_V(x),E]$.

(viii) If $x$ is in class 3A, $|C_V(x)| = 4$, if $x$ is in class 3C, $|C_V(x)| = 1$ and, if $x$ is in class 3D, $|C_V(x)| = 16$.

(ix) If $x$ is in class 3C, then $C_X(x)/(x) \cong \text{Sym}(3) \times \text{Sym}(3)$.

Proof. The facts about $U_4(2)$ are taken from the ATLAS [4]. The results about the module $V$ are either obtained from [14] or are calculated explicitly.

The following result is also well known.

Lemma 12. Suppose that $X$ is a group, $V$ is an elementary abelian normal 2-subgroup of $X$ and $x \in X$ is an involution. Set $C := C_X(x)$. Then there is a one to one correspondence between $VC$-orbits on the involutions in the coset $Vx$ and the $C$-orbits on the elements of $C_V(x)/[V,x]$. Furthermore, for $vx$ an involution in $Vx$, $|(vx)^VC| = |(v[V,x])C|.|[V,x]|$.

Proof. Set $W := [V,x]$. Then the map $(vx)^VC \mapsto (vW)^C$, where $vx \in Vx$ is an involution, is the required bijection.

3. Identifying $U_6(2)$

In this section we prove Theorem 1. So assume that $G$ satisfies the hypothesis of Theorem 1, $S \in \text{Syl}_3(G)$ and that $Z := Z(S)$ is not weakly closed in $S$ with respect to $G$. Set
Lemma 15. \(J(S) = \{1\}\) implies that \(J(S) = \{1\}\) has exponent 3.

Proof. From the above discussion, we know that \(S\) normalizes both \(Q_1\) and \(Q_2\). Since, for \(i \in \{1, 2, 3\}\), \(S/S_i\) has order \(3\) and \([S, S, S] \leq S_i\). Hence \([S, S, S] \leq Q_1 \cap Q_2 = Z\).

Lemma 14. \(N_M(S)/S \cong \text{Dih}(8)\).

Proof. We have \(N_M(S)/S\) has order 8, contains \(\langle t_1, t_2 \rangle\) and is non-abelian. It follows that \(N_M(S)/S \cong \text{Dih}(8)\).

Set \(J := C_S([Q, S])\).

Lemma 15. The following hold.

(i) \(J\) is abelian of order \(3^4\).

(ii) \(J = J(S)\).

Proof. Set \(A := [Q, S]\). Then \(A/Z = [Q_1, S][Q_2, S]/Z\) has order 9. Furthermore, for \(i \in \{1, 2\}\), \([Q_i, S]\) is elementary abelian and so \(A\) is elementary abelian. In particular, \(A \leq J\). Since \(Q\) is extraspecial, \(A = C_Q(A)\) and so \(|J| \leq 3^4\). By Lemma 13, \(A/Z \leq Z(S/Z)\), thus \(S/J\) is abelian and is isomorphic to a 3-subgroup of \(\text{GL}_3(3)\). It follows that \(|J| = 3^4\) and further, as \([J : A] = 3\), \(J\) is abelian. Therefore (i) holds.

Since the maximal order abelian subgroups of \(Q\) have order \(3^3\), \(J \leq A(S)\). Suppose that \(K \not\leq Q\) and thus \(KQ = S\). Hence \(K \cap Q = J \cap Q = A\) and \(K \leq C_S(A) = J\). Therefore \(K = J\) and so (ii) holds.

Lemma 16. \(Z\) is weakly closed in \(Q\) with respect to \(G\).
Proof. Assume that \( g \in G \) and \( X := Z^g \leq Q \) with \( X \neq Z \). Then \( Q \) normalizes \( ZX \) and \( C_Q(X) \) is non-abelian of order \( 3^4 \). Set \( R := O_3(M^g) \). Then, as \( C_Q(X) \) is non-abelian and \( M^g/R \) has cyclic Sylow \(-3\)-subgroups, \( Z = [C_Q(X), C_Q(X)] \leq R \) and \( C_Q(X)R \in \text{Syl}_3(M^g) \). Set \( T := C_Q(X)R \). Note that as \( Z \leq R \), we also have \( C_R(Z)Q \in \text{Syl}_3(M) \) and so we may suppose that \( S = QC_R(Z) \).

Set \( H := (S, T) \). Then, as \( S = QC_R(Z) \) and \( T = C_Q(X)R \), \( H \) normalizes \( ZX \). Since \( C_S(XZ) \) has index 3 in \( S \), we have \( C_S(XZ) = C_Q(X)C_R(Z) = CT(XZ) \) and as this subgroup is normal in \( H \), \([Q, S] \leq C_G(XZ) \). Since \([(Q, S), XZ] = 1 \), Lemma 15 implies \( J \leq C_S(XZ) \). Thus \( J = (C_S(XZ)) \) is normalized by \( H \). Again using \( S = QC_R(Z) \) and \( T = C_Q(X)R \) as well as \( XZ \leq Q \cap R \), we have \( Q \cap R \) is normalized by \( H \). Now \(|Q \cap R| = 3^3 \), \(|J| = 3^4 \) and \(|C_S(XZ)| = 3^5 \), so \( O^3(H) \) centralizes \( C_S(XZ)/(Q \cap R) \). In particular, \( O^3(H) \) normalizes \( C_Q(X) \geq Q \cap R \). But then \( O^3(H) \) normalizes \( [C_Q(X), C_Q(X)] = Z \) and so \( Z \) is normal in \( H \). However this means that \( X = Z(T) \geq Z \), which is a contradiction. \( \square \)

**Lemma 17.** Assume that \( g \in G \) and \( X := Z^g \leq S \). Then \( X \leq J \).

**Proof.** Let \( g \in G \) with \( X := Z^g \leq S \). Set \( A := [Q, S] \). Recall that \( A \) is elementary abelian of order \( 3^3 \) and that \([A, X] \leq [A, S] = Z \). Set \( R := O_3(M^g) \). As \( X \leq S \), \( Z \leq M^g \) and by Lemma 16, \( Z \not
leq R \). Hence \( T := ZR \in \text{Syl}_3(M^g) \). Put \( B := [T, R] \) and define

\[
F := (A, B).
\]

Then, as \([A, X] \leq Z \leq XZ \) and \([B, Z] \leq X \leq XZ \), \( F \) normalizes \( XZ \).

Assume that \( X \not
leq J \). Then, as \( J = C_S(A) \) by Lemma 15, \([A, X] \neq 1 \). If \( Z \) centralizes \( B \), then \( B \leq QX \) and \( A = (B \cap Q)Z = C_Q(X) \), which is a contradiction. Therefore, \([B, Z] \neq 1 \). It follows that \( F/C_F(XZ) \cong \text{SL}_2(3) \). As \(|BC_F(XZ)/C_F(XZ)| = 3 \), \(|C_F(Z)| = 3^2 \). Therefore, \( 1 \neq B \cap Q \leq C_Q(X) \leq A \). Hence \([A \cap B] = 3 \). Note that because \( A \) and \( B \) are abelian, \( A \cap B \leq Z(F) \). Let \( P \in \text{Syl}_3(F) \) with \( P \geq AX \). Then \( P = A(P \cap C_F(XZ)) \). Now \( C_F(XZ) \geq XZ \) and \( P \leq M (= N_G(Z)) \). So \( P \leq XQ \) and hence \( P \leq XC_Q(X) \leq AX \). Therefore \( P = AX \). Let \( f \in F \) be such that \( fC_F(XZ) \in Z(F/C_F(XZ))^g \). Then \( f \in N_F(PC_F(XZ)) \). Hence, by the Frattini Argument, we may suppose that \( f \in N_F(P) \) and that \( f \) has order coprime to 3. Since \( fC_F(XZ) \in Z(F/C_F(XZ)) \), \( f \) inverts \( XZ \) and hence \( f \in M \). On the other hand, \( f \) centralizes \( P/C_P(XZ) \) and \( A \cap B \). Therefore, \( f \) centralizes \( P/XZ \). Since \( P = AX \) and \( A \) is \( f \)-invariant (\( f \) normalizes \( S = QX \) and hence normalizes \( A = [Q, S] \)), we have that \( f \) centralizes \( A/Z \). Since \( f \) inverts \( Z \), \( f \notin O^2(M) \) and so \( f \) swaps \( Q_1 \) and \( Q_2 \), but \( f \) centralizes \( (A \cap Q_1)/Z \) which is impossible. This contradiction shows that \([A, X] = 1 \) and that \( X \leq J \) as claimed. \( \square \)

**Lemma 18.** The following hold.

(i) \( J \) is elementary abelian.

(ii) There are exactly ten \( G \)-conjugates of \( Z \) contained in \( S \).

(iii) \( N_M(J)/J \sim 3^2 \cdot \text{Dih}(8) \) and \( N_G(J)/J \cong \text{Sym}(6) \).
Proof. Since $Z$ is not weakly closed in $S$, there exists a $G$-conjugate $X$ of $Z$ contained in $S$ with $X \neq Z$. By Lemma 17, all such $X$ are contained in $J$. By Lemma 16, $X \not\leq Q$ and so $J = X(J \cap Q)$ and, in particular, by Lemma 15(i), $J$ is elementary abelian. Hence (i) holds.

By Lemmas 3 and 15(ii), $X$ and $Z$ are conjugate in $N_G(J)$. Suppose that $T \in \text{Syl}_3(N_G(J)) \setminus \{S\}$. Then $S \cap T \geq J$ and

$$J = (J \cap Q)Z(T) = (J \cap O_3(C_G(Z(T))))Z.$$  

Hence, by Lemma 15, $[J, S \cap T] \leq Z \cap Z(T) = 1$. Therefore $S \cap T = J$ again by Lemma 15. It follows that $|\text{Syl}_3(N_G(J))| \equiv 1 \pmod{|S/J|} = 1 \pmod{9}$. Furthermore, if $T \in \text{Syl}_3(N_G(J))$ and $Z(T) = Z$, then $T \leq M \cap N_G(J)$. Hence $T$ normalizes $JQ = S$ and so $T = S$. Thus there is a one to one correspondence between conjugates of $Z$ contained in $J$ and Sylow 3-subgroups of $N_G(J)$. Because of Lemma 16, $Z$ is the unique conjugate of $Z$ in $J \cap Q$ and so $|\text{Syl}_3(N_G(J))| \leq 27$. Since $J$ is elementary abelian, $N_G(J)/C_G(J)$ is isomorphic to a subgroup of $\text{GL}_4(3)$ and so the number of Sylow 3-subgroups of $N_M(J)$ must be a divisor of $|\text{GL}_4(3)|$. The only possibility is that $|\text{Syl}_3(N_G(J))| = 10$. So (ii) holds.

Now, by Lemma 14, $N_M(J) = N_M(QJ) = N_M(S)$ and $N_M(S)/S \cong \text{Dih}(8)$. It follows that $N_{N_G(J)}(Z) = N_M(S)$ has index 10 in $N_G(J)$. Hence $|N_G(J)/J| = 720$. Let $T \in \text{Syl}_2(N_M(S))$ and $\Omega = \text{Syl}_3(N_G(J))$. Then $S/J$ fixes one member of $\Omega$ and permutes the remaining 9 members regularly. Therefore the action of $T$ on $S/J$ is the same as the action of $T$ on $\Omega \setminus \{S\}$. Thus the involutions in $T$ have cycle shapes $2^4$ and $2^3$ and the elements of order 4 have cycle shape $4^2$ on $\Omega$. It follows that $N_M(J)/J$ has a subgroup of index 2 consisting of even permutations of $\Omega$. Call this subgroup $Y$. Then an elementary argument shows that $Y/J$ is a simple group which has order 360. Hence $Y/J \cong \text{Alt}(6)$. Because $T \cong \text{Dih}(8)$, we infer from the structure of $\text{Aut(Alt}(6))$ (see [4, p. 4]) that (iii) holds. □

Corollary 19. Suppose that $X \leq J$ has order 3 and is not 3-central. Then $N_{N_G(J)}(X)/J \cong \text{Sym}(4) \times 2$. In particular, $J = [J, C_{N_G(J)}(X)]X$.

Proof. $J$ is the $O_4^-(3)$-module for $\text{Sym}(6)$ and so the result follows. □

Let $\mathcal{J} := \{Z^g \mid g \in G$ and $Z^g \leq S\}$. By Lemmas 17 and 18 every element of $\mathcal{J}$ is contained in $J$ and $|\mathcal{J}| = 10$.

Lemma 20. Suppose that $B \leq J$ with $|J/B| = 3$, then there is a member of $\mathcal{J}$ contained in $B$.

Proof. Assume that $B$ contains no element from $\mathcal{J}$. Then, as $C_J(S) = Z(S) = Z$, $B$ does not contain any non-trivial $S$-invariant subgroups. So, as $|B| = 3^3$, $|B^S| = 9$ and no $S$-conjugate of $B$ contains any member of $\mathcal{J}$.

Now suppose that $B_1, B_2, B_3, B_4$ is a collection of any four distinct maximal subgroups of $J$. Then, by the inclusion exclusion principle, $|\bigcup_{i=1}^4 B_i| \geq 63$. Therefore, $81 = |J| \geq |\bigcup_{S \in \mathcal{S}} B^S| + 2|\mathcal{J}| \geq 63 + 20 = 83$, which is absurd. Thus the lemma holds. □
Lemma 21. Suppose that \( x \in (Q \cap J) \setminus Z \) is not 3-central. If \( R \) is a 3'-subgroup of \( C_G(x) \) which is normalized by \( J \), then \( R = 1 \).

Proof. Since \( J \) is elementary abelian and \( R \) is a 3'-group, using [17, 8.3.4] we have
\[
R = \left\{ C_R(B) \mid |J/B| = 3 \right\}.
\]
Therefore, by Lemma 20,
\[
R = \left\{ C_R(h) \mid h \in J \right\}.
\]
Now, for \( h \in J \), \( C_G(h) \) is 3-constrained and \( C_R(h) = R \cap C_G(h) \) is normalized by \( C_{CG(h)}(x) \). Thus Lemma 4 implies \( C_R(h) = 1 \). We conclude that \( R = 1 \) as claimed.

Set \( X := \left[ Q_1 \cap J, t_2 \right] \).

Lemma 22. The following hold.

(i) \( C_G(X) \cong 3 \times PSp_4(3) \cong 3 \times U_4(2) \).
(ii) \( N_G(X) \cong \text{Sym}(3) \times PSp_4(3) \cong \text{Sym}(3) \times U_4(2) \).
(iii) \( C_G(t_2) \supseteq C_G(X)' \cong PSp_4(3) \).

Proof. As \( t_2 \) inverts \( Q_1/Z \) and \( Q_1 \cap J \) is elementary abelian of order 9, we have \( |X| = 3 \) and, by Lemma 16, \( X \) is not 3-central. Hence, by considering orders we have
\[
C_S(X) = C_Q(X)J = Q_2JC_Q(X) \in \text{Syl}_3(C_G(X)).
\]
Let \( T := Q_2J \). From the structure of \( M \) given in Definition 1, \( C_M(X)/X \) has shape \( 3^{1+2}.Q_8.3 \). Now let \( zX \in Z(T/X) = ZX/X \) and assume that \( zX \) and \( (zX)^{-1} \) are conjugate in \( C_G(X)/X \) by the element \( gX \). Without loss of generality we may assume that \( z \in Z \). Then \( z^g \in XZ \) and so, by Lemma 16, \( z^g \in Z \). Hence \( z^g = z^{-1} \). But then \( g \in C_M(X) \leq C_G(Z) \) and we have a contradiction. Hence, as \( J/X \) is elementary abelian of order 27, using Lemmas 6 and 21, we have that the hypothesis of Theorem 5 holds. Thus either \( C_G(X)/X \) has a normal subgroup at index 3 or \( C_G(X)/X \cong PSp_4(3) \). Suppose that the former possibility holds. Then \( C_G(X)'/X \) does not contain \( T \). On the other hand, the structure of \( C_M(X) \) indicates that \( Q_2 \leq C_M(X)' \leq C_G(X)' \) and Corollary 19 implies that \( J = [C_{N_G(J)}(X), J]/X \leq C_G(X)' \). Therefore \( T = JQ_2 \leq C_G(X)'X \) and we have a contradiction. Thus \( C_G(X)/X \cong PSp_4(3) \). Since \( C_S(X) \) splits over \( X \), we apply Gashütz Splitting Theorem [17, Satz 3.3.2] to get \( C_G(X) \cong 3 \times PSp_4(3) \). Hence (i) holds.

Since \( t_2 \) inverts \( X \), we have \( [N_G(X) : C_G(X)] = 2 \). Now \( N_M(X)/X \cong 2 \times 3^{1+2}.Q_8.3 \). Using [4, p. 26], \( \text{Aut}(PSp_4(3)) \) does not contain such a subgroup, and we therefore infer that \( N_G(X) \cong \text{Sym}(3) \times PSp_4(3) \) as claimed in (ii).

Let \( F \leq N_G(X) \) with \( F \cong \text{Sym}(3) \). Then, as \( C_{N_G(X)/F}(t_2F) \) contains a Sylow 3-subgroup of \( N_G(X)/F \cong PSp_4(3) \), we conclude \( t_2F = F \) and so (iii) holds.

\( \square \)
We set $K := C_G(X)'$ and $J_K := J \cap K$. Then, by Lemma 22(iii), we have the important inclusions

$$J_K \leq K \leq C_G(t_2).$$

Note that $J_K$ is elementary abelian of order $3^3$ and $N_K(J_K) / J_K \cong \text{Sym}(4)$ (see [4, p. 26]). Note also that $Z \leq J_K$ and that $Z$ is 3-central in $K$. Thus all 3-central subgroups of $K$ are 3-central in $G$.

**Proposition 23.** Up to conjugacy in $N_K(J_K)$ the subgroups of $J_K$ of order 9 are of the following types.

- **Type I** These subgroups contain one 3-central subgroup of $G$ and three cyclic subgroups $Y$ with $C_G(Y) \cong 3 \times \text{PSp}_4(3)$. There are four of these subgroups.
- **Type II** These subgroups contain exactly two 3-central subgroups of $G$. There are six of these subgroups.
- **Type III** These subgroups contain no 3-central elements and have two cyclic subgroups $Y$ with $C_G(Y) \cong 3 \times \text{PSp}_4(3)$. There are three of these subgroups.

**Proof.** This follows from Lemma 8 after we note, using the notation from Lemma 8, that the three central subgroups are conjugates of $\langle v_1 + v_2 \rangle$ and the subgroups from $(Q \cap J_K) \setminus Z$ are conjugates of $\langle v_1 + v_2 \rangle$. \(\square\)

We use the notation from Proposition 23 in the next two lemmas.

**Lemma 24.** Suppose that $[J_K : A] = 3$.

(i) If $A$ is of Type I,

$$O_3(p) \langle A, t_2 \rangle \cong Q_8$$

and for $b \in A$, $b$ not 3-central,

$$O_3(p) \langle A, t_2 \rangle \leq O_3(p) \langle b, t_2 \rangle \cong 2^{1+4}.$$ 

(ii) If $A$ is of Type II or Type III,

$$O_3(p) \langle A, t_2 \rangle = \langle t_2 \rangle.$$ 

**Proof.** Assume that $A$ has Type I. Let $a \in A$ be 3-central in $G$ and $b \in A \setminus \langle a \rangle$. Then $C_G(b) \cong 3 \times \text{PSp}_4(3)$ by Proposition 23. Let $L := C_G(b)'$. Then, as $t_2$ centralizes $K$ and hence also centralizes $b$, $t_2 \in L$. Since $t_2$ commutes with $J_K \cap L$ which has order 9, $t_2$ is 2-central in $L$ (see [4, p. 26]). In particular, we have $C_G(\langle b, t_2 \rangle) \sim 3 \times 2^{1+4}.3^2.2$. Thus the second claim in (i) holds. Now $C_{C_G(a)}(b) \sim 3 \times 3^{1+2}.Q_8.3$ and so we see that $a \in L$ and $a$ is 3-central therein. It follows that $C_{L}(a) \cong 3^{1+2}.\text{SL}_2(3)$ and that $C_{L}(\langle a, t_2 \rangle) = 3 \times \text{SL}_2(3)$. Hence $O_3(p) \langle A, t_2 \rangle \cong Q_8$. 


Suppose that $A$ has Type II and let $a_1$ and $a_2$ be 3-central elements of $G$ which generate $A$. Then $N_G(\langle a_1 \rangle)$ is similar to a 3-normalizer in $U_6(2)$. By Lemma 16, $a_2 \notin O_3(N_G(\langle a_1 \rangle))$. Let $T := \langle a_2, O_3(N_G(\langle a_1 \rangle)) \rangle$. Then, by Lemma 17, $a_2 \in J(T)$ and, by Lemma 18, $J(T)$ is elementary abelian. It follows that $C_G(A)$ is 3-closed with $C_G(A)/J(T)$ elementary abelian of order 4. Since $t_2$ commutes with $JK \leq J(T)$, we finally get $C_G(\langle A, t_2 \rangle) = JK \langle s \rangle \times \langle t_2 \rangle$ where $s$ has order 2 and acts non-trivially on $JK$. Thus (ii) holds when $A$ has Type II.

Let $T := \langle a_2, O_3(N_G(\langle a_1 \rangle)) \rangle$. Then, by Lemma 17, $a_2 \in \mathbb{J}(T)$ and, by Lemma 18, $\mathbb{J}(T)$ is elementary abelian. It follows that $CG(A)$ is 3-closed with $CG(A)/\mathbb{J}(T)$ elementary abelian of order 4. Since $t_2$ commutes with $JK \leq J(T)$, we finally get $CG(\langle A, t_2 \rangle) = JK \langle s \rangle \times \langle t_2 \rangle$ where $s$ has order 2 and acts non-trivially on $JK$. Thus (ii) holds.

Lemma 25. Suppose that $R \in \mathbb{I}^*_G(\langle t_2 \rangle, 3')$. Then

(i) $\mathbb{I}^*_G(\langle t_2 \rangle, 3') = \mathbb{I}^*_G(\langle t_2 \rangle, 3') = \{ R \}$;
(ii) $N_{CG(\langle t_2 \rangle)}(R) = RK$; and
(iii) $R$ is extraspecial of order $2^9$ and plus type and $K$ acts irreducibly on $R/\langle t_2 \rangle$.

Proof. Let $R \in \mathbb{I}^*_G(\langle t_2 \rangle, 3')$. Then $R \nmid \langle t_2 \rangle$. Assume that $R \nmid \langle t_2 \rangle$. Let $p$ be a prime dividing $|R|$ and $R_p \in \text{Syl}_p(R)$ be $JK$-invariant. Then

$$R_p = \{ C_{R_p}(A) \mid [J_K : A] = 3 \}.$$ 

Now $C_{R_p}(A) \in \text{I}(C_G(\langle A, t_2 \rangle), 3')$ and so $R_p$ is a 2-group by Lemma 24. It follows that $R$ is a 2-group. Using Lemma 24 again we have

$$R = \{ C_{R}(A) \mid A \text{ is of Type I in } J_K \}.$$ 

Let $L \in \mathbb{I}^*_G(\langle t_2 \rangle, 3') \setminus \{ R \}$ and $A, B$ be subgroups of $J_K$ of Type I such that $C_L(A) \nmid \langle t_2 \rangle$ and $C_R(B) \nmid \langle t_2 \rangle$. Let $Y := A \cap B$. Then $|Y| \geq 3$,

$$C_L(A) \leq O_2(C_G(\langle Y, t_2 \rangle))$$

and

$$C_R(B) \leq O_2(C_G(\langle Y, t_2 \rangle)).$$

By Lemma 24, $O_2(C_G(\langle Y, t_2 \rangle))$ is extraspecial and so $C_R(B)$ is normalized by $C_L(A)$. Since this is true for all $B \leq J_K$ of Type I, we infer that $C_L(A)$ normalizes $R$ and then, as $R \in \mathbb{I}^*_G(\langle t_2 \rangle, 3')$, we get that $C_L(A) \leq R$. It follows that

$$\mathbb{I}^*_G(\langle t_2 \rangle, 3') = \{ R \}.$$
In particular, $R$ is normalized by $N_K(J_K) \sim 3^3 : \text{Sym}(4)$. By Proposition 23, $N_K(J_K)$ acts transitively on the Type I subgroups of $J_K$ and by Lemma 24(i), $C_R(A)$ is isomorphic to a subgroup of $Q_8$. Since $J_K$ acts irreducibly on $O_2(C_G((A, t_2)))$ and $C_R(A) > \langle t_2 \rangle$ by assumption, $C_R(A) \cong Q_8$ for all subgroup of $J_K$ of Type I. Further, for $A$ and $B$ of Type I in $J_K$, $\langle C_R(A), C_R(B) \rangle \leq O_2(C_G((A \cap B, t_2)))$ which is extraspecial of type $2^{1+4}$. Thus either $C_R(A)$ and $C_R(B)$ commute or are equal. This latter possibility leads to $C_R(J_K) > \langle t_2 \rangle$ and this contradicts Lemma 24. Therefore, $C_R(A)C_R(B) = O_2(C_G((A \cap B, t_2))) \cong 2^{1+4}$. Since there are exactly four subgroups of Type I in $J_K$ and $N_K(J_K)$ acts transitively on them, we get that $|R| = 2^9$ and $R$ is of plus type. Moreover, we have that $N_K(J_K)$ acts irreducibly on $R/\langle t_2 \rangle$.

Since $R$ is normalized by $N_K(J_K)$, $Q_1$ normalizes $R$. Let $k \in N_K(Q_1) \setminus J_K$. Then, as $N_K(J_K)$ is a maximal subgroup of $K$, $K = \langle k, N_K(J_K) \rangle$. Furthermore, $R^k$ is normalized by $Q_1$. Hence $R^k$ is normalized by $Q_1 \cap J_K$ which is of Type I in $J_K$.

Because of the structure of $R$, for $x \in Q_1 \cap J_K$, we have $C_R(x) = O_2(C_G((x, t_2)))$. Since $R^k = \langle C_R^k(x) \mid x \in Q_1 \cap J_K \rangle$, we deduce that $R^k \leq R$. Thus $R^k = R$ and $R$ is normalized by $K$.

We have shown that $\mathcal{U}^*(J_K, 3')$ has one element and that it is either $\langle t_2 \rangle$ or is a subgroup $R$ which extraspecial of order $2^9$ of plus type, normalized by $K$ with $K$ acting irreducibly on $R/\langle t_2 \rangle$.

Let $z$ be 3-central in $K$. Then $C_{C_G(t_2)}(z)R/R = 3_+^{1+2} : \text{SL}_2(3)$ and, by part (i), $J_K R/R$ normalizes no non-trivial 3'-subgroup of $N_{C_G(t_2)}(R)$. Since $K$ contains a Sylow 3-subgroup of $C_G(t_2)$, we have that $N_{C_G(t_2)}(R)$ does not contain a normal subgroup of index 3. Thus Theorem 5 implies that $RK = N_{C_G(t_2)}(R)$.

Finally consider the possibility that $R = \langle t_2 \rangle$. Then, as $N_{C_G(t_2)}(R) = RK$, $C_G(t_2) \cong 2 \times \text{PSp}_4(3) \langle \leq N_G(X) \rangle$. On the other hand, we have seen that $t_2$ is $G$-conjugate to $t_1$ which is a 2-central involution in $K$ and so $t_2 \in C_G(t_2)'$, which is a contradiction. Hence $R > \langle t_2 \rangle$ and all the claims in the lemma now follow. \qed

We now set $L := RK = N_{C_G(t_2)}(R)$. Then, by Lemma 25(ii) and (iii), $L \sim 2_+^{1+8} : \text{U}_4(2)$. Since $K$ acts irreducibly on $R/\langle t_2 \rangle$, we have that $R/\langle t_2 \rangle$ is isomorphic to the natural GF(4)$\text{U}_4(2)$-module considered as a GF(2)-module (see [14, p. 60]). In particular, the results in Lemma 11 are available to us.

**Lemma 26.** $N_{C_G(t_2)}(J_K) \leq L$.

**Proof.** Set $N := N_{C_G(t_2)}(J_K)$. Obviously $N$ permutes the members of $\mathcal{U}^*_{C_G(t_2)}(J_K, 3')$. Hence $N \leq N_{C_G(t_2)}(R) = L$ by Lemma 25. \qed

**Lemma 27.** The following hold.

(i) $N_{C_G(t_2)}(Z) \leq L$.

(ii) If $Y \leq J_K$ is $G$-conjugate to $X$, then $N_{C_G(t_2)}(Y) \leq L$. 
Proof. This is proved by considering orders. Since $Z$ is 3-central in $K$, Lemma 11(viii) implies that $C_R(Z) \cong Q_8$ and $C_K(Z) \sim 3^{1+2}.\text{SL}_2(3)$. On the other hand, $|C_M(t_2)| = 2^6.3^4$. Hence $C_M(t_2) \leq L$.

Similarly for $Y$, we have $|C_R(Y)| = 2^5$ and $|C_K(Y)| = 54$ whereas we know $C_{CG(Y)}(t_2) \sim 3 \times 2^{1+4}.3^2.2$ and so once again we have equality. Hence $N_{CG(t_2)}(Y) \leq L$.

Recall that for a prime $p$, a subgroup $Y$ of a group $H$ is strongly $p$-embedded if $p$ divides $|Y|$ and $p$ does not divide $|Y \cap Y^g|$ for all $g \in H \setminus Y$. This is equivalent to

$$1 \neq \{N_H(P) \mid P \leq Y \text{ and } |P|_p > 1\} \leq Y.$$  

Note that we have deliberately included that case that $H = Y$. We now further restrict the way that $L$ is contained in $C_G(t_2)$.

**Lemma 28.** $L$ is strongly 3-embedded in $C_G(t_2)$.

Proof. Assume that $D \leq L$ has order 3. Then, as $K$ contains a Sylow 3-subgroup of $L$, $D$ is conjugate to a subgroup of $K$. By Lemma 9, $D$ is conjugate to a subgroup of $J_K$. So we may assume that $D \leq J_K$. According to Lemma 8, there are three distinct possibilities for $D$ to investigate. Suppose first that $D$ is 3-central in $K$. Then $D$ is 3-central in $G$ and so, by conjugating by elements from $K$, we may suppose that $D = Z$ and $N_G(Z) = M$. So in this case we have $N_{CG(t_2)}(Z) = C_M(t_2) \leq L$ from Lemma 27. Next suppose that $D$ is conjugate to $X$ from Lemma 22. Then $N_{CG(t_2)}(D) \leq L$ again by Lemma 27.

Now suppose that $D$ is conjugate to neither $X$ nor $Z$. Then $D$ is in conjugacy class 3C and acts fixed point freely on $R/(t_2)$ by Lemma 11(viii). Thus $C_L(D) = (t_2) \times C_K(D) \sim 2 \times 3^3.2^2$. In particular, $J_K \in \text{Syl}_3(C_L(D))$ and $C_L(D)$ is 3-closed. Since $N_{CG(t_2)}(J_K) \leq L$ by Lemma 26, $J_K \in \text{Syl}_3(C_G((t_2, D)))$ and, using Lemma 11(ix),

$$N_{CG((t_2, D))}(J_K)/\langle t_2, D \rangle \cong \text{Sym}(3) \times \text{Sym}(3).$$

Set $N := N_{CG(t_2)}(D)$. Assume that $n \in N$ and $J^n_K \cap J_K > D$. Then there exists $A \leq J_K \cap J^n_K$ with $D < A$ and $A$ of order 9. It follows from Proposition 23 that there is a cyclic subgroup $B \leq A$ such that $B$ is $K$-conjugate to either $Z$ or $X$. However we have already seen that such subgroups have centralizers in $C_G(t_2)$ contained in $L$. Thus $J^n_K \leq C_{CG(t_2)}(B) \leq L$. But then $J^n_K = J_K$ as $C_L(D)$ is 3-closed. Therefore, the hypothesis of Lemma 7 is satisfied and we infer that $J_K \leq N$ and consequently $N \leq L$ by Lemma 26. So to summarize, if $D \leq L$ has order 3, then $C_{CG(t_2)}(D) \leq L$.

Now assume that $e \in C_G(t_2)$ and 3 divides $|L \cap L^e|$. Let $E \in \text{Syl}_3(L \cap L^e)$. If $E \notin \text{Syl}_3(L^e)$, then there exists a 3-group $E_0 \leq L^e$ such that $E_0 > E$ and $E_0 \leq C_{CG(t_2)}(d)$ for some $d \in Z(E)^\#$. But then $E_0 \leq L \cap L^e$ by the forgoing arguments. It follows that $E \in \text{Syl}_3(L) \cap \text{Syl}_3(L^e)$. Thus we may assume that $J_K \leq E$. Since $J_K$ is characteristic in $E$, we have

$$N_{CG(t_2)}(E) \leq N_{CG(t_2)}(J_K) \leq L.$$
by Lemma 26. By symmetry we also have $N_{CG(t_2)}(E) \leq L^e$. Therefore Sylow’s Theorem gives $L = L^e$ and we infer that $L$ is strongly 3-embedded in $CG(t_2)$. □

Because of Lemmas 25 and 28, we have that $C_G(t_2)/t_2$ satisfies the hypothesis of Theorem 37. Thus we deduce that $C_G(t_2) = L$. Finally Theorem 1 follows by applying theorem [22, Theorem A].

4. Identifying Fi$_{22}$

As promised in the introduction we begin this section by saying precisely what we mean when we say that a group $X$ is similar to a 3-normalizer in Fi$_{22}$.

**Definition 2.** Let $X$ be a group, $T \in \text{Syl}_2(X)$, $P \in \text{Syl}_3(X)$ and $Z := Z(P)$. Then $X$ is similar to a 3-normalizer in Fi$_{22}$ if

(i) $|X| = 2^8.3^9$;
(ii) $O_3(X)$ is extraspecial of order $3^7$ and exponent 3;
(iii) $O_2(X) = 1$;
(iv) $O^2(X) = C_X(Z)$ has index 2 in $X$;
(v) $C_X(Z)/O_3(X)$ has a normal subgroup which is elementary abelian of order 8 and $|Z(C_X(Z)/O_3(X))| = 2$;
(vi) $|Z(T)| = 4$; and
(vii) $J(P)$ is abelian.

Of course, if $M$ is the normalizer of a 3-central cyclic subgroup in Fi$_{22}$, then $M$ is similar to a 3-normalizer in Fi$_{22}$. From [4, p. 163] we have $M \sim 3^{1+6}:2^{3+4}:3^2:2$; a more precise description of $M$ is given in [2, 39.6].

Let $V$ be a 6-dimensional symplectic space over GF(2) with symplectic basis \{v$_1$, v$_2$, v$_3$, w$_1$, w$_2$, w$_3$\}, $K := \text{Sp}(V)$ and $H$ be the stabilizer in $K$ of the perpendicular decomposition $\langle v_1, w_1 \rangle \perp \langle v_2, w_2 \rangle \perp \langle v_3, w_3 \rangle$ of $V$. Then $H \cong \text{SL}_2(3) : \text{Sym}(3)$. Let $B$ be the base group of $H$ and $K$ be the complementary Sym(3) which permutes \{v$_1$, v$_2$, v$_3$\} and \{w$_1$, w$_2$, w$_3$\} in the obvious way. We pick out generators $s$ and $t$ of $K$ with $t$ of order 3 and $s$ of order 2. Write $B := X_1 \times X_2 \times X_3$ where, for $1 \leq i \leq 3$, $X_i \cong \text{SL}_2(3)$ acts naturally on $\langle v_i, w_i \rangle$. Let $X_1 := \langle a_1, t_1 \rangle$ with $a_1$ of order 4, $t_1$ of order 3 and $[v_1, t_1] = 0$. For $i = 2, 3$, set $a_i := a_i^{i-1}$ and $t_i := t_i^{i-1}$. Put $E := \langle t_1t_2t_3, t, [a_1, t] \rangle$. Then $|E| = 2^7.3^2$ and $|Z(E)| = 2$. Finally we set $u_1 := a_1^2$.

**Lemma 29.** Any subgroup of $H$ which contains $Z(B)$, has order $2^7.3^2$ and has centre of order 2 is conjugate in $H$ to $E$.

**Proof.** Suppose that $F \leq H$ with $|F| = 2^7.3^2$, $Z(B) \leq F$ and $|Z(F)| = 2$. Set $P := \langle t, t_1 \rangle$. Then $P \in \text{Syl}_3(H)$. Pick $R \in \text{Syl}_2(F)$ and $T \in \text{Syl}_3(F)$. We may assume that $T \leq P$. Since $Z(B)$ is extraspecial of order $3^7$ and exponent 3; $|Z(B)| = 4$ and $Z(B) \leq F$, $F \not\leq B\langle s \rangle$ and so also $T \not\leq B\langle s \rangle$. Thus $T \geq Z(P) = \langle t_1t_2t_3 \rangle$. Notice that $(O_2(B) \cap R)/Z(B)$ is normalized by $\langle t_1t_2t_3 \rangle$ and $\langle t_1t_2t_3 \rangle$
acts fixed point freely on $O_2(B)/Z(B)$. Therefore, $|(O_2(B) \cap R)/Z(B)| = 4^a$ for some $a \in \{1, 2, 3\}$. Since $|R| = 2^7$, we infer that $R \leq O_2(B)$ and $R$ is normal in $F$. Now $T$ acts on $R/Z(B)$ and since the cyclic subgroups of order 9 in $H$ act irreducibly on $O_2(B)/Z(B)$, we infer that $T$ is elementary abelian of order 9. Since all elementary abelian subgroups of order 9 contained in $P$ but not in $P \cap B$ are conjugate, we may assume that $T = \langle t_1t_2t_3, t \rangle$. Now

$$O_2(B)/Z(B) = C_{O_2(B)/Z(B)}(t) \oplus C_{O_2(B)/Z(B)}(tt_1t_2t_3) \oplus C_{O_2(B)/Z(B)}(t(t_1t_2t_3)^2),$$

where the summands in the decomposition are each of order 4 and form the complete set of minimal normal subgroups of $T O_2(B)/Z(B)$. Furthermore, these three summands are permuted transitively by $NP(T)$. Therefore, with out loss of generality we may assume that $R/Z(B) = C_{O_2(B)/Z(B)}(tt_1t_2t_3) \oplus C_{O_2(B)/Z(B)}(t(t_1t_2t_3)^2)$. But then $F = E$ and we are done. □

Lemma 30. The following hold.

(i) For $i = 1, 2, 3$, $[V, u_i] = \langle v_i, w_i \rangle$ is a non-degenerate symplectic 2-space and $C_V(u_i) = \langle v_j, w_j \mid 1 \leq j \leq 3, i \neq j \rangle$ is a non-degenerate symplectic 4-space.
(ii) $C_V(t_1t_2t_3) = \langle v_1, v_2, v_3 \rangle$.
(iii) $C_V(\langle t, t_1t_2t_3 \rangle) = \langle v_1 + v_2 + v_3 \rangle$.
(iv) $C_E(u_i)/\langle u_i \rangle$ acts faithfully as $(Q_8 \times Q_8)\langle t_1t_2t_3 \rangle$ on $C_V(u_1)$. Furthermore, $(C_E(u_i)/\langle u_i \rangle') \cong Q_8 \times Q_8$.
(v) $E$ acts irreducibly on $V$.

Proof. All of the statements follow simply from the description of $E$ given above. □

We now embark on the proof of Theorem 2. So suppose that $G$ is a group, $S \in Syl_3(G)$, $Z := Z(S)$ and $M := N_G(Z)$ is similar to a 3-normalizer in $Fi_{22}$. Set $J := J(S)$ and assume that $Z$ is not weakly closed in $J$ with respect to $G$. Set $N := N_G(J)$ and note that, by Lemma 3, $Z$ is not weakly closed in $J$ with respect to $N$. Set $Q := O_3(M)$. We begin by investigating the structure of $M$.

Lemma 31. $C_M(Z)/Q \cong E$.

Proof. Since $Q$ is extraspecial of order $3^7$ and exponent 3, we have that $C_{\text{Aut}(Q)}(Z) \cong 3^6:SP_6(3)$ by Lemma 10(i). Therefore, as $O_2(M) = 1$, $C_M(Z)/Q$ is isomorphic to a subgroup of $SP_6(3)$. Let $U$ be the normal subgroup of $C_M(Z)$ with $U/Q$ elementary abelian of order $2^3$. As the 2-rank of $SP_4(3)$ is 2, we have that $C_{Q/Z}(U) = 1$. Since $Q/Z$ is a symplectic space, for $U_0 \leq U$, we have $C_{Q}(U_0)$ is either $Z$ or is extraspecial. It follows that $Q = C_Q(A_1) \circ C_Q(A_2) \circ C_Q(A_3)$ where, for $i = 1, 2, 3$, $[U : A_i] = 2$, $C_Q(A_i) \cong 3_+^{1+2}$ and $\circ$ denotes the central product. As $U$ is normalized by $C_M(Z)$, it follows that $C_M(Z)$ embeds into the subgroup $SL_2(3) : \text{Sym}(3)$ of $SP_6(3)$. Now Lemma 29 gives $C_M(Z)/Q \cong E$. □
Spurred on by Lemma 31, we identify $C_M(Z)/Q$ with $E$ and select involutions $u_1, u_2$ and $u_3 \in M$ such that, for $i = 1, 2, 3, u_i Q/Q$ corresponds to the involutions $u_i \in E$ (hoping that the reader will not mind the abuse of notation). Set $U_i := \langle u_i \rangle$ and $U := \langle u_1, u_2, u_3 \rangle$. Additionally we assume, as we may, that $U$ has order 8.

**Lemma 32.** For $i = 1, 2, 3$, $C_M(u_i)/U_i$ is similar to a 3-normalizer in $U_6(2)$.

**Proof.** Since $u_1 Q, u_2 Q$ and $u_3 Q$ are conjugate in $C_M(Z)/Q$, $u_1, u_2$ and $u_3$ are conjugate in $C_M(Z)$ by Sylow’s Theorem. Thus it suffices to prove the result for $u_1$. From the structure of $H$, $U := \{u_1 Q, u_2 Q, u_3 Q\}$ is the set of elements $x Q$ of $M/Q$ with the property that $C_Q(x)$ is extraspecial of order $3^5$. Let $L$ be the kernel of the action of $M$ on $U$. Then Definition 2(v) implies $M/L \cong \text{Sym}(3)$, $|C_M(Z)/L| = 3$ and $|C_M(u_1)L/L| = 2$. It follows that $M = C_M(u_1)C_M(Z)$ and that there exists $x \in C_M(u_1)$ such that $u_1^x Q = u_3 Q$. Let $X := C_M(u_1)/U_1$. Then, by Lemma 30(iv), $O_3(X) \cong C_O(u_1)$ is extraspecial of order $3^5$. $O_2(X/O_3(X)) \cong Q_8 \times Q_8$ and $C_X(Z)/O_3(C_X(Z)) \sim (Q_8 \times Q_8).3$. Thus $|X/O^2(X)|$ has order 2. Finally, as there exists $x \in C_M(u_1)$ such that $u_1^x Q = u_3 Q$, we have that $O_3(X)/Z(O_3(X))$ is a chief factor for $X$. We have thus shown that $X$ satisfies Definition 1 and so $X$ is similar to a 3-normalizer in $U_6(2)$. \[\Box\]

Set $H := C_G(u_1)$ and $\bar{H} := H/U_1$. Then, by Lemma 32, $\bar{C}_M(u_1)$ is similar to a 3-normalizer in $U_6(2)$. Furthermore, $N_{\bar{H}}(\bar{Z}) = \bar{C}_M(u_1)$. Suppose that $\bar{Z}$ is not weakly closed in $\bar{C}_M(u_1)$. Then, by Theorem 1, $H \cong U_6(2)$ and, as $u_1 \in E'$, we infer that $H$ does not split over $U_1$. Thus $H$ is isomorphic to the unique group $2 \cdot U_6(2)$ [4, p. 164]. To apply the appropriate recognition theorem of Hunt [9], we need to show that $G$ is a simple group. So let $K$ be a minimal normal subgroup of $G$. Assume first that 3 divides $|K|$. Then $S \cap K \in \text{Syl}_3(K)$. Since $Z$ has order 3, we infer that $Z \leq K$ and, as $Z$ is not weakly closed $J$ with respect to $G$, we have that $S \cap K > Z$. Now $S \cap K \leq M \cap K \leq M$. Therefore $K \geq Q$, as $C_M(Z)/Q$ acts irreducibly on $Q/Z$ by Lemma 30(v). Hence $S \cap K$ has centre of order 3 and is non-abelian. It follows that $K$ is a non-abelian simple group. Furthermore the Frattini Argument (applied to $Z^K$) shows that $G = MK$. Hence $G/K$ is solvable. Therefore the perfect group $H$ is contained in $K$ and the theorem of Hunt [9] implies that $K \cong \text{Fi}_{22}$. Finally we have $G = K$ since $N_K(Z)$ has the same order as $M$. So assume that $K$ is a $3'$-group. Let $K^* := O_3(G)$. Then $K^* > 1$ and, as $O_3(M) = 1$, $C_K^*(Z) = 1$. Furthermore, $N_{G/K^*}(ZK/K) = MK^*/K^*$ and $ZU_1K^*/U_1K^*$ is not weakly closed in $C_M(U_1)U_1K^*/U_1K^*$. It follows from the previous case that $G/K^* \cong \text{Fi}_{22}$. We deduce a contradiction, for in $\text{Fi}_{22}$ there are subgroups of order 9 consisting only of cyclic subgroups which are 3-central (see [4, p. 163]). Thus it is impossible for $C_K^*(Z) = 1$. We have demonstrated that there are no minimal normal subgroups of $3'$-order and we infer that $G \cong \text{Fi}_{22}$.

We have shown that to prove Theorem 2 it suffices to contradict the assumption that $\bar{Z}$ is weakly closed in $\bar{H}$.

Since $Z$ is the unique cyclic subgroup of order 3 in $ZU_1$ we have that $Z$ is weakly closed in $H$. We continue via a series of four lemmas.
Lemma 33. \(|J| = 3^5, UQ\) normalizes \(J\) and \([[J, u_1]] = 3\).

**Proof.** Since \(J\) is abelian and \(Q\) is generated by abelian subgroups of order \(3^4\), we deduce that \(|J| \geq 3^5\) and \(|J \cap Q| \geq 3^3\). From Lemma 30(iii), we have \(|C_{Q/Z}(S)| = 3\). Thus \(|JQ/Q| = 3\) and \(|J \cap Q| = 3^4\). It now follows that \(JQ/Q\) acts like \(t_1 t_2 \bar{t}_3\). Therefore \(JQ\) is normalized by \(U\). Since \(J := J(JQ)\) and \([JQ/Q, U] = 1\), we have that \(UQ\) normalizes \(J\) and Lemma 30(i) and (ii) implies \([[J, u_1]] = 3\). \(\square\)

Lemma 34. Assume that \(g \in G, Y := Z^g \leq J \cap Q\) and \(YZ\) has order 9. Set \(L := \langle Q, Q^g \rangle\). Then

(i) \(L/C_L(YZ) \cong \text{SL}_2(3)\); and

(ii) \(C_{Q^g}(Z)Q \in \text{Syl}_3(M)\).

**Proof.** Since \(YZ \leq Q\) and \(Q' = Z\), we have that \(YZ\) is normalized by \(Q\). Also, as \(YZ\) has order 9, \(C_Q(Y)\) has order \(3^6\) and is non-abelian. Since \(M^g/Q^g\) has abelian Sylow 3-subgroups, we get \(Z = C_Q(Y) \leq Q^g\). Thus \(Q^g\) normalizes \(Y\). Therefore \(L\) normalizes \(ZQ\) and (i) follows as \([YZ, Q] \leq Z\) and \([YZ, Q^g] = Y\). Now \((Q^g \cap Q)' \leq Q' \cap (Q^g)' = Z \cap Y = 1\). It follows that \(|Q \cap Q'| \leq 3^4\) and so, as \(|C_{Q^g}(Z)| = 3^6\), we get (ii) by considering orders. \(\square\)

Lemma 35. For \(i = 1, 2, 3, Z\) is the only \(G\)-conjugate of \(Z\) contained in \(C_J(u_1)\). In particular, there are at most 37 conjugates of \(Z\) contained in \(J\).

**Proof.** Suppose that \(Y := Z^g \leq C_J(u_1)\) with \(Y \neq Z\). Then \(u_1 \in M^g = N_G(Y)\). Since \(|J \cap Q^g| = 3^4, [[J \cap Q^g, u_1]]\) has order at most 3 by Lemma 33 and \([Y, u_1] = 1\), we deduce that \(C_{Q^g}(u_1)\) is extraspecial of order \(3^5\). Since \(Z\) is weakly closed in \(C_M(u_1)\) and \(C_{Q^g}(u_1)\) normalizes \(J\), we infer that \(C_{Q^g}(u_1)\) centralizes \(Z\). Therefore, \(C_{Q^g}(u_1) \leq M\) and \(Y = C_{Q^g}(Y)Q \in \text{Syl}_3(M)\) by Lemma 34(ii). Since \(u_1\) normalizes \(C_{Q^g}(Y)Q\), this contradicts the structure of \(C_M(Z)/Q \cong E\). As \(u_1Q, u_2Q\) and \(u_3Q\) are all conjugate in \(M/Q\), the first part of the lemma holds.

Since \(|C_J((u_1, u_2, u_3))| = 3^2\), Lemma 33 implies that \(|C_J(u_1) \cup C_J(u_2) \cup C_J(u_3)| = 171\). So there are at most \((243 - 171)/2 + 1 = 19\) conjugates of \(Z\) contained in \(J\). \(\square\)

Lemma 36. We have \(\langle Z^g \mid g \in G, Z^g \leq J \rangle \leq Q\).

**Proof.** Assume that \(Y := Z^g \leq J\) and \(Y \not\leq Q\). Then, since \([YQ, U_1] \leq Q\), Lemma 35 implies that \(U_1\) does not normalize \(Y\). Therefore, \(N_{QU_1}(Y) = J \cap Q\) and so there are at least \(54 = |QU_1/(J \cap Q)|\) conjugates of \(Y\) in \(J\). On the other hand, Lemma 35 states that there are at most 37. Therefore, \(Y \leq Q\) and the result follows. \(\square\)

We now close in on the contradiction. By Lemma 36, we have \(W := \langle Z^N \rangle \leq Q\). Since \(US \leq N\), \(W\) is normalized by \(US\). So, as \(Z\) is not weakly closed in \(J\), we infer from Lemma 30 that \(W = Q \cap J\). Select \(g \in N\) such that \(Y := Z^g \leq W\) with \(Y \neq Z\). Note that by considering orders and using \(g \in N\), we have \(Q \cap Q^g = W = J \cap Q\). Set \(L := \langle Q, Q^g \rangle\).
Then, by Lemma 34(ii), \( L \cap M \) contains a Sylow 3-subgroup \( S_1 := C_{Q^g}(Z)Q \) of \( M \). Now \( C_Q(Y) \) has index 3 in \( Q \) and \( [C_Q(Y)/(Q \cap Q^g), S_1] = [C_Q(Y)/(Q \cap Q^g), C_{Q^g}(Z)Q] = 1 \). Thus, as \([J, S_1] \leq J \cap Q\), \( S_1 \) centralizes a subgroup of order \( 3^3 \) in \( C_{Q^g}(Z)C_Q(Y)/(Q \cap Q^g) \). Therefore \( S_1 \) normalizes a subgroup \( F \) of \( C_{Q^g}(Z) \) of order \( 3^5 \). Let \( f \in Q \setminus M^g \). Then \( f \) normalizes \( F \) and thus \( F \leq Q^g \cap Q^{gf} \). As \( Z^{gf} \neq Y \) (\( f \notin M^g \)), \( Q^g \cap Q^{gf} \) is abelian, however \( F \) is not! This contradiction shows that \( ZU_1/U_1 \) is not weakly closed in \( C_M(U_1)/U_1 \) and so we infer that \( G \cong \text{Fi}_{22} \) as claimed.

Appendix A

The objective of this appendix is to prove Theorem 37, which was used in the final stages of the proof of Theorem 1.

**Theorem 37.** Suppose that \( G \) is a group, \( R \leq G \) and \( L := N_G(R) \). Assume that

(i) \( L \) is strongly 3-embedded in \( G \); and

(ii) \( L = RK \) with \( K \cong \text{PSp}_4(3) \cong \text{U}_4(2) \), \(|R| = 2^8 \) and \( R \) is the unique minimal normal subgroup of \( L \).

Then \( G = O_{2^8}(G)L \).

**Proof.** We show that \( R \) is strongly closed in \( L \) with respect to \( G \) and then invoke a theorem of Goldschmidt’s.

(37.1) \( R \) is weakly closed in \( L \).

Let \( S \in \text{Syl}_2(L) \). Suppose that \( x \in G \) and \( R^x \leq S \) with \( R \neq R^x \). Then \( R^x R/R \) is an elementary abelian 2-subgroup of \( L/R \cong \text{U}_4(2) \) and so \(|R^x R/R| \leq 2^4 \) and \(|R \cap R^x| \geq 2^4 \). Since no four group of \( L/R \) has all its non-trivial elements 2-central, Lemma 11(ii) gives \(|C_R(R^x)| \leq 2^4 \). Hence \( R \cap R^x = C_R(R^x) \) has order \( 2^4 \) as does \( R^x R/R \). Because \( S/R \) has a unique elementary abelian subgroup of order \( 2^4 \), we get \( K := N_L(R^x R/R) \sim 2^8.2^4.\text{Alt}(5) \) (see [4, p. 26]). We claim that \( R \) and \( R^x \) are the unique elementary abelian subgroups of \( RR^x \) of order \( 2^8 \). Suppose that \( F \) is a further such subgroup. Then the argument above shows that \( F \cap R = R \cap R^x \) and \( FR = R^x R \). Select \( f \in F \) such that \( f \notin R^x \cup R \) and \( fR \) is not 2-central in \( L/R \). Then \( f = ab \) where \( a \in R \) and \( b \in R^x \) and \( f^2 = abab = 1 \). So \( a \in C_R(f) = R \cap R^x \) which means that \( f \in R^x \) which is a contradiction. Since \( F \) is generated by elements which project as non-2-central elements of \( L/R \), we conclude that \( F = R^x \). In particular we now have that \( R^x \) is normalized by \( K \) and \( K/R^x \sim 2^4.\text{Alt}(5) \). However, the chief factors for \( K \) in \( R \) are both \( \text{SL}_2(4) \)-type modules for \( \text{Alt}(5) \) and so therefore is \( R^x R/R^x \), but, as \( L^x/R^x \cong \text{U}_4(2) \), the chief factor for \( K \) in \( R^x R/R^x \) should be an \( O_4^- \)-type module for \( \text{Alt}(5) \). This is a contradiction and so we have that \( R \) is weakly closed in \( L \). \( \Box \)

We now move on to the main part of the proof of Theorem 37. Suppose that \( R \) is not strongly closed in \( L \). Select \( x \in G \) such that \( R^x \cap L \neq R \) and pick \( r \in R^x \) such \( rR \neq R \).
Now let $S \in \text{Syl}_2(L)$ be such that $C_S(r) \in \text{Syl}_2(C_L(r))$. As $R$ is weakly closed in $L$ by (ii), $S \in \text{Syl}_2(G)$. Select $T \in \text{Syl}_2(C_G(r))$ such that $C_S(r) \leq T$. Since $R^x \leq C_G(r)$, $T$ contains a conjugate of $R^x$ and so we may as well suppose that $R^x \leq T$. In particular, as $R^x$ is weakly closed in $L^x$, $T \leq L^x$ and $C_S(r)$ normalizes $R^x$.

Because of assumption (ii), $L \sim 2^8.U_4(2)$ and, as an $L/R$-module, $R$ is the natural $\text{GF}(4)U_4(2)$-module considered as a $\text{GF}(2)$-module. Therefore, since $r \in (R^x)^\#$, Lemma 11(i) provides the first parts of the following claim.

(37.2) One of the following holds:
(i) $C_L^x(r) \sim 2^8.2_1^{1+4}.\text{Sym}(3)$ and $C_G(r)$ has Sylow 3-subgroups of order 3; or
(ii) $C_L^x(r) \sim 2^8.3_1^{1+2}.Q_8$ and $C_G(r)$ has Sylow 3-subgroups of order 27.

For the second parts, we use the fact that $L^x$ is strongly 3-embedded in $G$ by (ii), to get that $\text{Syl}_3(C_L^x(r)) \subseteq \text{Syl}_3(C_G(r))$. So (37.2) holds. $\Box$

(37.3) $r$ is not conjugate to an element of $K$.

Suppose that $r$ is conjugate to an element of $K$. Then, by Lemma 11(ii) and (iii), either $r$ is in class 2A and $C_L^x(r) = C_R(r)C_K(r) \sim 2^6.2_1^{1+4}.(3 \times \text{Sym}(3))$ or $r$ is in class 2B and $C_L^x(r) = C_R(r)C_K(r) \sim 2^4.(2^4.\text{Sym}(3)) = 2^8.\text{Sym}(3)$. As $L$ is strongly 3-embedded in $G$, the first possibility implies that $C_G(r)$ has Sylow 3-subgroups of order 9, contrary to (37.2). Therefore, $C_L^x(r) \sim 2^8.\text{Sym}(3)$ and consequently $C_L^x(r) \sim 2^8.2_1^{1+4}.\text{Sym}(3)$. Let $D \in \text{Syl}_3(C_L^x(r))$. Then $C_R(D) = 1$ and, by Lemma 11(iv), $|C_K(D)| \leq 108$. On the other hand, for $D_1 \in \text{Syl}_3(C_L^x(r))$, we have $C_R^x(D_1) = 2^4$ by Lemma 11(v). Hence $C_G(D) \not\leq L$ and this contradicts $L$ being strongly 3-embedded in $G$. $\Box$

(37.4) $rR$ is in class 2A.

Suppose that $rR$ is in class 2B. Then $C_R^x(r) = [R, r]$ has order $2^4$. So all the involutions of $rR$ are conjugate by Lemma 12. In particular, $r$ is conjugate to an element of $K$, and this contradicts (37.3). $\Box$

Since $L/R$ contains no subgroups of order 4 all of whose non-trivial elements are in class 2A, we infer that

(37.5) $|(R^x \cap S)R/R| = 2$.

(37.6) $|C_S(r)| \geq 2^{11}$.

Since $rR$ is in class 2A, the involutions in $rR$ have centralizers of order $2^{12}, 2^{12}, 2^{11.3}$ and $2^{12}$ by Lemmas 11(vi) and 12. Since $C_S(r) \in \text{Syl}_2(C_L(r))$, the claim follows. $\Box$

Suppose that $C_R^x(r) \leq R^x$. Then $|R^x \cap R| = 2^6$ and $C_R^x(r) \subseteq R^x$ contains all the involutions in $Rr$ contrary to (37.3). Therefore, $C_R^x(r) \not\leq R^x$ and we infer from (37.4) that $|C_R^x(r)^{R^x}/R^x| = 2$. In particular, $|R \cap R^x| = 2^2$. It follows that $C_S(r) \cap R^x = (R \cap R^x)^r(r)$ has order $2^6$. Hence $|C_S(r)^{R^x}| \geq 2^{11+8-6} = 2^{14}$ by (37.6) and so $T_0 := C_S(r)^{R^x}$ has index at most 2 in $T$. 

Now $C^R_r(C_R(r)) = (R \cap R^x)(r)$ is normalized by $T_0$ and $r \in C^R_r(C_R(r))$. Thus, using Lemma 11(vii), we obtain the following contradiction:

$$r \in C^R_r(T_0) \leq [C^R_r(C_R(r)), T_0] = [(R \cap R^x)(r), R^x C_S(r)]$$

$$= [(R \cap R^x)(r), C_S(r)] = [R \cap R^x, C_S(r)] \leq R \cap R^x.$$ 

This contradiction proves that $R$ is strongly closed in $S$. Set $H := (R^G)$. Then, by Goldschmidt’s Theorem [10], $H$ contains no section isomorphic to PSp$(4, 3)$. Therefore, $L \cap H = R$ and, as $L = N_G(R)$, $R \in Syl_2(H)$. Hence $R \leq Z(N_H(R))$ and Burnside’s normal $p$-complement theorem gives $H = O_2(H)R$. Finally the Frattini Argument shows that $G = O_2(G)L$ as claimed. □

References

