Doubly Transitive Permutation Groups in Which the One-Point Stabilizer is Triply Transitive on a Set of Blocks

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Suppose that $G$ is a doubly transitive permutation group on a finite set $\Omega$ and that for $\alpha$ in $\Omega$ the stabilizer $G_{\alpha}$ has a set $\Sigma = \{B_1, \ldots, B_r\}$ of nontrivial blocks of imprimitivity in $\Omega - \{\alpha\}$, that is, $|B_i| > 1$ and $|\Sigma| > 1$. In two previous papers [5, 6] it was shown that apart from a few known groups, the setwise stabilizer of $B_i$ in $G_{\alpha}$ acts faithfully on $B_i$ if $G_{\alpha,\Sigma}$ is the alternating or symmetric group, one of the Mathieu groups, or a normal extension of $PSL(2, q)$ in their usual representations. The raises the question:

QUESTION. If $G_{\alpha,\Sigma}$ is multiply transitive, is it possible to characterize the groups $G$ for which the setwise stablizer in $G_{\alpha}$ of the block $B$ of $\Sigma$ does not act faithfully on $B$?

The only groups I know of in which $G_{\alpha}$ is 2-transitive on $\Sigma$ and for $B$ in $\Sigma$, the stablizer of $B$ is not faithful on $B$, are the following:

(i) $PSL(n, q) < G < PGL(n, q)$, for $n \geq 3$ in its natural representation. If $B \in \Sigma$ then $B \cup \{\alpha\}$ is a line and $PSL(n - 1, q) < G_{\alpha,\Sigma} < PGL(n - 1, q)$.

(ii) $PSU(3, q) < G < PSL(3, q)$ permuting the set of absolute points of the projective plane over a field of $q^2$ elements. If $B \in \Sigma$ then $B \cup \{\alpha\}$ is a non-absolute line, $|B| = q$, $|\Sigma| = q^2$, and $G_{\alpha,\Sigma}$ has a regular normal subgroup of order $q^2$.

(iii) $G = A_7$ acting on the 15 points of the projective geometry of dimension 3 over a field of two elements; $G_{\alpha} \simeq PSL(2, 7)$ acts 2-transitively on the set of lines containing $\alpha$.

(iv) $G$ has a regular normal subgroup.

If we drop the assumption that $G_{\alpha,\Sigma}$ is 2-transitive, then we have another family of examples.

(v) A group $G$ of Ree type $R(q)$ is 2-transitive on $q^3 + 1$ points. For any two distinct points $\alpha, \beta$, there is a unique nontrivial element $g$ in $G_{\alpha,\beta}$ which
fixes at least three points. The set of points distinct from α which are fixed by g
is a block of imprimitivity for \( G_\alpha \) which is fixed pointwise by g.

The aim of this paper is to give an affirmative answer to the question in the
case where \( G_\alpha \) is 3-transitive on \( \Sigma \), but not faithful on \( \Sigma \). Different methods will
be needed to deal with the case \( G_\alpha \) faithful on \( \Sigma \). Throughout the paper we
assume the following hypothesis.

**Hypothesis (\( \ast \)).** (a) \( G \) is a doubly transitive permutation group on a set \( \Omega \) of \( n \)
points. For \( \alpha \) in \( \Omega \) the stabilizer \( G_\alpha \) has a set \( \Sigma = \{ B_1, \ldots, B_t \} \) of nontrivial blocks of
imprimitivity, where \( |\Sigma| > 1 \), \( |B_i| > 1 \), and \( n = 1 + tb \).

(b) We denote by \( K_i \) the subgroup of \( G_\alpha \) fixing \( B_i \) setwise, by \( K_i \) the subgroup
fixing \( B_i \) pointwise, and by \( H \) the subgroup fixing all blocks of \( \Sigma \) setwise.

(c) \( H \neq 1, K_i \neq 1 \).

The result of this paper, which is stated below, is a generalization of [5, Theorems 1 and 2; 6].

**Theorem.** Assume Hypothesis (\( \ast \)) and assume that \( G_\alpha \) is 3-transitive of degree
\( t \geq 3 \). Then \( \text{PGL}(2, q) \leq G_\alpha \leq \text{PGL}(2, q) \) in its natural representation where
\( q = t - 1 \) is a prime power. Moreover, either

(a) \( \text{PSL}(3, q) \leq G \leq \text{PGL}(3, q) \) in its natural representation, or

(b) \( \Omega \) with the translates under \( G \) of \( B_1 \cup \{ \alpha \} \) as lines is an affine translation
plane of order \( q \), and \( G \) contains the translation group.

An immediate corollary of this result which follows from [5] is:

**Corollary.** If Hypothesis (\( \ast \)) is true and if \( G_\alpha \) is 4 transitive, then \( G_\alpha \cong S_t \),
where \( t \) is 4 or 5, and either

(a) \( \text{PSL}(3, t - 1) \leq G \leq \text{PGL}(3, t - 1) \) in its natural representation, or

(b) \( t = 4 \), \( G \) has a regular elementary abelian normal subgroup of order 9,
and \( H \) is cyclic of order 2.

**Notation.** Most of the notation used follows the conventions of [1, 7].
If a group \( G \) has a permutation representation on a set \( \Sigma \), the constituent of \( G \)
on \( \Sigma \) is denoted by \( G^\Sigma \); the set of fixed points of \( G \) in \( \Sigma \) is denoted by \( \text{fix}_G \) \( \Sigma \) and
simply by \( \text{fix} G \) if the set \( \Sigma \) is clear from the context; and orbits of \( G \) containing
more than one point are called long \( G \)-orbits.

**Definition.** A block design consists of a set of \( v \) points and a set of \( b \) blocks
with a relation called incidence between points and blocks, such that any block is
incident with \( k \) points and any two points with \( \lambda \) blocks, where \( \lambda > 0 \) and
2 \leq k < v - 1. The number \( r \) of blocks incident with a given point is also constant. If \( k > 2 \) it is called a proper design. By easy counting arguments we have

\[ vr = bk, \quad (v(\lambda - 1)) = bk(k - 1). \]

Also it is well known that \( b > v \), or equivalently \( r > k \).

**Proof of the Theorem.** Assume that Hypothesis \((\ast)\) is true and that \( G_{s}^{*} \) is 3-transitive of degree \( t > 3 \). It is easy to check that a group \( G \) satisfying \( PSL(mq) < G \leq PTL(m, q) \) in its natural representation for some \( m \geq 3 \) and prime power \( q \), satisfies Hypothesis \((\ast)\) where \( \{B_{i} \cup \{a\} | i = 1, \ldots, t\} \) is the set of lines containing \( \alpha \), and these \( B_{i} \) are the only nontrivial blocks for \( G_{s} \). Further, since \( PGL(m - 1, q) < G_{s} \leq PTL(m - 1, q) \) in its natural representation, then \( G_{s}^{*} \) is 3-transitive if and only if \( m = 3 \). Thus in this case the theorem is true. So we assume that \( G \) does not satisfy \( PSL(m, q) < G \leq PTL(m, q) \) for any \( m \geq 3 \). Then by \([6, Lemma 1.1]\), \( H \) is semiregular on \( \Omega - \{\alpha\} \) and \( G \) is a group of automorphisms of a block design with \( \lambda = 1 \), the blocks of which are the translates under \( G \) of the set \( B_{i} \cup \{\alpha\} \).

**Proposition 1.** If \( K_{1} \) has a normal subgroup which acts regularly on \( \Sigma - \{B_{1}\} \) then the theorem is true.

**Proof.** Assume that \( K_{1} \) has a normal subgroup which is regular on \( \Sigma - \{B_{1}\} \). Then by \([2, Theorem 1.1]\), the fact that \( G_{s}^{*} \) is 3-transitive, and \([7, 10.2 \text{ and } 11.3]\), either

(a) \( PGL(2, q) \leq G_{s}^{*} \leq PTL(2, q) \) in its natural representation where \( q = t - 1 \) is a prime power, or

(b) \( G_{s}^{*} \) has a regular elementary abelian normal subgroup of order \( t \), where \( t = 3 \) or \( t = 2^{a} \) for some integer \( a \geq 2 \) (since \( t \geq 3 \)).

In case (a) the theorem is true by \([6]\). If \( t = 3 \), the theorem is true by \([5, \text{ Theorem 1}]\), for \( G_{s}^{*} = S_{3} \simeq PSL(2, 2) \). So suppose that \( G_{s}^{*} \) has a regular normal subgroup which is elementary abelian of order \( t = 2^{a} \) for some \( a \geq 2 \). Then \( G_{a} \) has a normal subgroup \( N \) containing \( H \) such that \( N^{*} = N|H \) is elementary abelian and regular, and \( N^{*} \) is the unique minimal normal subgroup of \( G_{s}^{*} \), (by \([7, 11.5, 10.1]\)).

Let \( M = \langle K_{i} | i = 1, \ldots, t \rangle \). Since each \( K_{i} \) centralizes \( H \), (for \( H \) and \( K_{i} \) are normal subgroups of \( K_{1} \) with trivial intersection), then also \( M \) centralizes \( H \). If \( M \cap H \) is nontrivial, then it is an abelian normal subgroup of \( G_{a} \). On the other hand, if \( M \cap H \) is trivial, then \( M \) is isomorphic to \( M^{*} \), a nontrivial normal subgroup of \( G_{s}^{*} \). Hence \( M \) has a subgroup which is normal in \( G_{a} \), elementary abelian of order \( 2^{a} \), and acts regularly on \( \Sigma \). Thus in either case \( G_{a} \) has a non-
trivial abelian normal subgroup, and since the degree \( n = 1 + 2^m b \) is odd, it follows from [4, Theorem B] that \( G \) is a normal extension of one of

(a) \( \text{PSL}(m, q) \) where \((q^m - 1)/(q - 1)\) is odd and \( m \geq 2 \),
(b) \( \text{PSU}(3, q) \),
(c) \( S_n(q) \), in their natural representations, or
(d) \( G \) has a regular normal subgroup.

We are assuming that \( G \) does not satisfy (a) if \( m \geq 3 \), and \( \text{PSL}(2, q) \) is 2-primitive and so cannot satisfy Hypothesis (*). By considering the possible parameters of a block design, it is easy to show that \( S_n(q) \) cannot act as a group of automorphisms of a block design with \( \lambda = 1 \). Also no normal extension of \( \text{PSU}(3, q) \) can satisfy Hypothesis (*) with \( G \) 3-transitive. Hence \( G \) has a regular normal subgroup, \( R \), say.

Consider the group \( NR \), where \( N \) is the normal subgroup of \( G \) containing \( H \) such that \( N^2 = N/H \) is elementary abelian and regular. Since \( H \) is semiregular on \( \Omega - \{a\} \) and \( N \) is regular on \( \Sigma \) it is easy to see that \( N \) is semiregular on \( \Omega - \{a\} \). Thus \( NR \) is a Frobenius group. By [1, 10.3.1], the Sylow 2-subgroups of \( N \) are cyclic or generalized quaternion, and since \( N/H \) is elementary abelian of order \( 2^n \geq 4 \), it follows that \( 2^n = 4 \). Thus \( G_\alpha \cong S_n \cong \text{PGL}(2, 3) \), and the theorem follows in this case from [5, Theorem 1] or [6]. This completes the proof of Proposition 1.

Thus we may assume that \( K_1 \) has no normal subgroup acting regularly on \( \Omega - \{B_1\} \).

**Lemma 2.** (a) \( K_1 \cap K_2 \) fixes the blocks \( B_1 \) and \( B_2 \) setwise and has nontrivial orbits of equal length in \( \Omega - \{B_1, B_2\} \).

(b) The centralizer \( C_\alpha(K_1) \) of \( K_1 \) acts as a Frobenius group on \( B_1 \cup \{a\} \) of degree \( 1 + b = r^c \) for some prime \( r \) and positive integer \( c \). Moreover, \( C_\alpha(K_1) \) has a normal subgroup \( A \) which is elementary abelian of order \( r^c \) and acts regularly on \( B_1 \cup \{a\} \).

**Proof.** (a) Since \( K_1 \) is 2-transitive on \( \Omega - \{B_1\} \) and has no normal subgroup acting regularly on \( \Omega - \{B_1\} \) it follows from [7, 12.1] that \( K_1 \cap K_2 \) fixes only the blocks \( B_1 \) and \( B_2 \) setwise and has orbits of equal length in \( \Omega - \{B_1, B_2\} \).

(b) It is easy to show that \( K_1^{\alpha} \) has a trivial centralizer in \( G_\alpha \cong \text{G}(\Phi) \), (see [3]). It follows from [6, Lemma 1.4] that the centralizer \( C_\alpha(K_1) \) of \( K_1 \) acts as a Frobenius group on \( B_1 \cup \{a\} \) of degree \( 1 + b = r^c \) for some prime \( r \) and positive integer \( c \), and \( C_\alpha(K_1) \) has a normal subgroup \( A \) such that \( A^{B_1 \cup \{a\}} \) is elementary abelian and regular. Moreover, since \( A_\alpha \) fixes \( B_1 \cup \{a\} \) pointwise, then \( A_\alpha \) is a subgroup of \( K_1 \) which centralizes \( K_1 \). Since \( K_1^{\alpha} \cong K_1 \) has a trivial centralizer, and since \( A_\alpha \cong A_\alpha \) centralizes \( K_1^{\alpha} \), it follows that \( A_\alpha \) is trivial. Thus \( A \) is elementary abelian of order \( r^c \).
LEMMA 3. (a) All long orbits of $K_1$ have length $b'(t-1)$, where $b'$ is a divisor of $b = r^c - 1$, and consist of $b'$ points of each block of $\Sigma - \{B_i\}$.

(b) The group $A$ fixes some long $K_1$-orbit $\Gamma$ setwise.

Proof. (a) We show that $K_1 \cap K_2$ is transitive on $B_1$, and then it follows that its normal subgroup $K_1 \cap K_2$ has orbits of equal length, say $b'$, in $B_1$, for some $b'$ dividing $b$. Then since $K_1$ is transitive on $\Sigma - \{B_i\}$, it follows that all long orbits of $K_1$ have length $b'(t-1)$ and contain $b'$ points of each block of $\Sigma - \{B_i\}$.

Thus we need to show that $K_1 \cap K_2$ is transitive on $B_1$. Let $\beta$ be a point of $B_1$. Since $K_1$ is transitive on $B_1$, then $(K_1 \cap K_2)$ contains $\beta$, and so $(K_1 \cap K_2)$ is transitive on $\Sigma - \{B_1\}$ and $\{B_2\}$. It follows that $(K_1 \cap K_2)$ has index $b$ in $K_1$, that is, $K_1 \cap K_2$ is transitive on $B_1$.

(b) Since $A$ centralizes $K_1$, it permutes the orbits of $K_1$ among themselves. Since $A$ is an $r$ group and since the number of long $K_1$-orbits, namely, $b/b'$ is not divisible by $r$, we conclude that $A$ must fix one of these orbits, say $\Gamma$, setwise.

LEMMA 4. If $B_i$ and $B_j$ are (not necessarily distinct) blocks of $\Sigma - \{B_i\}$ and if $a$ is a nonidentity element of $A$, then $|B_i^a \cap B_j| \leq 1$.

Proof. Suppose to the contrary that for some $i \geq 2$, $j \geq 2$ and for some nonidentity element $a$ of $A$, we have $|B_i^a \cap B_j| \geq 2$. Then $(B_i \cup \{a\})^a$ and $B_j \cup \{a\}$ are blocks of a design with $\lambda = 1$ which have at least two points in common, and hence $(B_i \cup \{a\})^a = B_j \cup \{a\}$. Thus $a^a$ belongs to $B_j \cup \{a\}$, which is a contradiction, for $a^a$ is a point of $B_1$ since $A$ acts regularly on $B_1 \cup \{a\}$.

LEMMA 5. $b' \neq 1$.

Proof. Suppose that $b' = 1$ and let $\Gamma$ be a long orbit of $K_1$ fixed setwise by $A$ (by Lemma 3). By Lemma 2, $K_1 \cap K_2$ fixes only the blocks $B_1$ and $B_2$ of $\Sigma$ setwise. Hence $K_1 \cap K_2$ fixes exactly one point of $\Gamma$, namely the point $\gamma$ where $\Gamma \cap B_2 = \{\gamma\}$. Since $A$ centralizes $K_1 \cap K_2$ then $A$ fixes $\gamma$, and since $K_1$ is transitive on $\Sigma - \{R_i\}$ we conclude that $A$ fixes $\Gamma$ pointwise.

Since $b' = 1$, there is an orbit $\Gamma''$ of $K_1$ of length $t - 1$ distinct from $\Gamma$. Let $A'$ be the setwise stabilizer in $A$ of $\Gamma''$. The index of $A'$ in $A$ is at most the number of long $K_2$ orbits so that $|A : A'| \leq b < r^c = |A|$, that is, $A'$ is nontrivial. Let $a$ be a nonidentity element of $A'$. Then, as in the previous paragraph, we can show that fixes $\Gamma''$ pointwise. Thus $B_2^a \cap B_2$ contains $(\Gamma \cap B_2) \cup (\Gamma'' \cap B_2)$, a contradiction to Lemma 4. Hence $b' \neq 1$.

LEMMA 6. The group $A$ is semiregular on $\Gamma$, and if $\gamma$ is a point of $\Gamma$, then $\text{fix}_A(K_1)$, is a union of $x$ orbits of $A$ and hence contains $xm \gamma$ points for some $x \geq 1$. 
Proof. The group $A$ acts faithfully on $\Gamma$, for suppose that a nonidentity element $a$ of $A$ fixed $\Gamma$ pointwise. Then $B_a \cap B_2 \subseteq \Gamma \cap B_2$, which contradicts Lemmas 4 and 5.

Then it follows from [3] that $A$ is semiregular on $\Gamma$ and $\text{fix}_r(K_1)_\gamma$ is a union of say $x$ orbits of $A$ for any $\gamma \in \Gamma$. Thus $|\text{fix}_r(K_1)_\gamma| = x\gamma^r$.

**Lemma 7.** (a) $t - 1$ is divisible by $r^\gamma$.

(b) There are $xr^\gamma/m$ blocks of $\Sigma - \{B_1\}$ which contain a point of $\text{fix}_r(K_1)_\gamma$ where $\gamma \in \Gamma \cap B_2$ and each of these blocks contains exactly $m$ points of $\text{fix}_r(K_1)_\gamma$, for some positive integer $m$ dividing $x$.

Proof. (a) By [7, 3.6], the normalizer $N$ of $(K_1)_\gamma$ in $K_1$ is transitive on $\text{fix}_r(K_1)_\gamma$ and hence $x\gamma^r$ divides $|N : (K_1)_\gamma|$ which divides $|K_1 : (K_1)_\gamma| = b'(t - 1)$. Then since $b'$ divides $b = r^\gamma - 1$ it follows that $r^\gamma$ divides $t - 1$.

(b) Now $\{\Gamma \cap B_j | j \geq 2\}$ is a set of blocks of imprimitivity for $(K_1)_\gamma$, and since $N$ is transitive on $\text{fix}_r(K_1)_\gamma$, clearly $N$ is transitive on

$$X = \{\Gamma \cap B_j | B_j \cap \text{fix}_r(K_1)_\gamma \neq \emptyset\}.$$ 

It follows that each $\Gamma \cap B_j$ in $X$ contains the same number $m$ of points of $\text{fix}_r(K_1)_\gamma$, and hence, that $X$ has $|\text{fix}_r(K_1)_\gamma|/m = x\gamma^r/m$ members. Finally, $K_1 \cap K_2$ is transitive on $\Gamma \cap B_2$ and the stabilizer of the point $\gamma$ of $\Gamma \cap B_2$ is $(K_1)_\gamma$. Moreover, $(K_1)_\gamma$ fixes exactly $m$ points of $\Gamma \cap B_2$ as we have just shown, and so by [7, 3.6], the normalizer $N \cap K_2$ of $(K_1)_\gamma$ in $K_1 \cap K_2$ is transitive on $\text{fix}_{r \cap B_2}(K_1)_\gamma$. Hence $m$ divides $|N \cap K_2 : (K_1)_\gamma|$ which divides

$$|(K_1 \cap K_2) : (K_1)_\gamma| = |\Gamma \cap B_2| = b'.$$

Thus $m$ divides $b'$, and since $x\gamma^r/m$ is an integer then $m$ must divide $x$.

**Lemma 8.** (a) The orbits of $K_1 \cap K_2$ in $\Sigma - \{B_1, B_2\}$ all have length $b'$.

(b) For a point $\gamma$ of $\Gamma \cap B_2$, $(K_1)_\gamma$ fixes my blocks of $\Sigma - \{B_1, B_2\}$ setwise, and each of these blocks contains a point of $\text{fix}_r(K_1)_\gamma$.

Proof. Let $a$ be a nonidentity element of $A$. Then since $\Gamma \cap B_2$ is an orbit of $K_1 \cap K_2$ and since $A$ centralizes $K_1 \cap K_2$, then $(\Gamma \cap B_2)^a$ is also an orbit of $K_1 \cap K_2$. By Lemma 4, $(\Gamma \cap B_2)^a$ consists of one point from each of $b'$ blocks of $\Sigma$. Thus $(\Gamma \cap B_2)^a$ corresponds to an orbit $\Delta$ of $K_1 \cap K_2$ of length $b'$ in $\Sigma - \{B_1, B_2\}$. Hence by Lemma 2, all orbits of $K_1 \cap K_2$ in $\Sigma - \{B_1, B_2\}$ have length $b'$.

Now by Lemma 7, $(K_1)_\gamma$ fixes $m$ points of $\Gamma \cap B_2$ and permutes the remaining points nontrivially. Hence, since $A$ centralizes $(K_1)_\gamma$, $(K_1)_\gamma$ fixes exactly $m$ points.
of \((\Gamma \cap B_2)^a\) and permutes the remaining points nontrivially. It follows that there are exactly \(m\) blocks in the corresponding \((K_1 \cap K_2)\)-orbit \(\Delta\) in \(\Sigma\) which contain a point of \(\text{fix}_r(K_1)_r\), and the remaining blocks of \(\Delta\) are permuted nontrivially by \((K_1)_r\).

To complete the proof of Lemma 8 we show: if \(B_j\) is an arbitrary block of \(\Sigma - \{B_1, B_2\}\) which is fixed setwise by \((K_1)_r\), then

(a) the \((K_1 \cap K_2) - \text{orbit} \Delta'\) in \(\Sigma\) containing \(B_j\) contains exactly \(m\) blocks which contain a point of \(\text{fix}_r(K_1)_r\), and the remaining blocks of \(\Delta'\) are permuted nontrivially by \((K_1)_r\), and hence,

(b) \(B_j\) contains a point of \(\text{fix}_r(K_1)_r\). To show this, let \(B_i\) be a block of \(\Delta\) containing a point of \(\text{fix}_r(K_1)_r\) (where \(\Delta\) was defined above), and let \(B_j\) be an arbitrary block of \(\Sigma - \{B_1, B_2\}\) fixed setwise by \((K_1)_r\). Then \(B_i\) and \(B_j\) lie in orbits of \(K_1 \cap K_2\) in \(\Sigma\) of length \(b'\), and hence the groups \(K_1 \cap K_2 \cap K_i\) and \(K_1 \cap K_2 \cap K_j\) have index \(b'\) in \(K_1 \cap K_2\). Since \((K_1)_r\) fixes \(B_i\) and \(B_j\) setwise it is a subgroup of \(K_1 \cap K_2 \cap K_i\) and \(K_1 \cap K_2 \cap K_j\). Then since \((K_1)_r\) has index \(b'\) in \(K_1 \cap K_2\) it follows that \((K_1)_r = K_1 \cap K_2 \cap K_i = K_1 \cap K_2 \cap K_j\). Since \(K_1 \cap K_2\) is transitive on \(\Sigma - \{B_1, B_2\}\) there is an element \(g\) in \(K_1 \cap K_2\) such that \(B_i^g = B_j\). Hence \((K_1)_r^g = (K_1 \cap K_2 \cap K_i)^g = K_1 \cap K_2 \cap K_j = (K_1)_r\), that is, \(g\) normalizes \((K_1)_r\). It follows that the orbit of \(K_1 \cap K_2\) in \(\Sigma\) containing \(B_j\), namely \(\Delta^g\), contains exactly \(m\) blocks which contain a point of \(\text{fix}_r(K_1)_r\), and the remaining blocks are permuted nontrivially by \((K_1)_r\).

We now complete the proof of the theorem. By Lemmas 7 and 8 it follows that the number of blocks of \(\Sigma - \{B_i\}\) which contain a point of \(\text{fix}_r(K_1)_r\) is, on the one hand, \(x r^e/m\), and on the other, \(1 + my\). Hence \(1 + my = x r^e/m = O (\text{mod } r^e)\).

Consider the set \(X = \{K_1 \cap K_j \cap K_i | j > 2\}\). This is a conjugacy class of subgroups of \(K_1 \cap K_2\). Since \(K_1 \cap K_2\) is transitive on \(\Sigma - \{B_1, B_2\}\), then each group in \(X\) fixes the same number of blocks of \(\Sigma - \{B_1, B_2\}\) setwise. We showed in the proof of Lemma 8 that \((K_1)_r\) belongs to \(X\) and fixes exactly \(my\) blocks of \(\Sigma - \{B_1, B_2\}\) setwise. Hence \(|X| = (t - 2)/my\).

On the other hand, \(K_1 \cap K_2\) fixes \(B_2\) as a set and is transitive on it. Hence, since the group \((K_1)_r\), of \(X\) fixes points of \(B_2\), it follows that each group in \(X\) fixes the same number, say \(z\), of points of \(B_2\). Thus \(b = z |X|\), and so \(my = (t - 2)z/b\). Hence we have \(1 + my = 1 + (t - 2)z/b = 0 (\text{mod } r^e)\), and so \(b + (t - 2)z = 0 (\text{mod } r^e)\). By Lemma 7, \(t - 2 = -1 (\text{mod } r^e)\), and so \(b = z (\text{mod } r^e)\). However \(z \leq b < r^e\). Hence \(b = z\), and so \(my = t - 2\). This means that \((K_1)_r\) fixes all blocks of \(\Sigma\) setwise, and hence that \(K_1 \cap K_2\) acts semiregularly on \(\Sigma' - \{B_1, B_2\}\). Thus \(K_1\) acts on \(\Sigma - \{B_1\}\) as a Frobenius group. By [7, 5.1], \(K_1\) has a characteristic subgroup acting regularly on \(\Sigma - \{B_1\}\), a contradiction to our assumption that \(K_1\) has no normal subgroup acting regularly on \(\Sigma - \{B_1\}\). This completes the proof of the theorem.
References

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