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Attainability of Systems of Identities on Semigroups

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1. INTRODUCTION

A decomposition Δ of a semigroup S is the partition of all elements of S due to a congruence relation ξ, that is, $\Delta : S = \bigcup_{\alpha \in S'} S_\alpha$ where $S_\alpha \cap S_\beta = \square$, $\alpha \neq \beta$, $S' \cong S/\xi$. Let ξ_1 and ξ_2 be two congruences corresponding to the decompositions Δ_1 and Δ_2, respectively. As usual we say Δ_1 is greater than Δ_2 if and only if $\xi_1 \subsetneq \xi_2$. Consider all congruences ξ on S such that S/ξ satisfies a given condition $\mathscr{T}$. Then we say that S/ξ is of given type $\mathscr{T}$, Δ is a $\mathscr{T}$-decomposition, and ρ is a $\mathscr{T}$-congruence. Throughout this paper $\mathscr{T}$ means a system of identities, which will be strictly defined later, something like $\{x^2 = x, yz = zx\}$. As is well known, for any $\mathscr{T}$, there is the smallest $\mathscr{T}$-congruence on any semigroup S, equivalently, the greatest $\mathscr{T}$-decomposition of S [*1*, *4*, *8*, *9*, *10*, *12*]. If there is a $\mathscr{T}$-congruence ξ on S such that $\iota \subseteq \xi \subset \omega$ where ι is the equality relation and ω is the universal relation [p. 13, *1*], then S is called $\mathscr{T}$-decomposable; otherwise S is $\mathscr{T}$-indecomposable. In the greatest semilattice-decomposition of any semigroup, each congruence class is semilattice-indecomposable [*7*, *9*, *11*]. On the other hand, in the case of the greatest idempotent-decomposition or commutative-decomposition, a congruence class which is a subsemigroup is not necessarily indecomposable with respect to the type. The following question arises: Under what condition on $\mathscr{T}$ does the same situation occur for any semigroup S as in the case of the greatest semilattice-decomposition? This paper proves that the question is confirmative if and only if $\mathscr{T}$ is equivalent to semilattice $\{x^2 = x, xy = yx\}$ except for two trivial cases.

2. PRELIMINARIES

A word, $f(x_1, \cdots, x_n)$ or simply f, is a finite sequence of letters $x_1, \cdots, x_n$ in which same letters may be used repeatedly. Throughout this paper, x, y, z, $x_i, y_i \cdots$ denote letters, and $f, g, h, k, w \cdots$ denote words. A sequence of n

identical letters, $xx \cdots x$, is symbolized as x^n, $n \geq 1$, where n is called the exponent of x. Generally f has the form

$$f(x_1, \cdots, x_n) = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_m}^{n_m},$$

x_{i_j} being one of x_i, $1 \leqslant i \leqslant n$. A pair (f, g) of arbitrary words is called an identity. The equality of pairs is defined to be $(f, g) = (h, k)$ if and only if there is a one-to-one mapping $x_i \to x_i'$ between letters $x_1, \cdots, x_n$ used in (f, g) and $x_1', \cdots, x_n'$ in (h, k) such that either h is obtined from f, and k from g, or h from g, and k from f by replacing x_i by x_i'. Hereafter we shall denote an identity by $f = g$ instead of (f, g). To avoid confusion, the equality expressing definition is written as $=$, that is, $f = x^2y$ means that a word f is defined to be x^2y.

Let $\mathfrak{T}$ be the family of all systems $\mathscr{T}$ of identities such that each element $\mathscr{T}$ of $\mathfrak{T}$ is composed of finite or infinite number of identities, and $\mathscr{T}$ contains $x = x$. Containing $x = x$ is for theoretical convenience; and we will not explicitly write $x = x$ in expressing $\mathscr{T}$, except the case of $\mathscr{T}$ consisting of $x = x$ alone. Let $\mathscr{T} = \{T_\lambda; \lambda \in \Lambda\}$, $\mathscr{T}' = \{T_{\lambda'}; \lambda' \in \Lambda'\}$ where T_λ is $f_\lambda = g_\lambda$ and $T_{\lambda'}$ is $f_{\lambda'} = g_{\lambda'}$. The equality $\mathscr{T} = \mathscr{T}'$ is defined as follows: There is a one-to-one mapping $\lambda \to \lambda'$ between Λ and Λ' such that $f_\lambda = g_\lambda$ is equal to $f_{\lambda'} = g_{\lambda'}$. A system $\mathscr{T}$ is associated with a statement that each $f_\lambda = g_\lambda$ of $\mathscr{T}$ identically holds when all letters are replaced by elements of S. Then we say that S satisfies $\mathscr{T}$ or S is of type $\mathscr{T}$, and a congruence ξ is a $\mathscr{T}$-congruence if S/ξ satisfies $\mathscr{T}$. As was stated, for any $\mathscr{T}$ and for any semigroup S, there is the smallest $\mathscr{T}$-congruence on S. The greatest $\mathscr{T}$-homomorphic image of S is denoted by $S/\mathscr{T}$. Also we can say that given $\mathscr{T}$, there are semigroups satisfying $\mathscr{T}$. These semigroups may happen to be one-element semigroups. The two quasiorderings ρ and σ on $\mathfrak{T}$ are defined as follows:

(1) $\mathscr{T} \rho \mathscr{S}$ if and only if $\mathscr{S}$ is a subsystem of $\mathscr{T}$.

We also use the symbols, union $\cup$, intersection $\cap$, as usual. $\mathscr{T} \rho \mathscr{S}$ if and only if $\mathscr{T} \cup \mathscr{S} = \mathscr{T}$ equivalently $\mathscr{T} \cap \mathscr{S} = \mathscr{S}$.

(2) $\mathscr{T} \sigma \mathscr{S}$ if and only if $\mathscr{T}$ implies $\mathscr{S}$, that is, a semigroup satisfies $\mathscr{S}$, whenever it satisfies $\mathscr{T}$.

In other words, $\mathscr{T} \sigma \mathscr{S}$ if and only if $\mathscr{S}$ is obtained from $\mathscr{T}$ by a combination of a finite number of the following procedures besides those defining the equality. Let f_i, f and g be words.

(3) $(f_1 f_2) f_3$ is replaced by $f_1(f_2 f_3)$ and vice versa.

(4) $f = g$, $g = h$ imply $f = h$.

(5) $f = g$ implies $fh = gh$ and $hf = hg$ where h is any word.

(6) The letters x_i involved in $f = g$ are replaced by words $h_i(y_{i_1}, \cdots, y_{i_l})$, that is, from $f(x_1, \cdots, x_n) = g(x_1, \cdots, x_n)$ we have

$$f(h_1, \cdots, h_n) = g(h_1, \cdots, h_n).$$

As a special case of (6) there exists:

(6′) For any partition Δ of the set $\{x_1, \cdots, x_n\}$ of all letters in $f = g$, all the letters in a same equivalent class are replaced by one letter, and those in distinct classes are replaced by distinct letters. We call it the procedure by a partition Δ. Also we can say

(7) $\mathscr{T} \sigma \mathscr{S}$ if and only if $S/\mathscr{T}$ satisfies $\mathscr{S}$ for any semigroup S.

Let $\hat{\sigma}$ denote the equivalence derived from σ in such a way that $\mathscr{T} \hat{\sigma} \mathscr{S}$ if and only if $\mathscr{T} \sigma \mathscr{S}$ and $\mathscr{S} \sigma \mathscr{T}$. Clearly $\rho \subseteq \sigma$ and $\hat{\sigma} \subseteq \sigma$. When $\mathscr{F} \hat{\sigma} \mathscr{S}$ we say $\mathscr{T}$ and $\mathscr{S}$ are equivalent. We easily have

LEMMA 1. *If* $\mathscr{T} \sigma \mathscr{S}$, *then* $\mathscr{T} \hat{\sigma} (\mathscr{T} \cup \mathscr{S})$. *If* $\mathscr{T}_\alpha \sigma \mathscr{S}_\alpha$, $\alpha \in \Xi$, *then* $(\bigcup_{\alpha \in \Xi} \mathscr{T}_\alpha) \sigma (\bigcup_{\alpha \in \Xi} \mathscr{S}_\alpha)$.

Generally let η be a binary relation on a set E and ξ an equivalence on E which need not be included in η. Then a relation η/ξ on the set E/ξ is defined as follows: Let a^* denote the equivalence class containing $a \in E$. For a^*, $b^* \in E/\xi$, $a^* \eta/\xi\ b^*$ if and only if there are $a_1 \in a^*$, $b_1 \in b^*$ such that $a_1 \eta b_1$.

We denote $\sigma/\hat{\sigma}$ by σ^*, $\mathfrak{T}/\hat{\sigma}$ by $\mathfrak{T}^*$, and the set union of ρ and $\hat{\sigma}$ by $\rho \cup \hat{\sigma}$; the transitive closure of $\rho \cup \hat{\sigma}$ by $\overline{\rho \cup \hat{\sigma}}$.

THEOREM 1.

$$\sigma^* = \frac{\sigma}{\hat{\sigma}} = \frac{\rho}{\hat{\sigma}} = \frac{\rho \cup \hat{\sigma}}{\hat{\sigma}} = \frac{\overline{\rho \cup \hat{\sigma}}}{\hat{\sigma}},$$

and $\mathfrak{T}^*$ *is a complete lattice with respect to* σ^*.

Proof. Since $\rho \subseteq \sigma$ and $\hat{\sigma} \subseteq \sigma$, we get $\overline{\rho \cup \hat{\sigma}} \subseteq \sigma$ and hence $(\sigma \cup \hat{\sigma})/\hat{\sigma} \subseteq \sigma/\hat{\sigma} = \sigma^*$. Suppose $\mathscr{T}^* \sigma^* \mathscr{S}^*$. By the definition there are $\mathscr{T}_1 \in \mathscr{T}^*$ and $\mathscr{S}_1 \in \mathscr{S}^*$ such that $\mathscr{T}_1 \sigma \mathscr{S}_1$. By Lemma 1, $\mathscr{T}_1 \hat{\sigma} (\mathscr{T}_1 \cup \mathscr{S}_1)$ and $(\mathscr{T}_1 \cup \mathscr{S}_1) \rho \mathscr{S}_1$. This means that $\mathscr{T}^*\ \rho/\hat{\sigma}\ \mathscr{S}^*$. Therefore we have $\sigma^* \subseteq \rho/\hat{\sigma} \subseteq (\rho \cup \hat{\sigma})/\hat{\sigma} \subseteq (\overline{\rho \cup \hat{\sigma}})/\hat{\sigma}$. Thus we have proved all the equalities in this theorem. Certainly σ^* is a partially ordering since σ is a quasiordering. Let $\{\mathscr{T}_\alpha^*\}$ be any subset of $\mathfrak{T}^*$. Then, by Lemma 1, we can easily show that $(\cup_\alpha \mathscr{T}_\alpha)^*$ is the greatest lower bound of $\{\mathscr{T}_\alpha^*\}$. Since $\mathfrak{T}^*$ contains the greatest and smallest elements it is complete lattice.

Remark. Since each $\mathscr{T}_\alpha$ contains $x = x$, the intersection is not empty.

When $\mathscr{T}\rho\mathscr{S}$, we say that $\mathscr{S}^*$ is greater than $\mathscr{T}^*$ with respect to σ^*. The greatest lower bound of $\{\mathscr{T}_\alpha{}^*\}$ is denoted by $\bigwedge_\alpha \mathscr{T}_\alpha{}^*$.

COROLLARY 1.

$$\bigwedge_\alpha \mathscr{T}_\alpha{}^* = (\bigcup_\alpha \mathscr{T}_\alpha)^*$$

The smallest $\mathscr{T}$- and $\mathscr{S}$-congruences on a same semigroup S are denoted by $\rho_{\mathscr{T}}$ and $\rho_{\mathscr{S}}$, respectively. Another quasiordering π on $\mathfrak{I}$ is defined as follows:

(8) $\mathscr{T}\pi\mathscr{S}$ if and only if $\rho_{\mathscr{S}} \subseteq \rho_{\mathscr{T}}$ on any semigroup.

LEMMA 2. *$\sigma = \pi$ and hence $\hat{\sigma} = \hat{\pi}$.*

Proof. Clearly $\sigma \subseteq \pi$. We may show $\pi \subseteq \sigma$. Suppose $\mathscr{T}\pi\mathscr{S}$ and let S be any semigroup satisfying $\mathscr{T}$. Consider the smallest $\mathscr{T}$- and $\mathscr{S}$-congruences on S. Then $\iota \subseteq \rho_{\mathscr{S}} \subseteq \rho_{\mathscr{T}}$ on S. Since $\rho_{\mathscr{T}} = \iota$, we have $\rho_{\mathscr{S}} = \iota$ on S. In other words S satisfies $\mathscr{S}$. Therefore $\mathscr{T}\sigma\mathscr{S}$.

Let $\mathscr{T}$ be a system of identities, $\mathfrak{S}$ be the family of semigroups. A semigroup S which belongs to $\mathfrak{S}$ is called a semigroup of type $\mathfrak{S}$.

DEFINITION. *$\mathscr{T}$ is universality on all semigroups of type $\mathfrak{S}$ if and only if $\mathscr{T}$ implies $\{x = y\}$ under the condition that all the letters are regarded as elements of $S \in \mathfrak{S}$. $\mathscr{T}$ is equality on all semigroups of type $\mathfrak{S}$ if and only if $\mathscr{T}$ is equivalent to $\{x = x\}$ under the same condition. In particular, if $\mathfrak{S}$ is characterized by a system $\mathscr{S}$ of identities, then $\mathscr{T}$ is universality on all semigroups of type $\mathfrak{S}$ if and only if $(\mathscr{T} \cup \mathscr{S})\sigma\{x = y\}$, equivalently, $(\mathscr{T} \cup \mathscr{S})\hat{\sigma}\{x = y\}$; $\mathscr{T}$ is equality on all semigroups of type $\mathfrak{S}$ if and only if $(\mathscr{T} \cup \mathscr{S})\hat{\sigma}\{x = x\}$. Also $\mathscr{T}$ is universality on all semigroups if and only if $\mathscr{T}\sigma\{x = y\}$, equivalently $\mathscr{T}\hat{\sigma}\{x = y\}$; $\mathscr{T}$ is equality on all semigroups if and only if $\mathscr{T}\hat{\sigma}\{x = x\}$. We say that $\mathscr{T}$ is universal on a semigroup S if and only if S is $\mathscr{T}$-indecomposable.*

DEFINITION. *Let ξ_S be the smallest $\mathscr{T}$-congruence on any semigroup S belonging to $\mathfrak{S}$. $\mathscr{T}$ is called attainable on all semigroups of $\mathfrak{S}$ (or simply "on $\mathfrak{S}$") if, for each $S \in \mathfrak{S}$, the following condition is satisfied: If a congruence class of S modulo ξ_S is a subsemigroup, then it is $\mathscr{T}$-indecomposable. In particular if $\mathfrak{S}$ is the family of all semigroups (without restriction) and if $\mathscr{T}$ is attainable on $\mathfrak{S}$, $\mathscr{T}$ is called attainable (on all semigroups). If $\mathscr{T}$ is attainable on $\mathfrak{S}$ and if $\mathfrak{S}$ consists of a semigroup S alone, we say that $\mathscr{T}$ is attainable on S.*

As the trivial cases, if $\mathscr{T}$ is universality or equality, then $\mathscr{T}$ is attainable. We shall call $\mathscr{T}$ trivial if and only if $\mathscr{T}$ is either universality or equality.

The following lemma is very fundamental:

LEMMA 3. *If $\mathscr{T}$ is universal on a semigroup S then $\mathscr{T}$ is universal on any homomorphic image of S.*

The following lemma is an immediate consequence from Lemma 2.

LEMMA 4. *Let $\mathscr{T}_1 \hat{\sigma} \mathscr{T}_2$. Then $\mathscr{T}_1$ is attainable on semigroups of type $\mathscr{S}$ if and only if $\mathscr{T}_2$ is attainable on the semigroups of type $\mathscr{S}$.*

Let w be a word and let $E(w)$ denote the set of all letters which appear in w. The sum of all exponents of the same letter x is called the total exponent of x in w, for example, if $w = x^2yzx$, that of x is 3. The sum of the total exponents of all letters in w is called the length of w, denoted by $\| w \|$.

Let T be an identity $f = g$ and let $E(f) \cup E(g) = \{x_1, \cdots, x_m\}$. The total exponents of x_i in f and g are denoted by s_i and t_i, respectively ($i = 1, \cdots, m$), $s_i \geqslant 0$, $t_i \geqslant 0$; in detail, $s_i \neq 0$ and $t_i = 0$ if and only if x_i is in f but not in g; but s_i and t_i are not simultaneously zero. If $s_i = t_i$ ($i = 1, \cdots, m$) then $f = g$ is called equiexponential; otherwise heteroexponential.

For a given identity $f = g$, the sequences $\{F_i; i = 0, 1, 2 \cdots\}$ and $\{G_i; i = 0, 1, 2 \cdots\}$ of sets are associated with f and g, respectively as follows. For each nonnegative integer i,

$$F_i = \{x \in E(f) \cup E(g); \text{the total exponent of } x \text{ in } f \text{ is } i\}$$
$$G_i = \{x \in E(f) \cup E(g); \text{the total exponent of } x \text{ in } g \text{ is } i\}.$$

F_i and G_i may happen to be empty. Clearly there is a positive integer i_1 such that

$$F_i = G_i = \Box, \qquad \text{for all} \quad i > i_1$$

and

$$\bigcup_{i=0}^{i_1} F_i = \bigcup_{i=0}^{i_1} G_i = E(f) \cup E(g).$$

An identity $f = g$ is equiexponential if and only if $F_i = G_i$ for all non-negative integers i.

3. On Commutative Semigroups

Let I be the semigroup of all positive integers with usual addition. Any congruence ξ on I different from the equality relation is uniquely determined by its homomorphic image, a finite cyclic semigroup, and hence by the index a and the period r [1]; ξ is denoted by $\xi\ (a, r)$, where a is still called the index of ξ and r the period of ξ. For convenience let $\xi(\infty, r) = \xi(a, \infty)$ for all r, a, and denote it by $\xi(\infty)$, corresponding to the equality relation. The universal relation on I is $\xi(1, 1)$. The complete lattice L of all congruences on I consists of all $\xi(a, r)$ and $\xi(\infty)$. The following lemma is easily obtained.

LEMMA 5. *The join and meet of a subset* $\mathscr{A} = \{\xi(a_\lambda, r_\lambda); \lambda \in \Lambda\}$ *are given as follows:*

$$\text{join } \xi(\bar{a}, \bar{r}), \quad \text{where} \quad \bar{a} = \text{Min.}\{a_\lambda; \lambda \in \Lambda\}, \quad \bar{r} = \text{g.c.d.}\{r_\lambda; \lambda \in \Lambda\},$$

$$\text{meet } \xi(\underline{a}, \underline{r}), \quad \text{where} \quad \underline{a} = \text{Max.}\{a_\lambda; \lambda \in \Lambda\}, \quad \underline{r} = \text{l.c.m.}\{r_\lambda; \lambda \in \Lambda\}.$$

We consider now the smallest T-congruence on I. If T is equiexponential, the congruence is the equality relation.

LEMMA 6. *Let T be a heteroexponential identity $f = g$. The smallest T-congruence on I is given by $\xi(a_0, r_0)$, where*

$$a_0 = \begin{cases} \text{Min.}\{\|f\|, \|g\|\}, & \text{if} \quad \|f\| \neq \|g\| \\ \|f\| + i_0,\ i_0 = \text{Min.}\{i \geqslant 0; F_i \neq G_i\}, & \text{if} \quad \|f\| = \|g\| \end{cases}$$

$$r_0 = \text{g.c.d.}\{|s_i - t_i|; 1 \leqslant i \leqslant m\}.$$

Proof. The proof of a_0 is given below in the case where $\|f\| = \|g\|$, while the remaining part is easily obtained by recalling the method of construction of the smallest congruence [10]. By the definition of i_0, we may assume there is a letter $x_i \in F_{i_0}$ but $x_i \notin G_{i_0}$ such that $s_i = i_0$. Then $i_0 < t_i$ by the minimality of i_0. This shows that $\|f\| + s_i < \|g\| + t_i$, where $\|f\| + s_i$ is the value of f obtained by replacing x_i by 2 and all the other by 1 under addition. Immediately we have

$$a_0 = \|f\| + s_i .$$

Let $\mathscr{T}$ be a system of identities: $\mathscr{T} = \{T_\lambda; \lambda \in \Lambda\}$, where T_λ is $f_\lambda = g_\lambda$; and let $E(f_\lambda) \cup E(g_\lambda) = \{x_{\lambda 1}, \cdots, x_{\lambda m_\lambda}\}$; and let $s_{\lambda i}$ and $t_{\lambda i}$ be the total exponents of $x_{\lambda i}$ in f_λ and g_λ, respectively. Let Λ_0 be the set of all indices λ such that T_λ is heteroexponential. Λ_0 may happen to be empty.

LEMMA 7. *If $\mathscr{T}$ is attainable on I, then either*

(9.1) *All T_λ, $\lambda \in \Lambda$, are equiexponential, that is, $\mathscr{T}$ is the equality on I;*

or

$$(9.2) \quad \begin{cases} (9.2.1) & \text{Min.}\{\|f_\lambda\|, \|g_\lambda\|; \lambda \in \Lambda_0\} = 1, \quad \Lambda_0 \neq \square \\ (9.2.2) & \text{g.c.d.}\{|s_{\lambda i} - t_{\lambda i}|; 1 \leqslant i \leqslant m_\lambda, \lambda \in \Lambda_0\} = 1. \end{cases}$$

Conversely if $\mathscr{T}$ includes heteroexponential identities and satisfies (9.2) then $\mathscr{T}$ is universal on I.

Proof. For simplicity the left sides of (9.2.1) and (9.2.2) are denoted by $\theta(\mathscr{T})$ and $\pi(\mathscr{T})$, respectively. Let $\xi(a_0, r_0)$ be the smallest $\mathscr{T}$-congruence on the additive semigroup I of all positive integers. Recalling the method of construction of the smallest congruence and Lemma 6, we have $r_0 = \pi(\mathcal{T})$ and $a_0 \geqslant \theta(\mathcal{T})$. To show $\theta(\mathcal{T}) = 1$, suppose $\theta(\mathcal{T}) \neq 1$. Then $a_0 \geqslant 2$ and the congruence class I_0 which is a subsemigroup of I consists of

$$ir_0 , (i + 1) r_0 , \cdots \qquad \text{for some } i,$$

where $ir_0 \geqslant a_0$. Consider the smallest $\mathcal{T}$-congruence on I_0. Then ir_0 composes a class by itself alone. Therefore I_0 is $\mathcal{T}$-decomposable, contradicting the assumption that $\mathcal{T}$ is attainable. Thus $\theta(\mathcal{T}) = 1$. Accordingly $\mathcal{T}$ contains an identity $f_\lambda = g_\lambda$, $\lambda \in \Lambda_0$, such that $\| f_\lambda \| = 1$. If $\| f_\lambda \| = \| g_\lambda \| = 1$, $f_\lambda = g_\lambda$ is $x = y$, universality, which satisfies $\theta(\mathcal{T}) = \pi(\mathcal{T}) = 1$. Therefore we may assume $\| f_\lambda \| < \| g_\lambda \|$. We have $a_0 = \| f_\lambda \| = 1$ and $r_0 = \pi(\mathcal{T})$ by Lemmas 5 and 6. Suppose $r_0 > 1$. Then $\iota \subset \xi(1, r_0) \subset \omega$, $I/\xi(1, r_0)$ being the group mod. r_0, and the congruence class which is a subsemigroup I_0 consists of r_0, $2r_0$, $\cdots$, which is isomorphic to I. Thus I_0 is $\mathcal{T}$-decomposable, a contradiction. Therefore $\pi(\mathcal{T}) = r_0 = 1$. The proof of the remaining part is already included above.

The following lemma is discussed on all semigroups.

LEMMA 8. *If $\mathcal{T}$ includes heteroexponential identities and satisfies (9.2), then there is $\mathcal{T}'$ such that $\mathcal{T}\hat{\sigma}\mathcal{T}'$ and $\mathcal{T}'\rho\{x = x^2\}$.*

Proof. Let S be any semigroup and let a be any element of S. Since $\mathcal{T}$ satisfies (9.2.1) and (9.2.2), $\mathcal{T}$ is universal on I by Lemma 7; and hence, by Lemma 3, $\mathcal{T}$ is universal on the cyclic subsemigroup $[a]$ of S generated by a. Therefore the image a' of a into $S/\mathcal{T}$ satisfies $a' = a'^2$, that is, $S/\mathcal{T}$ satisfies $x = x^2$. Thus $\mathcal{T}\sigma\{x = x^2\}$. By Lemma 1 and (7), this lemma has been proved.

Remark. Lemma 8 can also be proved directly by procedures (3) through (6) without using (7), but the above proof is simpler.

THEOREM 2. *Let $\mathcal{T}$ be a system including heteroexponential identities. If $\mathcal{T}$ is attainable on all (commutative) semigroups, then*

$$\mathcal{T}\hat{\sigma}\mathcal{T}', \qquad \mathcal{T}'\rho\{x = x^2\} \qquad \textit{for some} \qquad \mathcal{T}'.$$

Proof. Immediately from Lemmas 7 and 8.

4. ON GROUPS

Let J be the group of all integers with addition. Analogously to Lemma 7, and with the same notations, LEMMA 9. *If $\mathscr{T}$ is attainable on J, then either*

(11.1) *all T_λ, $\lambda \in \Lambda$, are equiexponential;*

or

(11.2) g.c.d. $\{|s_{\lambda_i} - t_{\lambda_i}|; 1 \leqslant i \leqslant m_\lambda, \lambda \in \Lambda\} = 1$.

Conversely if $\mathscr{T}$ includes heteroexponential identities and satisfies (11.2) then $\mathscr{T}$ is universal on J.

THEOREM 3. *If $\mathscr{T}$ is a system including heteroexponential identities and if $\mathscr{T}$ is attainable on all groups, then $\mathscr{T}$ is universal on all groups.*

Proof. By using Lemmas 9 and 3, we get $\mathscr{T}\sigma\{x = x^2\}$ in the same way as the proof of Lemma 8. On the other hand $x = x^2$ on groups implies that $\mathscr{T}$ is universal on all groups.

For a word $h = x_1 \cdots x_n$ we introduce the inverse word as follows

$$h^{-1} = x_n^{-1}x_{n-1}^{-1} \cdots x_1^{-1},$$

and for an identity $f = g$, the word fg^{-1}, which has been reduced by the procedure $wxx^{-1}v = wv$, is called the induced word of $f = g$. The terminology "total exponent" is defined even if negative exponents are admitted. A word in which the total exponent of each letter is 0 is called null-exponential.

Let $\mathscr{T} = \{T_\lambda; \lambda \in \Lambda\}$ be a system of equiexponential identities $T_\lambda : f_\lambda = g_\lambda$. Accordingly the induced word of T_λ

(10) $f_\lambda g_\lambda^{-1} = x_{i_1}^{n_i} \cdots x_{i_\lambda}^{n_{i\lambda}}$, where n_{i_j} are integers is null-exponenital. Let G be a group. The smallest $\mathscr{T}$-congruence on G is determined by the normal subgroup N_G of G which is defined in the following way: N_G is a subgroup of G generated by (10), precisely speaking, by the subset

$$\{x_{i_1}^{n_{i_1}} \cdots x_{i_\lambda}^{n_{i\lambda}}; x_{i_1}, \cdots, x_{i_\lambda} \in G, \lambda \in \Lambda\}.$$

N_G is called the normal subgroup of G associated with $\mathscr{T}$.

THEOREM 4. *If $\mathscr{T}$ is a system of nontrivial equiexponential identities, then $\mathscr{T}$ is not attainable on all groups.*

Proof. Let F be a free group such that the number of the generators of F is equal to the number of letters contained in a particular identity T_λ. Let N_F be the normal subgroup of F associated with $\mathscr{T}$. Then $N_F \subset F$ because each element of N_F is null-exponential with respect to the generators of F. Also since N_F contains the word $f_\lambda g_\lambda^{-1}$, $\{\epsilon\} \subset N_F$, that is, N_F is a proper normal subgroup of F. According to the theory of free groups [6], N_F is also a free group. Consider the normal subgroup $N_{N_F} \subset N_F$. This means that N_F is $\mathscr{T}$-decomposable.

5. On Idempotent Semigroups

As a consequence of sections 3 and 4, we may proceed under the assumption that $\mathscr{T}$ consists of heteroexponential identities including $x^2 = x$. In other words we may find a system $\mathscr{T}$ of identities which are attainable on idempotent semigroups. The purpose of this section is to prove

Theorem 5. *Let $\mathscr{T}$ contain $x = x^2$. $\mathscr{T}$ is attainable if and only if $\mathscr{T}$ is equivalent to either universality or $\{x = x^2, xy = yx\}$.*

As the first step

Lemma 10. *If $\{xy = y\}\, \sigma \mathscr{T}$, then $\mathscr{T}$ is not attainable.*

Proof. Let R be a right zero semigroup, ($xy = y$ for all $x, y \in R$) with the cardinal number $|R| \geqslant 2$. Let $S = R \cup \{p, q\}$, $p \notin R$, $q \notin R$, $p \neq q$, and let a and b be fixed elements of R. A binary operation in S is defined as follows:

$$\begin{aligned} qp &= b \\ xp &= a, \quad \text{if} \quad q \neq x \in S \\ xy &= y, \quad \text{if} \quad p \neq y \in S. \end{aligned} \tag{12}$$

Associativity is easily checked. Define a congruence ξ on S as follows: $x\xi y$ if and only if $x = y$ or $x, y \in \{a, b, p\}$. This ξ is the smallest idempotent-congruence on S and S/ξ is of type $\{xy = y\}$. Hence ξ is the smallest $\mathscr{T}$-congruence. The congruence class $\{a, b, p\}$, however, is $\{xy = y\}$-decomposable, hence $\mathscr{T}$-decomposable.

As the dual case.

Lemma 10′. *If $\{x = x^2, xy = x\}\, \sigma \mathscr{T}$, then $\mathscr{T}$ is not attainable.*

Lemma 11. *If $\{x^2 = x, xy = yx\}\, \hat{\sigma} \mathscr{T}$, then $\mathscr{T}$ is attainable.*

Lemma 11 was proved in [*7*, *9*, *11*].

To prove the main theorem, we may show

Lemma 12. *If $\mathscr{T}$ contains $x = x^2$, then $\mathscr{T}$ satisfies exactly one of the following:*

(13.1) $\mathscr{T} \hat{\sigma} \{x = x^2, xy = yx\}$.

(13.2) $\mathscr{T} \hat{\sigma} \{x = y\}$.

(13.3) $\{xy = y\}\, \sigma \mathscr{T}$, or $\{xy = x\}\, \sigma \mathscr{T}$, *or both.*

To prove Lemma 12 we need to investigate the concept of single identities on idempotent semigroups. The following terminology is essentially due to Kimura [5].

Recall that $E(f)$ denotes the set of all letters contained in a word f. Let $H(f)$ be the head of f, that is, the first letter in f; $L(f)$ the tail of f, that is, the final letter of f. For example if $f = x_1^2x_2x_3^2$, $H(f) = x_1$, $L(f) = x_3$. Let f have the form: $f = x_{i_1}x_{i_2} \cdots x_{i_l}$, $E(f) = \{x_1, \cdots, x_k\}$. For f, another word f' is defined as follows:

$$f' = x'_{i_1}x'_{i_2} \cdots x'_{i_k},$$

$$x'_{i_1} = x_{i_1};$$

assuming that $x'_{i_1}, \cdots, x'_{i_j}$ are selected, $x'_{i_{j+1}}$ is determined in such a way that $x'_{i_{j+1}}$ is the letter in f which is distinct from any one of $x'_{i_1}, \cdots, x'_{i_j}$ and appears for the first time from the left. The word f' is called the initial part of f, denoted by $I(f)$. Dually we can define the final part $F(f)$ of f. If $I(f) = I(g)$, then $f = g$ is called coinitial; if $F(f) = f(g)$, then $f = g$ is called confinal.

Most parts of the following lemma, (14.1) through (14.7), are due to Kumura [3, 13, 14]. We shall state the sketch of the proof of (14.1) through (14.7) and will give only the proof of the remaining part.

Let $\mathscr{T}_0 = \{x = x^2, f = g\}$ where we may assume f and g are different words.

LEMMA 13.

(14.1) *$E(f) = E(g)$, $H(f) \neq H(g)$, and $L(f) \neq L(g)$ imply $\mathscr{F}_0\hat{\sigma}\{x = x^2, xy = yx\}$, semilattice.*

(14.2) *$E(f) \neq E(g)$, $H(f) \neq H(g)$, and $L(f) \neq L(g)$ imply $\mathscr{T}_0\hat{\sigma}\{x = y\}$ universality.*

(14.3) *$E(f) \neq E(g)$, $H(f) = H(g)$, and $L(f) \neq L(g)$ imply $\mathscr{T}_0\hat{\sigma}\{xy = x\}$, left zero semigroup.*

(14.4) *$E(f) \neq E(g)$, $H(f) \neq H(g)$, and $L(f) = L(g)$ imply $\mathscr{T}_0\hat{\sigma}\{xy = y\}$, right zero semigroup.*

(14.5) *$E(f) \neq E(g)$, $H(f) = H(g)$, and $L(f) = L(g)$ imply $\mathscr{T}_0\hat{\sigma}\{xyx = x\}$, rectangular band.*

(14.6) *$E(f) = E(g)$, $H(f) = H(g)$, $L(f) \neq L(g)$ and $I(f) = I(g)$ imply $\mathscr{T}_0\hat{\sigma}\{x = x^2, xyx = xy\}$, left regular band.*

(14.7) *$E(f) = E(g)$, $H(f) \neq H(g)$, $L(f) = L(g)$ and $F(f) = F(g)$ imply $\mathscr{T}_0\hat{\sigma}\{x = x^2, xyx = yx\}$, right regular band.*

(14.8) *$E(f) = E(g)$, $H(f) = H(g)$, $L(f) \neq L(g)$, and $I(f) \neq I(g)$ imply $\mathscr{F}_0\hat{\sigma}\{x = x^2, xyz = xzy\}$, left normal band.*

(14.9) *$E(f) = E(g)$, $H(f) \neq H(g)$, $L(f) = L(g)$ and $F(f) \neq F(g)$ imply $\mathscr{T}_0 \hat{\sigma} \{x = x^2, xyz = yxz\}$, right normal band.*

(14.10) *$E(f) = E(g)$, $H(f) = H(g)$, $L(f) = L(g)$ imply*

$$\{x = x^2, xyzx = xzyx\} \, \sigma \mathscr{T}_0 .$$

$\{x = x^2, xyzx = xzyx\}$ is called a normal band.

Proof. To change identities, in addition to the procedures defining the equality and the procedures (3) through (6), we shall use the following procedures:

(15.1) A word f^2 is replaced by f,

(15.2) A word f is replaced by f^2.

For (14.1) through (14.5), if $| E(f) \cup E(g) | \geqslant 2$, by means of suitable partitions (6′) of the set of the letters, $f = g$ is reduced to $f' = g'$ where $| E(f') \cup E(g') | = 2$, in which the conditions on E, H, and L are preserved. For (14.6) and (14.7), if $| E(f) \cup E(g) | \geqslant 3$, $f = g$ is reduced to $f' = g'$ where $| E(f') \cup E(g') | = 3$, in which the conditions on E, H, L, I, and F are preserved. This gives $\mathscr{T}_0 \sigma \{f = g\}$. The proof of $\{f = g\} \, \sigma \mathscr{T}_0$ is easy.

Below we use for simplicity the terminology "by $\{x, \cdots\} \cup \{y, \cdots\}$," which means "with the procedure (6′) by a partition of the set of all letters included in $f = g$ such that a class contains x, the complementary class contains y"; also "$\{x, y\} \cup \{z\} \cup \{u, \cdots\}$" is a partition of the set into the three classes: the first contains only x, y, the second z alone, the third all other elements containing u.

Proof of (14.8) through (14.10). The identity $f = g$ has the form $x \cdots y = x \cdots u$, $y \neq u$. By $\{x, u, \cdots\} \cup \{y, \cdots\}$ we have $xyx = xy$. Let $I(f) = x_1 \cdots x_i \cdots x_m$, $I(g) = y_1 \cdots y_i \cdots y_m$. By the assumption, $x_1 = y_1$, $\cdots$, $x_{i-1} = y_{i-1}$, $x_i \neq y_i$, $i > 1$. By $\{x_1, \cdots, x_{i-1}\} \cup \{x_i\} \cup \{y_i \cdots\}$, we get $xyz = xzy$. Conversely $xyz = xzy$ enables us derive $f = g$ by permuting letters other than heads of f and g. Thus we have (14.8).

(14.9) is obtained as the dual of (14.8).

(14.10) According to Kimura and Yamada [*13*, *14*], a normal band can be embedded into the direct product of a left normal band a right normal band. Immediately we see

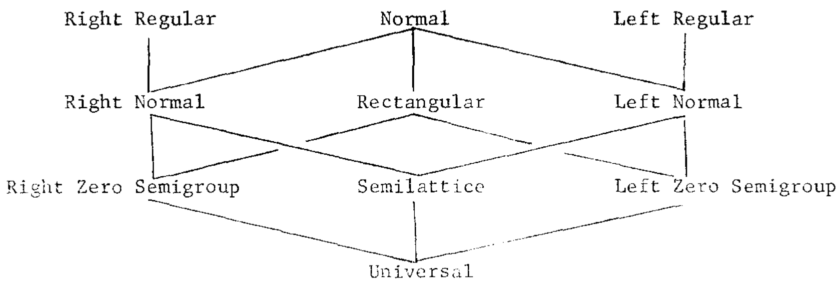
$$\{x = x^2, xyzx = xzyx\} \, \hat{\sigma} \{x = x^2, xyzu = xzyu\}.$$

It is easy to show that $x = x^2$ and $xyzu = xzyu$ imply $f = g$. Thus Lemma 13 has been proved.

Recall the ten cases of bands which appear in Lemma 13:

(15) semilattice, universality, left zero semigroup, right zero semigroup, rectangular band, left regular band, right regular band, left normal band, right normal band, normal band.

Using Kimura and Yamada's result [*3*, *14*], we can show that any two of (15) are not $\hat{\sigma}$-equivalent. Let $\mathfrak{U}^*$ be the set of the ten $\hat{\sigma}$-equivalence classes such that each class contains exactly one of (15). According to Yamada [*10*], $\mathfrak{U}^*$ is a subsemilattice of $\mathfrak{T}^*$ with respect to the greatest lower bound. The ordering σ^* of $\mathfrak{U}^*$ is shown in the following diagram. The readers can verify the relation σ immediately from the definition without any references.



To make the readers understand immediately, we give examples as follows:

Example 1. Normal band

	a	*b*	*c*	*d*
a	*a*	*b*	*a*	*a*
b	*a*	*b*	*a*	*a*
c	*a*	*b*	*c*	*c*
d	*a*	*b*	*d*	*d*

To show $xyzx = xzyx$, we can check only the cases $x = c$ or d; and $yz = a$, and $zy = b$; or $yz = c$ and $zy = d$. However, it is not left regular since $dbd \neq db$; not right regular since $cdc \neq dc$; not rectangular since $cac \neq c$; not left normal since $aab \neq aba$; not right normal since $cdc \neq dcc$.

EXAMPLE 2. Left regular band

	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

It is easy to show $xyx = xy$; but it is not right regular since $aba \neq ba$; not normal since $cabc \neq cbac$.

EXAMPLE 2′. Right regular band: the dual of Example 2. It is neither left regular nor normal.

EXAMPLE 3. Left normal band

	a	b	c
a	a	a	a
b	b	b	b
c	a	a	c

It is easy to show $xyz = xzy$; but not right regular since $bcb \neq cb$; not rectangular since $cbc \neq c$.

EXAMPLE 3′. Right normal band; the dual of Example 3.

EXAMPLE 4. Rectangular band

	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

This is not left regular since $bab \neq ba$; not right regular since $cac \neq ac$. It is obvious that there is an example of a semilattice which is not rectangular.

LEMMA 14. *The converse of Lemma 13, that is of each of (14.1) through (14.10), is true.*

Proof. The conditions in (14.1) through (14.10) with respect to f and g are all possible cases any two of which do not ceoxist. Let us start with the proof of the converse of (14.10). Suppose that $\{x = x^2, xyzx = xzyx\}\ \sigma\mathscr{T}_0$ implies one of the conditions of (14.1) through (14.9). Then normal band would imply one of (15) except normal band. This is impossible by Example 1. Therefore the converse of (14.10) has been proved. By examples 1 though 4, we see that any two of (15) are not $\hat{\sigma}$-equivalent. Immediately we can easily show that each converse of (14.1) through (14.9) is true. Q.E.D.

Let $\mathfrak{J}$ be the set of all systems $\mathscr{T}$ of identities such that each identity $f = g$ belonging to $\mathscr{T}$ satisfies $E(f) = E(g)$, $H(f) = H(g)$, $L(f) = L(g)$, and let $\mathfrak{J}^* = \mathfrak{J}/\hat{\sigma}$. Below the symbol $(x = x^2, f = g)$ denotes the $\hat{\sigma}$-class containing $\{x = x^2, f = g\}$. Clearly

$$\mathfrak{T}^* = \mathfrak{U}^* \cup \mathfrak{J}^*, \qquad \mathfrak{U}^* \cap \mathfrak{J}^* = (x = x^2, xyzx = xzyx),$$

and $\mathfrak{J}^*$ is a subsemilattice of $\mathfrak{T}^*$ with respect to greatest lower bound, and $(x = x^2, xyzx = xzyx)$ is the smallest element of $\mathfrak{J}^*$. For any $\mathscr{T}^* \in \mathfrak{J}^*$

$$\mathscr{T}^* \bigwedge (x = x^2, xyx = xy) = \text{either } (x = x^2, xyx = xy) \text{ or } (x = x^2, xyz = xzy),$$

$$\mathscr{T}^* \bigwedge (x = x^2, xyx = yx) = \text{either } (x = x^2, xyx = yx) \text{ or } (x = x^2, xyz = yxz).$$

If $\mathscr{S}^* \in \mathfrak{U}^*$ *and if* $\mathscr{S}^*$ is neither left regular nor right regular then $\mathscr{T}^* \bigwedge \mathscr{S}^* = \mathscr{S}^*$ for every $\mathscr{T}^* \in \mathfrak{J}^*$. Now we have arrived in the step of the proof of Lemma 12.

Proof of Lemma 12. Let $\mathscr{T} = \{x = x^2, f_\lambda = g_\lambda;\ \lambda \in \Lambda\}$. Then $\mathscr{T} = \bigcup_\lambda \mathscr{T}_\lambda$, where $\mathscr{T}_\lambda = \{x = x^2, f_\lambda = g_\lambda\}$. By Corollary 1, $\mathscr{T}^* = \bigwedge_\lambda \mathscr{T}_\lambda$. By Lemma 13, each $\mathscr{T}_\lambda^*$ belongs to either $\mathfrak{U}^*$ or $\mathfrak{J}^*$, and hence, by the above statement, either $\mathscr{T}^* \in \mathfrak{U}^*$ or $\mathscr{T}^* \in \mathfrak{J}^*$. This concludes that $\mathscr{T}$ satisfies one of (13.1), (13.2), and (13.3). Thus Lemma 12 has been proved.

After all, gathering Lemmas 12, 10, 10′, 11, we have completed the proof of Theorem 5.

6. CONCLUSION AND GENERALIZED PROBLEMS

In consequence of the discussions through this paper, we get not only the results concerning attainability on all semigroups but also those concerning attainability on commutative semigroups, idempotent semigroups and groups. Thus we summarize them as follows:

Let $\mathscr{T}$ be a nontrivial system of identities.

THEOREM. *$\mathscr{T}$ is attainable on all semigroups if and only if $\mathscr{T}$ is equivalent to $\{x = x^2, xy = yx\}$.*

$\mathscr{T}$ is attainable on all commutative semigroups if and only if $\mathscr{T}$ is equivalent to $\{x = x^2\}$.

$\mathscr{T}$ is attainable on all idempotent semigroups if and only if $\mathscr{T}$ is equivalent to $\{xy = yx\}$ on all idempotent semigroups.

No nontrivial system of identities is attainable on all groups, even on all abelian groups.

Finally we propose a few unsolved generalized problems. We can consider identities $f_\lambda = g_\lambda$ not only on semigroups but also on groupoids and algebraic systems with more than one binary operations. In such cases, the words f_λ, g_λ are regarded as the sequences of letters which are connected by parentheses and the binary operations. For a given system $\mathscr{T}$ of identities, there is the greatest $\mathscr{T}$-decomposition of an algebraic system of a fixed type [*10*].

PROBLEM 1. Determine all attainable systems of identities on all groupoids.

PROBLEM 2. Find all attainable systems of identities on all rings, lattices, or semirings.

PROBLEM 3. If identities admit constant elements, how can we study the problem of attainability of systems of identities on all semigroups? In this case the sense of attainability has to be modified, if necessary.

An implication has the form $f_\mu = g_\mu$, $\mu \in M$, $\Rightarrow h = k$ where f_μ, g_μ, h and k are words. We know that for a given system $\mathscr{S}$ of implications there is the greatest $\mathscr{S}$-decomposition of any semigroup or groupoid.

PROBLEM 4. Let $\mathscr{S}$ be a system of implications. Under what conditions on $\mathscr{S}$, is $\mathscr{S}$ attainable on all semigroups? What about the case where the words contain constant elements?

We note that the condition "weak reductivity" is an implication. Let S be a semigroup.

$$x_\xi y = x_\xi z, \qquad yx_\xi = zx_\xi, \qquad \text{for all} \qquad x_\xi \in S$$

imply

$$y = z,$$

where each x_ξ is a constant element, and the caridnal number of the set of

$x_\xi y = x_\xi z$ and $yx_\xi = zx_\xi$ is equal to $2 | S |$. Clearly weak reductivity is attainable on all semigroups. Is there any attainable implication besides it?

From another generalized view, we have the following problem:

Problem 5. Let $\mathscr{T}$ and a semigroup S be fixed. Under what conditions on $\mathscr{T}$ and S, is $\mathscr{T}$ attainable on the semigroup S?

Remark. The brief announcement of this paper was published in [*15*].

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