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ON THE DECOMPOSITION OF SETS OF REALS TO BOREL SETS

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1. Introduction and statement of the results

Our results will be about sets of real numbers and how they are related to Borel sets of real numbers, and to sets of real numbers which are in the classical projective hierarchy. In order to facilitate our treatment we shall deal with the space ${}^\omega\omega$ of all number theoretic functions rather than with the space R of all real numbers. All our results will be proved about the space ${}^\omega\omega$, but they will apply equally well to the space R since R is homeomorphic to the open unit interval $(0, 1)$, ${}^\omega\omega$ is homeomorphic to the set I of all irrational numbers in $(0, 1)$ and $(0, 1)$ and I differ only by the set of rationals in $(0, 1)$, which is a countable set. Bearing these facts in mind the reader will have no difficulty in transfering our results about ${}^\omega\omega$ to the corresponding results about R. The reader can look in [8, 3.5] for some of the details. In the light of what was said till now, we shall take the liberty of exclusively using the term *real numbers* for the members of ${}^\omega\omega$.

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α, β, γ will vary over real numbers. We shall use the standard notations $\mathbf{\Sigma}^1_k$ and $\mathbf{\Pi}^1_k$ for the classes of the classical projective hierarchy over ${}^\omega\omega$.

We shall say that a set A of reals has the *decomposition property* if it is the union of $\aleph_1$ Borel sets. If we assume the continuum hypothesis $2^{\aleph_0} = \aleph_1$ then every set of reals is of cardinality $\leq \aleph_1$, and hence has, trivially, the decomposition property. As a consequence we shall devote our attention to the decomposition property in the set theory ZF for those cases where $2^{\aleph_0} \neq \aleph_1$. These will be: the case where we have also the axiom of choice, and $2^{\aleph_0} = \aleph_\alpha$ for some $\alpha > 1$, and the case where $2^{\aleph_0}$ is not an aleph at all (in which case the axiom of choice does not hold).

It is by now a classical result in ZF that every $\mathbf{\Sigma}^1_2$ set is the union of $\aleph_1$ Borel sets. Since the authors know of no place in the literature (except [4, §3]) where this is explicitly proved, we shall outline the proof here. Given a $\mathbf{\Sigma}^1_2$-set A, it is a projection of a planar $\mathbf{\Pi}^1_1$-set B. We uniformize B by the Novikoff–Kondo–Addison uniformization theorem [7, p. 188] obtaining thereby a uniform planar $\mathbf{\Pi}^1_1$-set C whose projection is A (by C being uniform we mean that $\langle x, y\rangle \in C \wedge \langle x, z\rangle \in C \to y = z$). By the theorem that every $\mathbf{\Pi}^1_1$-set is the union of $\aleph_1$ Borel sets ([3, p. 52], [8, §1]) we can write $C = \bigcup_{\alpha<\omega_1} C_\alpha$, where the C_α's are Borel sets. Let A_α be the projection of C_α then, since A is the projection of C and $C = \bigcup_{\alpha<\omega_1} C_\alpha$, we get $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Since $C_\alpha \subseteq C$ for $\alpha < \omega_1$, also C_α is a uniform set. By a theorem of Lusin [3, p. 59] the projection of a uniform Borel set is a Borel set, hence A_α is a Borel set. Since $A = \bigcup_{\alpha<\omega_1} A_\alpha$, A has the decomposition property.

The result that every $\mathbf{\Sigma}^1_2$-set has the decomposition property cannot be improved to hold also for $\mathbf{\Pi}^1_2$-sets. It is shown in [5, §3] that it is consistent with ZFC to assume that the union of $\aleph_1$ Borel sets is always a $\mathbf{\Sigma}^1_2$-set (this follows also from the axiom of determinacy (see [6, (8.2.4)])), but this means that it is exactly the $\mathbf{\Sigma}^1_2$-sets that have the decomposition property, and those $\mathbf{\Pi}^1_2$-sets which are not $\mathbf{\Sigma}^1_2$-sets (such as any universal $\mathbf{\Pi}^1_2$-set) do not have the decomposition property. Decomposition results for sets higher up in the projective hierarchy than the $\mathbf{\Sigma}^1_2$-sets are proved in Martin [4]. One of his results is that if there is a measurable cardinal than every $\mathbf{\Sigma}^1_3$-set is the union of $\aleph_2$ Borel sets.

We shall now prove in ZFC that if $2^{\aleph_0} > \aleph_1$ then there are sets which do not have the decomposition property (where ZFC is ZF with the axiom of choice).

By a theorem of Alexandroff and Hausdorff every Borel set is countable (i.e., finite or denumerable) or of the cardinality $2^{\aleph_0}$ [3, p. 29]. Therefore every set of reals which has the decomposition property is of cardinality $\leq \aleph_1$ or $2^{\aleph_0}$. As a consequence, if $2^{\aleph_0} > \aleph_2$ then no set A of reals such that $\aleph_2 \leq |A| < 2^{\aleph_0}$ (where $|A|$ denotes the cardinality of A) has the decomposition property. To cover also the case where $2^{\aleph_0} = \aleph_2$, we prove the following theorem.

Theorem. *There is a set A of cardinality $2^{\aleph_0}$ such that every Borel set which is a subset of A is countable. Hence every decomposable subset of A is of cardinality $\leq \aleph_1$, and if $2^{\aleph_0} > \aleph_1$ then A itself, in particular, does not have the decomposition property.*

Proof. Let $2^{\aleph_0} = \aleph_\lambda$. Since there are $2^{\aleph_0}$ uncountable Borel sets let $\{C_\alpha \mid \alpha < \omega_\lambda\}$ be the set of all uncountable Borel sets. We define sequences $\langle a_\alpha \mid \alpha < \omega_\lambda\rangle$ and $\langle b_\alpha \mid \alpha < \omega_\lambda\rangle$ of real numbers by transfinite induction as follows:

$$a_\alpha, b_\alpha \in C_\alpha \sim \{a_\beta, b_\beta \mid \beta < \alpha\}, \qquad a_\alpha \neq b_\alpha .$$

Since C_α is an uncountable Borel set its cardinality is $2^{\aleph_0} = \aleph_\lambda$ and therefore there are a_α and b_α as required. Let $A = \{a_\alpha \mid \alpha < \omega_\lambda\}$. Let C be a Borel set which is a subset of A. If C were uncountable then $C = C_\alpha$ for some $\alpha < \omega_\lambda$, but this is a contradiction since $b_\alpha \in C_\alpha$ but $b_\alpha \notin A \supseteq C$. □

The sets A which were proved above not to have the decomposition property were obtained in a very non-constructive way, since their construction essentially involved a well ordering of the set of all real numbers. The following Theorem 1 will assert that one cannot prove in ZFC the existence of sets which do not have the decomposition property and which are constructive in any meaningful sense.

Theorem 1. *If* ZF *is consistent with the existence of an inaccessible cardinal then* ZFC *is consistent with* $2^{\aleph_0} > \aleph_1$ *together with the statement "every set of reals definable from a countable sequence of ordinals has the decomposition property".*

In fact, we can replace here $2^{\aleph_0} > \aleph_1$ by $2^{\aleph_0} = \aleph_\Lambda$, where Λ is any "reasonably" defined ordinal (see [9, Th. 3, Remark 1]).

Abandoning the axiom of choice we get the following theorem.

Theorem 2. *If* ZF *is consistent with the existence of an inaccessible cardinal then* ZF *is consistent with* DC (*the axiom which admits countably many dependent choices*) *together with "every well-ordered set of reals is countable" and "every set of reals has the decomposition property."*

2. Proof of Theorem 1

We shall use here the model and methods used by Solovay in the proof of [9, Th. 3]. There a model is constructed in which $2^{\aleph_0}$ has a prescribed value and in which every set of real numbers which is definable from a countable sequence of ordinals is Lebesgue measurable, has the Baire property, and is either countable or includes a perfect set. We shall show here that in the same model also every set of reals which is definable from a countable sequence of ordinals has the decomposition property.

From now on we shall use, without further mention, the terminology and notation of [9].

Let $\mathcal{M}$ be a countable transitive model of ZFC which contains an ordinal Ω which is inaccessible in $\mathcal{M}$, and such that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds in $\mathcal{M}$ for $\alpha \geq \Omega$. Such a model is usually obtained by taking a countable transitive model of ZFC + GCH with Ω an inaccessible cardinal of the model. Let Θ be a cardinal of $\mathcal{M}$ with cofinality $\geq \Omega$ in $\mathcal{M}$. Let $\mathcal{P}^\Omega$ be the partially ordered set appropriate to collapsing all cardinals of $\mathcal{M}$ below Ω, i.e., $f \in \mathcal{P}^\Omega$ iff f is a function whose domain is a finite subset of $\Omega \times \omega$ and such that $f(\alpha, n)$ is an ordinal $< \alpha$ whenever defined. Let $\mathcal{P}'_\Theta$ be the partially ordered set appropriate to adding Θ generic subsets of ω, i.e., $\mathcal{P}'_\Theta$ is the set of all functions from finite subsets of $\Theta \times \omega$ into $\{0, 1\}$. Let $\mathcal{P}$ be $\mathcal{P}^\Omega \times \mathcal{P}'_\Theta$, and let G be an $\mathcal{M}$-generic filter on $\mathcal{P}$.

By [9, I, Lemma 2.3], $G = G' \times G''$, where G' is a generic filter on $\mathcal{P}^{\Omega}$ and G'' is a generic filter on $\mathcal{P}'_{\Theta}$. $F' = \bigcup G'$ is a function on $\Omega \times \omega$ such that for $\alpha < \Omega$, $\{\langle n, F(\alpha, n)\rangle \mid n < \omega\}$ is a (generic) map of ω on α. $F'' = \bigcup G''$ is a function on $\Theta \times \omega$ into $\{0, 1\}$ such that $\{\{n \in \omega \mid F''(\alpha, n) = 1\} \mid \alpha < \Theta\}$ is a set of Θ (generic) subsets of ω. Let $\mathfrak{N}_2$ be $\mathfrak{M}[G]$. In the model $\mathfrak{N}_2$ we have $\aleph_1 = \Omega$, $2^{\aleph_0} = \Theta$ [9, III, 3.3]. We quote now two lemmas from [9, III, 3.4 and 3.6].

Lemma 3. *Let* $f: \omega \to \mathrm{OR}$, $f \in \mathfrak{N}_2$. *Then there is a* $\xi < \Omega$ *and a subset* A *of* Θ *such that:*

(a) $A \in \mathfrak{M}$, *and* $\mathrm{card}^{\mathfrak{M}}(A) < \Omega$ (*where* $\mathrm{card}^{\mathfrak{M}}(A)$ *denotes the cardinality of* A *in* $\mathfrak{M}$).

(b) $f \in \mathfrak{M}[G \cap (\mathcal{P}^{\xi} \times \mathcal{P}'_A)]$ (*where* $\mathcal{P}'_A$ *is the partially ordered set appropriate for adding generic subsets of* ω *corresponding to the ordinals in* A, *i.e.*, $\mathcal{P}'_A = \{f \in \mathcal{P}'_{\Theta} \mid \mathrm{domain}(f) \subseteq A \times \omega\}$).

Lemma 4. *Let* $f: \omega \to \mathrm{OR}$, $f \in \mathfrak{N}_2$. *Then there is an* $\mathfrak{M}[f]$*-generic filter* G_1 *on* $\mathcal{P}$ *such that* $\mathfrak{M}[f][G_1] = \mathfrak{N}_2$.

We shall prove first that the following Lemma 5 implies Theorem 1, and then we shall turn to the proof of this lemma.

Lemma 5. *In* $\mathfrak{N}_2$ *every* $\mathfrak{M}$*-definable set of reals* (*i.e., every set of reals of* $\mathfrak{N}_2$ *which is definable in* $\mathfrak{N}_2$ *by means of constants taken from* $\mathfrak{M}$) *is the union of* $\aleph_1$ *Borel sets.*

Proof of Theorem 1 from Lemma 5. In order to get Theorem 1 we need the result of the lemma to hold not only for every $\mathfrak{M}$-definable set of reals but for every set of reals $\mathfrak{M}$-definable from a countable sequence of ordinals in $\mathfrak{N}_2$. Suppose $E \in \mathfrak{N}_2$ is a set of reals definable from members of $\mathfrak{M}$ and from $f: \omega \to \mathrm{OR}$ which is in $\mathfrak{N}_2$. By Lemma 4 $\mathfrak{N}_2 = \mathfrak{M}[f][G_1]$, where G_1 is an $\mathfrak{M}[f]$-generic filter on $\mathcal{P}$. E is obviously an $\mathfrak{M}[f]$-definable set in $\mathfrak{N}_2$. Since $\mathfrak{N}_2 = \mathfrak{M}[f][G_1]$ we can substitute $\mathfrak{M}[f]$ for $\mathfrak{M}$ and G_1 for G in Lemma 5 without replacing $\mathfrak{N}_2$, and the lemma will assert that E is the union of $\aleph_1$ Borel sets. In order to justify this substitution of $\mathfrak{M}[f]$ for $\mathfrak{M}$ in the lemma we still have to show that all the assumptions we made concerning $\mathfrak{M}$, Ω and Θ

hold also for $\mathfrak{M}[f]$; i.e., we have to show that Ω is inaccessible in $\mathfrak{M}[f]$, that $\mathfrak{M}[f]$ satisfies the generalized continuum hypothesis from Ω upwards and that $\mathrm{cf}(\Theta) \geq \Omega$ in $\mathfrak{M}[f]$.

By Lemma 3, $f \in \mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$ for some ξ and A as required there. By [9, I, Lemma 2.3], $G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)$ is an $\mathfrak{M}$-generic filter over $\mathcal{P}^\xi \times \mathcal{P}'_A$ (since $\mathcal{P}$ is isomorphic to $(\mathcal{P}^\xi \times \mathcal{P}'_A) \times (\mathcal{R}^\xi \times \mathcal{P}'_{\Theta\sim A})$; where $\mathcal{R}^\xi = \{p \in \mathcal{P}^\Omega \mid \mathrm{domain}(p) \cap (\xi \times \omega) = \emptyset\}$). Since the set $\mathcal{P}^\xi \times \mathcal{P}'_A$ is of cardinality $< \Omega$ in $\mathfrak{M}$ we get, by [9, I, Lemma 1.11], that Ω is inaccessible in $\mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$ and therefore Ω is also inaccessible in $\mathfrak{M}[f]$ which is a submodel of $\mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$. The fact that $\mathrm{card}^{\mathfrak{M}}(\mathcal{P}^\xi \times \mathcal{P}'_A) < \Omega$ implies also, by the standard arguments, that the cardinals of $\mathfrak{M}$ above Ω are also cardinals of $\mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$, that for every $\lambda \in \mathfrak{M}$ if $\mathrm{cf}^{\mathfrak{M}}(\lambda) \geq \Omega$ then $\mathrm{cf}^{\mathfrak{M}}(\lambda)$ is the cofinality of λ also in $\mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$, and that also in $\mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$ we have $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for $\alpha \geq \Omega$. Since that $\mathfrak{M} \subseteq \mathfrak{M}[f] \subseteq \mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$ this proves that the cardinals of $\mathfrak{M}$ above Ω are also cardinals of $\mathfrak{M}[f]$, that if $\mathrm{cf}^{\mathfrak{M}}(\lambda) \geq \Omega$ then $\mathrm{cf}^{\mathfrak{M}}(\lambda)$ is the cofinality of λ also in $\mathfrak{M}[f]$ and that also in $\mathfrak{M}[f]$, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for $\alpha \geq \Omega$. This completes the proof of Theorem 1 from Lemma 5.

We shall now show that Lemma 5 follows rather easily from the following Lemma 6, with the proof of which we shall be concerned for the rest of the present section.

Lemma 6. *Let U be an $\mathfrak{M}$-definable set of reals in $\mathfrak{N}_2$ and let $s \in U$. Then there is a Borel set W with a code in $\mathfrak{M}[G']$ such that $s \in W \subseteq U$. (G' is the $\mathfrak{M}$-generic filter on $\mathcal{P}^\Omega$ obtained from G).*

Proof of Lemma 5 from Lemma 6. It follows directly from Lemma 6 that an $\mathfrak{M}$-definable set U of reals in $\mathfrak{N}_2$ is the union of Borel sets of reals with codes in $\mathfrak{M}[G']$. Since G' is an $\mathfrak{M}$-generic filter on $\mathcal{P}^\Omega$ we have, by [9, I, Corollaries 3.3 and 3.4], that the set of all reals of $\mathfrak{M}[G']$ is of cardinality $\aleph_1 (= \Omega)$ in $\mathfrak{M}[G']$, and hence is of cardinality $\leq \aleph_1$ in $\mathfrak{M}[G]$. Thus U is the union of (at most) $\aleph_1$ Borel sets of reals in $\mathfrak{N}_2$.

Proof of Lemma 6. Let $s \in U$. By Lemma 3 there is a $\xi < \Omega$ and a subset $A \in \mathfrak{M}$ of Θ such that $\mathrm{card}^{\mathfrak{M}}(A) < \Omega$ and $s \in \mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$.

Without loss of generality we can assume that ξ is a successor ordinal and that ξ is greater than the order type of A (in the natural order of the ordinals), since otherwise ξ can be replaced by an ordinal $\mu > \xi$ which satisfies these requirements and which is still $< \Omega$ (since $\operatorname{card}^{\mathcal{M}}(A) < \Omega$); we have $s \in \mathcal{M}[G \cap (\mathcal{P}^{\xi} \times \mathcal{P}'_A)] \subseteq \mathcal{M}[G \cap (\mathcal{P}^{\mu} \times \mathcal{P}'_A)]$ since $G \cap (\mathcal{P}^{\xi} \times \mathcal{P}'_A)$ is obviously a member of $\mathcal{M}[G \cap (\mathcal{P}^{\mu} \times \mathcal{P}'_A)]$. Since ξ is now a successor ordinal let $\xi = \eta + 1$. Throughout the rest of this section we shall write Q for $\mathcal{P}^{\xi} \times \mathcal{P}'_A$. By [9, I, Lemma 2.3] (and our remarks above in the proof of Theorem 1 from Lemma 5), $G \cap Q$ is an $\mathcal{M}$-generic filter on Q. Let g be any $\mathcal{M}$-generic filter on Q which belongs to $\mathcal{N}_2$. We shall first show that there is a real $r : \omega \to 2$ such that $\mathcal{M}[g] = \mathcal{M}[r]$. For this purpose we code g by r in such a way that each one of g and r is obtainable from the other by means of absolute operations. Let F' be the union of the first components of the members of g ($\subseteq \mathcal{P}^{\xi} \times \mathcal{P}'_A$) and let $f = \{\langle n, \beta\rangle \mid F'(\eta, n) = \beta\}$, where $\eta + 1 = \xi$. Since g is $\mathcal{M}$-generic on Q, f is a mapping of ω on η ([9, I, Lemma 3.2]). Let h be the function on η defined by $h(\lambda) =$ *the least k such that $f(k) = \lambda$*. Let j be the function on A given as follows: If α is the λ^{th} member of A then $j(\alpha) = h(\lambda)$. Take now r to be the characteristic function of the set

$$\begin{aligned}
&\{3^k \cdot 5^l \mid k, l \in \omega, f(k) \leq f(l)\} \cup \\
&\{2 \cdot 3^{h(\lambda)} \cdot 5^n \cdot 7^{h(F'(\lambda, n))} \mid \lambda < \xi, n < \omega\} \cup \\
&\{2^2 \cdot 3^{j(\alpha)} \cdot 5^n \mid \alpha \in A, F''(\alpha, n) = 1\}\,,
\end{aligned}$$

where F'' is the union of the second components of the members of g ($\subseteq \mathcal{P}^{\xi} \times \mathcal{P}'_A$). It is clear that r is defined by absolute operations from ξ and A, which are in $\mathcal{M}$, and from g; therefore $r \in \mathcal{M}[g]$. An inspection of our definition of r will immediately show that from r and A we can reconstruct, by absolute operations, the generic filter g, and hence $g \in \mathcal{M}[r]$; therefore $\mathcal{M}[g] = \mathcal{M}[r]$, which is what we set out to prove at this point. By Lemma 4 there is an $\mathcal{M}[r]$-generic filter H on $\mathcal{P}$ such that $\mathcal{M}[r][H] = \mathcal{N}_2$. Since $\mathcal{M}[g] = \mathcal{M}[r]$ we have that H is also $\mathcal{M}[g]$-generic and that $\mathcal{M}[g][H] = \mathcal{N}_2$. Since, as we mentioned above, $G \cap Q$ is $\mathcal{M}$-generic over Q, we have an $\mathcal{M}[G \cap Q]$-generic filter H^* on $\mathcal{P}$ such that $\mathcal{N}_2 = \mathcal{M}[G \cap Q][H^*]$.

Since U is an $\mathcal{M}$-definable set of reals there is a formula $\Gamma(x)$ of $\mathcal{L}$ which mentions no constants other than ones denoting members of $\mathcal{M}$ such that in $\mathcal{N}_2$

(1) for every t, $\Gamma(t) \leftrightarrow t \in U$,

The forcing language $\mathcal{L}'$ ([9, I, 1.9]) over any countable transitive model $\mathcal{M}^*$ of ZF contains an individual constant $\underline{t}$ for every member t of $\mathcal{M}^*$; let us assume that $\underline{t}$ is formally defined as $\langle 7, t\rangle$. Thus the constant for s in the forcing language $\mathcal{L}'$ over $\mathcal{M}[G \cap Q]$ is $\langle 7, s\rangle$, and we have therefore $\mathcal{M}[G \cap Q][H] = \mathcal{N}_2 \models \Gamma(\langle 7, s\rangle)$. $\Gamma(\langle 7, s\rangle)$ does not contain a symbol which stands for H^*, therefore, by the usual symmetry arguments (as in [9, I, 3.5]) we get that

(2) $0 \Vdash \Gamma(\langle 7, s\rangle)$.

Since forcing is absolute, (2) is true also in $\mathcal{M}[G \cap Q]$. By [9, I, 1.7] there is a formula $\Phi(x, y, z)$ of $\mathcal{L}'$ such that for every member t of $\mathcal{M}[G \cap Q]$ there is a $u \in \mathcal{M}$ such that t is the unique member of $\mathcal{M}[G \cap Q]$ such that u, $G \cap Q$, t satisfy $\Phi(x, y, z)$ in $\mathcal{N}_2$. u should be thought of as a name, in $\mathcal{M}$, for t which is in $\mathcal{M}[G \cap Q]$. Since $s \in \mathcal{M}[G \cap Q]$ let $u \in \mathcal{M}$ be a name for s and thus, since (2) is true in $\mathcal{M}[G \cap Q]$ we get that

$$\exists s\, \forall t[(\Phi(u, G \cap Q, t) \leftrightarrow t = s) \wedge 0 \Vdash \Gamma(\langle 7, s\rangle)] \ .$$

Since "whatever holds is forced", we use $\langle 7, u\rangle$ as a name for u and $\underline{g}$ as a name for the $\mathcal{M}$-generic filter on Q and we get that there is a $p_0 \in G \cap Q$ such that

(3) $p_0 \Vdash \exists s\, \forall t[(\Phi(\langle 7, u\rangle, \underline{g}, t) \leftrightarrow t = s) \wedge 0 \Vdash \Gamma(\langle 7, s\rangle)]$,

where the two $\Vdash$ symbols refer to forcing over Q and $\mathcal{P}$, respectively.

Let g be an $\mathcal{M}$-generic filter on Q which contains p_0. By (3) we have

(4) $\mathcal{M}[g] \models \exists s\, \forall t[(\Phi(u, g, t) \leftrightarrow t = s) \wedge 0 \Vdash \Gamma(\langle 7, s\rangle)]$.

Let $s(g)$ be the unique s which satisfies $\Phi(u, g, s)$ in $\mathcal{M}[g]$, then by (4) we have in $\mathcal{M}[g]$

(5) $0 \Vdash \Gamma(\langle 7, s(g)\rangle)$.

We have shown above that there is an $\mathfrak{M}[g]$-generic filter H such that $\mathfrak{N}_2 = \mathfrak{M}[g][H]$, therefore, by (5), we have $\mathfrak{N}_2 \models \Gamma(s(g))$. By (1) every such $s(g)$ is in U, i.e.,

$$s \in \{s(g) \mid g \in \mathfrak{N}_2, g \text{ is } \mathfrak{M}\text{-generic over } Q, p_0 \in g\} \subseteq U .$$

We denote the set

$$\{s(g) \mid g \in \mathfrak{N}_2, g \text{ is } \mathfrak{M}\text{-generic over } Q, p_0 \in g\}$$

by T; we have $s \in T \subseteq U$. It can be shown that T is a $\mathbf{\Sigma}_1$-set with a code in $\mathfrak{M}[G']$, which establishes Lemma 6 with W a $\mathbf{\Sigma}_1$-set instead of a Borel set. From this version of Lemma 6, Lemma 5 follows as above, except that now we have to use also the classical theorem that every $\mathbf{\Sigma}_1$-set is the union of $\aleph_1$ Borel sets. We choose however to prove directly, by a somewhat longer proof, that T is the union of $\aleph_1$ Borel sets, using the method of [9, I, 4.4], since we shall need this proof anyway for the proof of Theorem 2.

Let Q^* be the set of all members $\langle y, z\rangle$ of Q such that for some $n < \omega$ we have

$$\operatorname{dom}(y) = (\{\eta\} \cup \{y(\eta, k) \mid y(\eta, k) > 0 \wedge k < n\}) \times n ,$$

where $\xi = \eta + 1$, and

$$\operatorname{dom}(z) = \{\text{the } y(\eta, k)^{\text{th}} \text{ member of } A \mid k < n, y(\eta, k) < \rho\} \times n ,$$

where ρ is the order type of A. n is, obviously, uniquely determined by $\langle y, z\rangle$ and we call it the length, $\operatorname{lh}(\langle y, z\rangle)$, of $\langle y, z\rangle$. As easily seen, every member of Q can be extended to a member of Q^* of arbitrarily large length and of any two compatible members of Q^* the one with the greater length, if any, is $>$ the other, and if they are of equal length they are equal.

Let S be the set of all pairs $\langle p, w\rangle$ such that

(6) $\qquad p \in Q^*$ and $w : \operatorname{lh}(p) \to \omega$,

(7) p is compatible with p_0,

(8) if $k < \mathrm{lh}(p)$, $l < \omega$ and $p \Vdash \forall x(\Phi(\underline{u}, \underline{g}, x) \to x(\underline{k}) = \underline{l})$ then $w(k) = l$ (i.e., the information concerning $s(g)$ forced by p is compatible with w).

It is clear that $S \in \mathfrak{M}$ since it is defined from Q^* which is in $\mathfrak{M}$ (since $A \in \mathfrak{M}$) in terms of forcing which is defined in $\mathfrak{M}$. It is easily seen that

(9) if $\langle p, w\rangle$, $\langle p', w'\rangle$ are two pairs as in (6) and $\langle p', w'\rangle$ is an extension of $\langle p, w\rangle$ in the sense that $p' \geq p$ and $w' \supseteq w$ and if $\langle p', w'\rangle \in S$ then also $\langle p, w\rangle \in S$.

Let g be an $\mathfrak{M}$-generic filter on Q. Since every member of Q can be extended to a member of Q^* of length $\geq n$, and every member of Q^* of length $\geq n$ is an extension of a member of Q^* of length n, the set of members of Q which are $\geq$ than a member of Q^* of length n is a set of $\mathfrak{M}$ dense in Q. Therefore g, being $\mathfrak{M}$-generic over Q, contains a member q' such that for some $q \in Q^*$ of length n, $q' \geq q$. Since $q' \geq q$ and $q' \in g$ also $q \in g$. q is uniquely determined by g and n as the member of $g \cap Q^*$ of length n, since any other member q'' of g must be compatible with q and if q'' is also a member of Q^* of length n it is equal to q, as we saw above. Therefore we can denote this q by g^*n. We have, by the definition of g^*n, $g \cap Q^* = \{g^*n \mid n \in \omega\}$. If D is a dense subset of Q in $\mathfrak{M}$ then we shall see that for some n, $g^*n \in D$. Since $\{p \in D \mid (\exists q \in D \cap Q^*)(p \geq q)\} \in \mathfrak{M}$ is a dense set, and as g is generic there are $p \in D$ and $q \in D \cap Q^*$ such that $p \geq q$ and $p \in g$. Since $p \geq q$ also $q \in g$, and q, being in Q^*, is of the form g^*n.
$g^*0 = \langle 0, 0\rangle$; for the sake of convenience we denote the condition $\langle 0, 0\rangle$ of Q with 0.

Lemma 7. *In $\mathfrak{N}_2$ we have*

$$T = \{t \in {}^\omega\omega \mid (\exists g \in \mathfrak{N}_2)(g \text{ is an } \mathfrak{M}\text{-generic filter on } Q \wedge (\forall n \in \omega)(\langle g^*n, t \upharpoonright n\rangle \in S))\}\ .$$

Proof. Let T' be the right-hand side of the equality in Lemma 7, and let us start by proving $T \subseteq T'$. Let $t \in T$, then $t = s(g)$ for some $\mathfrak{M}$-generic filter $g \in \mathcal{N}_2$ on Q, with $p_0 \in g$. In order to prove $t \in T'$ it will suffice to show that for every $n \in \omega$, $\langle g^*n, t \upharpoonright n\rangle \in S$. $\langle g^*n, t \upharpoonright n\rangle$ obviously satisfies (6). (7) holds since both p_0 and g^*n are members of g. To see that (8) holds notice that if $g^*n \Vdash \forall x(\Phi(\underline{u}, \underline{g}, x) \to x(\underline{k}) = \underline{l})$ then, since $g^*n \in g$, we have $\mathcal{M}[g] \models \forall x(\Phi(u, g, x) \to x(k) = l)$ hence $t(k) = s(g)(k) = l$ ($s(g)$ being the unique x such that $\Phi(u, g, x)$ holds in $\mathcal{M}[g]$). Therefore $(t \upharpoonright n)(k) = l$ as required in (8).

To prove that $T' \subseteq T$, let $t \in T'$ and let $g \in \mathcal{N}_2$ be an $\mathcal{M}$-generic filter on Q such that $(\forall n \in \omega)(\langle g^*n, t \upharpoonright n\rangle \in S)$. In order to prove $t \in T$ it suffices to show that $p_0 \in g$ and that $s(g) = t$. The set of all $p \in Q$ such that the domains of the two components of p include the respective domains of the components of p_0 is obviously a dense subset of Q in $\mathcal{M}$ and hence it contains some g^*n. Since by our assumptions about t and g and by (7), g^*n is compatible with p_0 and since the domains of the components of g^*n include those of p_0 we have $g^*n \geq p_0$, and hence also $p_0 \in g$. To prove $s(g) = t$ we shall show that for every $k \in \omega$, $s(g)(k) = t(k)$. Suppose that for some k, $s(g)(k) = l$, while $t(k) \neq l$. Since the set of conditions $p \in Q$ which decide the statement $\forall x(\Phi(\underline{u}, \underline{g}, x) \to x(\underline{k}) = \underline{l})$ is dense, and since the statement is true in $\mathcal{M}[g]$, there is an $n > k$ such that $g^*n \Vdash \forall x(\Phi(\underline{u}, \underline{g}, x) \to x(\underline{k}) = \underline{l})$. Since we assumed $t(k) \neq l$ we have $\langle g^*n, t \upharpoonright n\rangle \notin S$, since requirement (8) fails for this pair, but this contradicts our assumptions about t and g. □

For $p \in Q^*$ and $t \in {}^\omega\omega$ we define $\Psi_\gamma(p, t)$ by induction on γ as follows:

(10) $\Psi_0(p, t)$ iff $\langle p, t \upharpoonright \mathrm{lh}(p)\rangle \notin S$.

(11) For $\gamma > 0$, $\Psi_\gamma(p, t)$ iff there is a dense subset D of Q in $\mathcal{M}$ such that whenever $p' \in Q^* \cap D$ and $p' \geq p$ then $\Psi_\lambda(p', t)$ holds for some $\lambda < \gamma$.

It is very easy to see that if $\delta < \gamma$ then $\Psi_\delta(p, t) \to \Psi_\gamma(p, t)$ (for $\delta = 0$ take D in (11) to be Q^*).

Lemma 8. *For every $t \in {}^\omega\omega \cap \mathcal{N}_2$, $t \in T$ iff for no γ does $\Psi_\gamma(0, t)$ hold.*

Proof. Assume $t \in T$, and let $g \in \mathcal{N}_2$ be an $\mathcal{M}$-generic filter on Q as in Lemma 7. We have to prove that $\Psi_\gamma(0, t)$ holds for no γ, so let us assume, in order to get a contradiction, that $\Psi_\gamma(0, t)$ holds for some γ. Let γ

be the least ordinal such that $\Psi_\gamma(p, t)$ holds for some $p \in g \cap Q^*$ (there is such a γ since $0 \in g \cap Q^*$). γ cannot be 0 since by Lemma 7, $\langle p, t \restriction \mathrm{lh}(p)\rangle \in S$, $p \in Q^*$ being one of the g^*n's, and thus (10) fails. Since now $\gamma > 0$ and $\Psi_\gamma(p, t)$ holds, there is, by (11) a dense subset D of Q in $\mathcal{M}$ such that if $p' \in Q^* \cap D$ and $p' \geq p$ then $\Psi_\delta(p', t)$ for some $\delta < \gamma$. By what we said above about Q^*, right before Lemma 7, there is an n such that $g^*n \in D$. Let $p' = g^*\max(n, \mathrm{lh}(p))$. p' is obviously in $Q^* \cap g$, and since p' extends g^*n we have $p' \in D$. Therefore we have $\Psi_\delta(p', t)$ for some $p' \in g \cap Q^*$ and some $\delta < \gamma$, contradicting the minimality of γ.

To prove the other direction of Lemma 8 let us first prove:

(12) For every $p \in Q^*$, if for every γ we have $\neg\, \Psi_\gamma(p, t)$, then for every dense subset D of Q^* in $\mathcal{M}$ there is a $p' \in D \cap Q^*$ such that $p' \geq p$ and for every γ we have $\neg\, \Psi_\gamma(p', t)$.

Let D be a dense subset of Q^* in $\mathcal{M}$. By definition of $\Psi_\gamma(p, t)$, since $\neg\, \Psi_\gamma(p, t)$ there is a $p' \in Q^* \cap D$, $p' \geq p$, such that $\Phi_\lambda(p', t)$ holds for no $\lambda < \gamma$. Let us denote this p' with $p'(\gamma)$, since it depends on γ. Since $Q^* \cap D$ is a set there is a $p' \in Q^* \cap D$ such that $p' = p'(\gamma)$ for an unbounded class of γ's. For this p' we have for every λ, $\neg\, \Psi_\lambda(p', t)$, which establishes (12).

Let us now prove the other direction of Lemma 8. Let $t \in {}^\omega\omega$ be such that for no γ, $\Psi_\gamma(0, t)$. Using (12) we shall now construct an $\mathcal{M}$-generic filter g on Q^* as in Lemma 7. Since Ω is inaccessible in $\mathcal{M}$, the cardinality in $\mathcal{M}$ of $Q^* \subseteq \mathcal{P}^\xi \times \mathcal{P}'_A$ is $< \Omega$ and also the cardinality of its power set in $\mathcal{M}$ is $< \Omega$. Since ordinals $< \Omega$ are countable in $\mathcal{N}_2$ there are in $\mathcal{N}_2$ only $\aleph_0$ dense subsets of Q^* which are in $\mathcal{M}$; let $D_1, D_2, \ldots$ be a sequence in $\mathcal{N}_2$ of all these subsets. We shall now define in $\mathcal{N}_2$ a sequence $\{q_n \mid n < \omega\}$ of members of Q^* as follows. $q_0 = 0$; by our assumption we have for every γ, $\neg\, \Psi_\gamma(q_0, t)$. We assume, as an induction hypothesis, that for every γ, $\neg\, \Psi_\gamma(q_n, t)$. By (12) there is a $q_{n+1} \geq q_n$ such that $q_{n+1} \in Q^* \cap D_{n+1}$ and every γ, $\neg\, \Psi_\gamma(q_{n+1}, t)$. Thus $\{q_n \mid n < \omega\}$ is an ascending sequence of members of Q^* such that for every n, $n > 0 \to q_n \in D_n$, and for all γ, $\neg\, \Psi_\gamma(q_n, t)$. Let

$$(13) \qquad g = \{q \in Q \mid q \text{ is compatible with every } q_n\} \ .$$

We shall now prove that g is as required in Lemma 7, and first we shall prove that g is an $\mathfrak{M}$-generic filter over Q. By (13) if $q \in g$ and $q' \leq q$ then also $q' \in g$. If $q', q'' \in g$ then the set of all members p of Q such that the domains of two components of p include the respective domains of the components of q' and q'' is a dense subset of Q in $\mathfrak{M}$, and is therefore D_m for some $0 < m < \omega$. Since $q_m \in D_m$, the domains of the components of q_m include the respective domains of the components of q' and q'' and since q', q'' are, by (13), compatible with q_m we have $q', q'' \leq q_m \in g$. Since $g \supseteq \{q_n \mid n < \omega\}$, g has a common member with each D_n, and thus g is $\mathfrak{M}$-generic. We still have to prove that for every $n < \omega$, $\langle g^*n, t \restriction n \rangle \in S$. As we saw above with respect to q' and q'', since $g^*n \in Q^*$ there is a q_m such that $q_m \geq g^*n$, and then $\mathrm{lh}(q_m) \geq n$. If $\langle g^*n, t \restriction n \rangle \notin S$ then by (9) also $\langle q_m, t \restriction \mathrm{lh}(q_n) \rangle \notin S$. By (10) we have $\Psi_0(q_m, t)$, contradicting our proof above that for all n and γ, $\neg \Psi_\gamma(q_n, t)$.

Lemma 9. *If γ is such that* $(\forall p \in Q^*)[\Psi_\gamma(p, t) \rightarrow (\exists \lambda < \gamma)\Psi_\lambda(p, t)]$ *then* $\forall \delta (\forall p \in Q^*)[\Psi_\delta(p, t) \rightarrow (\exists \lambda < \gamma)\Psi_\lambda(p, t)]$.

Proof. The conclusion of the lemma has to be proved for $\delta > \gamma$ since it is obvious for $\delta < \gamma$, and for $\delta = \gamma$ it is our assumption. We shall prove it by induction on δ, thus our induction hypothesis is

(14) $\qquad (\forall p \in Q^*)[\Psi_\beta(p, t) \rightarrow (\exists \lambda < \gamma)\Psi_\lambda(p, t)] \quad \text{for} \quad \beta < \delta.$

Assume $\Psi_\delta(p, t)$. Since $\delta > \gamma \geq 0$ we have, by (11) that there is a dense subset D of Q in $\mathfrak{M}$ such that for every $p' \in Q^* \cap D$ if $p' \geq p$ then $\Psi_\lambda(p', t)$ for some $\lambda < \delta$. By (14), if $\Psi_\lambda(p', t)$ for $\lambda < \delta$ then there is a $\lambda < \gamma$ such that $\Psi_\lambda(p', t)$. Thus we have now that for every $p' \in Q^* \cap D$ if $p' \geq p$ then $\Psi_\lambda(p', t)$ for some $\lambda < \gamma$. By (11) we have $\Psi_\gamma(p, t)$, and hence by the hypothesis of the lemma $(\exists \lambda < \gamma)\Psi_\lambda(p, t)$.

Lemma 10. *For every $t \in {}^\omega\omega \cap \mathfrak{N}_2$ there is a $\gamma < \Omega$ such that* $(\forall p \in Q^*)[\exists \delta \Psi_\delta(p, t) \rightarrow (\exists \delta < \gamma)\Psi_\delta(p, t)]$.

Proof. Let t be given. By Lemma 3 there is an ordinal $\zeta > \xi$ and a subset B of Θ, $B \in \mathfrak{M}$ such that $t \in \mathfrak{M}[G \cap (\mathcal{P}^\zeta \times \mathcal{P}'_B)]$. As we saw in the proof of Theorem 1 from Lemma 5, Ω is still inaccessible in $\mathcal{M}[G \cap (\mathcal{P}^{\xi} \times \mathcal{P}'_B)]$. Hence if we take γ to be the first uncountable ordinal in $\mathcal{M}[G \cap (\mathcal{P}^{\xi} \times \mathcal{P}'_B)]$ then $\gamma < \Omega$. Since $\xi < \zeta$, ξ is countable in $\mathcal{M}[G \cap (\mathcal{P}^{\xi} \times \mathcal{P}'_B)]$ and hence also Q, and Q^* are countable in $\mathcal{M}[G \cap (\mathcal{P}^{\xi} \times \mathcal{P}'_B)]$. To show that γ satisfies the requirements of the lemma we shall prove

$$(15) \qquad \Psi_\gamma(p, t) \to (\exists\, \delta < \gamma)\Psi_\delta(p, t)\,.$$

By Lemma 9, (15) implies Lemma 10. To prove (15) assume $\Psi_\gamma(p, t)$, then, by (11), there is a dense subset D of Q^* in $\mathcal{M}$ such that $(\forall p' \in D \cap Q^*)(p' \geq p \to (\exists\, \lambda < \gamma)\Psi_\lambda(p', t))$. For every $p' \in D \cap Q^*$ such that $p' \geq p$ let $\lambda(p')$ be the least ordinal λ such that $\Psi_\lambda(p', t)$. The set $\{\lambda(p') \mid p' \in D \cap Q^* \wedge p' \geq p\}$ is a set of $\mathcal{M}[G \cap \mathcal{P}^{\xi} \times \mathcal{P}'_B]$ (being obtained from t, together with S, Q and $\mathcal{M}$, by absolute definitions) and is in that model a countable set of countable ordinals (since Q is countable there and γ is the $\aleph_1$ of that model). Therefore the set $\{\lambda(p') \mid p' \in D \cap Q^* \wedge p' \geq p\}$ has a strict bound δ which is countable in that model, i.e., $\delta < \gamma$. By (11) we have $\Psi_\delta(p, t)$, and thus (15) holds.

Lemma 11. *For all $t \in {}^{\omega}\omega \cap \mathcal{N}_2$, $t \in T$ iff*

$$(16) \qquad (\exists\, \gamma < \Omega)[(\forall p \in Q^*)(\Psi_\gamma(p, t) \to (\exists\, \lambda < \gamma)\Psi_\lambda(p, t)) \wedge (\forall \lambda < \gamma) \neg \Psi_\lambda(0, t)]\,.$$

Proof. Assume first that (16) holds. We claim that for all λ $\neg \Psi_\lambda(0, t)$ and hence, by Lemma 8, $t \in T$. Let γ be as in (16). Suppose $\Psi_\lambda(0, t)$ holds, then by the first part of (16) and Lemma 9, there is a $\lambda < \gamma$ such that $\Psi_\lambda(0, t)$, but this contradicts the second part of (16).

For every $t \in {}^{\omega}\omega \cap \mathcal{N}_2$ there is, by Lemma 10, a $\gamma < \Omega$ such that $(\forall p \in Q^*)(\Psi_\gamma(p, t) \to (\exists\, \lambda < \gamma)\Psi_\lambda(p, t))$. If also $t \in T$ then by Lemma 8, $\forall \lambda \neg \Psi_\lambda(0, t)$. (16) follows trivially from these two facts. □

We set now

$$(17) \qquad B_\gamma = \{t \in {}^{\omega}\omega \cap \mathcal{N}_2 \mid (\forall p \in Q^*)(\Psi_\gamma(p, t) \to (\exists\, \lambda < \gamma)\Psi_\lambda(p, t)) \wedge (\forall \lambda < \gamma) \neg \Psi_\lambda(0, t)\}\,.$$

By Lemma 11, $T = \bigcup_{\gamma<\Omega} B_\gamma$. Since s, of Lemma 6, is a member of T, it is a member of some B_γ. We shall now prove that each B_γ is a Borel set with a code in $\mathfrak{M}[G']$.

By a *description* of a set $C \subseteq {}^\omega\omega$, $C \in \mathfrak{N}_2$ we shall mean a pair $\langle l, a\rangle$, where l is one of 0, 1, 2, 3, 4, 5 and such that the following recursive conditions hold:

(a). If $n \in \omega$ and $h : n \to \omega$ then $\langle 0, h\rangle$ describes the set $\{f \in {}^\omega\omega \cap \mathfrak{N}_2 \mid f \supseteq h\}$.

(b). If c describes C then $\langle 1, c\rangle$ describes ${}^\omega\omega \cap \mathfrak{N}_2 \sim C$.

(c). If c_1, c_2 describe C_1, C_2 then $\langle 2, \{c_1, c_2\}\rangle$ describes $C_1 \cup C_2$ and $\langle 3, \{c_1, c_2\}\rangle$ describes $C_1 \cap C_2$.

(d). If C and h are functions on the same domain such that for every x in their domain $h(x)$ describes $C(x)$, then $\langle 4, h\rangle$ describes $\bigcup_{x\in\mathrm{dom}(C)} C(x)$ and $\langle 5, h\rangle$ describes $\bigcap_{x\in\mathrm{dom}(C)} C(x)$.

By the *history* $H(d)$ of a description d we mean the set of all functions h as in (d) which belong to the transitive hull of d. We can define $H(d)$ inductively as follows:

$$H(d) = \begin{cases} 0 & \text{if } d = \langle 0, h\rangle\,, \\ H(c) & \text{if } d = \langle 1, c\rangle\,, \\ H(c_1) \cup H(c_2) & \text{if } d = \langle 2, \{c_1, c_2\}\rangle \text{ or } \langle 3, \{c_1, c_2\}\rangle, \\ \{h\} \cup \bigcup_{x\in\mathrm{dom}(h)} H(h(x)) & \text{if } d = \langle 4, h\rangle \text{ or } \langle 5, h\rangle\,. \end{cases}$$

It can be easily shown, by induction on the description d, that for every transitive model $\mathfrak{M}'$ of ZF, if $H(d) \in \mathfrak{M}'$ then $d \in \mathfrak{M}'$. Another fact, which can again be shown by induction on d, is that if the domains of all the members of $H(d)$ are countable then d describes a Borel set. Moreover, if the domains of all members of $H(d)$ are ω or finite ordinals then d can be translated by means of an absolute function to a Borel code for the set of reals described by d, as given in [9, II, 1.1] (Here we have to use a slightly different definition since [9] deals with Borel sets of genuine reals, while we deal with Borel subsets of ${}^\omega\omega$. All we have to do is replace (1) of [9, II, Definition 1.1] by: (1) If $h : n \to \omega$ then α codes $\{x \in {}^\omega\omega \mid x \supseteq h\}$ if $\alpha(0) = 3n$, and for all $k < n$ $\alpha(k+1) = h(k)$.)

We shall now obtain descriptions of the sets B_γ as in (17) and by means of these descriptions we shall show that they are Borel sets with codes in $\mathfrak{M}[G']$. We shall first define a function $\psi_\gamma(p)$ in $\mathfrak{M}$ such that

$\psi_\gamma(p)$ will be a description in $\mathfrak{N}_2$ of $\{x \in {}^\omega\omega \cap \mathfrak{N}_2 \mid \Psi_\gamma(p, x)\}$.

$$\psi_0(p) = \langle 4, \{\langle h, \langle 0, h\rangle\rangle \mid h : lh(p) \to \omega \wedge \langle p, h\rangle \not\in S\}\rangle$$

since by (10)

$$\{x \in {}^\omega\omega \cap \mathfrak{N}_2 \mid \Psi_0(p, x)\} = \bigcup_{h : lh(p) \to \omega \wedge \langle p,h\rangle \not\in S} \{x \in {}^\omega\omega \cap \mathfrak{N}_2 \mid x \supseteq h\},$$

and

$$\psi_\gamma(s) = \langle 4, \{\langle D, \langle 5, \{\langle p', \langle 4, \{\langle \lambda, \psi_\lambda(p')\rangle \mid \lambda < \gamma\}\rangle\rangle \mid p' \in Q^* \cap D \wedge p' \geq p\}\rangle \mid D \in \mathfrak{M} \wedge D \text{ is a dense subset of } Q\}\rangle$$

since by (11)

$$\{x \in {}^\omega\omega \cap \mathfrak{N}_2 \mid \Psi_\gamma(p, x)\} = \bigcup_{\substack{D \in \mathfrak{M} \\ D \text{ is a dense} \\ \text{subset of } Q}} \bigcup_{\substack{p' \in Q^* \cap D \\ p' \geq p}} \bigcup_{\lambda < \gamma} \{x \in {}^\omega\omega \cap \mathfrak{N}_2 \mid \Psi_\lambda(p', x)\}.$$

By (17) the following b_γ is a description of B_γ

$$b_\gamma = \langle 3, \{\langle 5, \{\langle p, \langle 2, \{\langle 1, \psi_\gamma(p)\rangle, \langle 4, \{\langle \lambda, \psi_\lambda(p)\rangle \mid \lambda < \gamma\}\rangle\}\rangle\rangle \mid p \in Q^*\}\rangle, \langle 5, \{\langle \lambda, \langle 1, \psi_\lambda(0)\rangle\rangle \mid \lambda < \gamma\}\rangle\}\rangle .$$

The definition of b_γ is thus an absolute definition in $\mathfrak{M}$ (since A, Q, $Q^* \in \mathfrak{M}$). It can also easily be seen from the definition of b_γ that the domains of all functions in $H(b_\gamma)$ are the following: $\{h : lh(p) \to \omega \mid \langle p, h\rangle \not\in S\}$, for $p \in Q^*$; δ, for $\delta \leq \gamma$; $\{D \in \mathfrak{M} \mid D$ is a dense subset of $Q\}$; $\{p' \in Q^* \cap D \mid p' \geq p\}$, where D is a dense subset of Q in $\mathfrak{M}$ and $p \in Q^*$. As we have already seen, the inaccessibility of Ω in $\mathfrak{M}$ implies that all these sets are of cardinality $< \Omega$ in $\mathfrak{M}$, and are therefore countable in $\mathfrak{M}[G']$ and in $\mathfrak{N}_2$. Since in $\mathfrak{N}_2$ the domain of each function in $H(b_\gamma)$ is countable, b_γ is a Borel set in $\mathcal{N}_2$. Since the domains of these functions are already countable in $\mathcal{M}[G']$ we can replace, in $\mathcal{M}[G']$, these functions by functions on ω and on finite ordinals (with the same values) and thus obtain in $\mathcal{M}[G']$ a description c_γ for B_γ such that the domains of all functions in $H(c_\gamma)$ are ω or finite ordinals. Since, as remarked above, such a c_γ can be translated in an absolute way, to a code of the Borel set described by it, we get such a code for B_γ inside $\mathcal{M}[G']$, which is what we set out to prove.

3. Proof of Theorem 2

In order to prove Theorem 2 we look first at the model $\mathcal{N} = \mathcal{M}[G]$ as in [9, I, §3], where G is an $\mathcal{M}$-generic filter over the set $\mathcal{P}^\Omega$ of all functions p from finite subsets of $\Omega \times \omega$ into Ω such that $p(\alpha, n) < \alpha$ for $\langle \alpha, n \rangle \in \mathrm{dom}(p)$. We pass now to the submodel $\mathcal{N}_1$ of $\mathcal{N}$ which consists of all the members of $\mathcal{N}$ which are hereditarily $\mathcal{M}$-definable in $\mathcal{N}$ from a sequence of ordinals. What we did here is slightly different from what was done in [9, III, §2] since there $\mathcal{N}_1$ is taken to be the submodel of $\mathcal{N}$ consisting of all members of $\mathcal{N}$ which are definable in $\mathcal{N}$ from a sequence of ordinals, whereas we replace "definable" by "$\mathcal{M}$-definable", i.e., we allow also constants for members of $\mathcal{M}$ in the definition. Everything said in [9] about $\mathcal{N}_1$ holds also for our $\mathcal{N}_1$ by the same proof. In particular, $\mathcal{N}_1$ is a model of ZF + DC, in $\mathcal{N}_1$ every set of reals is Lebesgue measurable and has the Baire property, and also other statements hold in $\mathcal{N}_1$ as given in [9, Theorem 1]. A proof that every well-ordered set of reals in $\mathcal{N}_1$ is countable in $\mathcal{N}_1$ is given in [2, §4] (That proof has to undergo trivial modifications owing to the slightly different definitions which we use here).

All which we still have to prove is that in $\mathcal{N}_1$ every set U of reals is the union of $\aleph_1$ Borel sets. This will follow from the following lemma, which will be proved later.

Lemma 12. *Every set U of reals in $\mathcal{N}_1$ which is $\mathcal{M}$-definable in $\mathcal{N}$ is the union of $\aleph_1$ Borel sets in $\mathcal{N}_1$.*

If U is a set of reals in $\mathcal{N}_1$ then U is $\mathcal{M}$-definable in $\mathcal{N}$ from a sequence of ordinals, hence U is $\mathcal{M}[f]$-definable in $\mathcal{N}$. As we saw in the proof of Theorem 1 from Lemma 5, $\mathcal{M}[f]$ satisfies all our assumptions about $\mathcal{M}$, hence we can replace $\mathcal{M}$ by $\mathcal{M}[f]$ in Lemma 12. This replacement will not change $\mathcal{N}_1$ since $\mathcal{N}_1$ is, trivially, also the class of all members of $\mathcal{N}$ which are hereditarily $\mathcal{M}[f]$-definable in $\mathcal{N}$ from a sequence of ordinals. Therefore, by the conclusion of Lemma 12, U is the union of $\aleph_1$ Borel sets in $\mathcal{N}_1$.

Proof of Lemma 12. In $\mathcal{N}$, U is a union of $\aleph_1$ Borel sets as shown in the proof of Theorem 1. In the proof of Theorem 1 we worked with the model $\mathcal{M}[G]$ where G was an $\mathcal{M}$-generic filter on $\mathcal{P}^{\Omega} \times \mathcal{P}'_{\Theta}$, whereas now G is an $\mathcal{M}$-generic filter on $\mathcal{P}^{\Omega}$. However, the method used there to decompose U works equally well in our case, since the addition of the factor $\mathcal{P}'_{\Theta}$ did not facilitate the proof at any point; it only resulted in a slightly longer proof because there was one extra thing to take care of. All the Borel sets of reals in $\mathcal{N}$ are also Borel sets of reals in $\mathcal{N}_1$ [9, III, Lemma 2.5 and II, Theorem 1.4]. To show that the set Z of Borel sets which we constructed and into which U decomposes is indeed in $\mathcal{N}_1$ and has cardinality $\aleph_1$ in $\mathcal{N}_1$, it is enough to prove that Z has a well ordering in $\mathcal{N}_1$. If f is a one-one map of Z on some cardinal $\aleph_\alpha^{\mathcal{N}_1}$ of $\mathcal{N}_1$, f is also a one-one map of Z on $\aleph_\alpha^{\mathcal{N}}$ in $\mathcal{N}$ (since $\aleph_\alpha^{\mathcal{N}_1} = \aleph_\alpha^{\mathcal{N}}$ for every α, because of the following facts: $\mathcal{M} \subset \mathcal{N}_1 \subset \mathcal{N}$, all the cardinals of $\mathcal{M}$ which are $\geq \Omega$ are also cardinals of $\mathcal{N}$ (and, a fortiori, of $\mathcal{N}_1$) and the cardinals of $\mathcal{M}$ which are $< \Omega$ are countable ordinals in $\mathcal{N}$ and $\mathcal{N}_1$); but since Z is of cardinality $\aleph_1$ in $\mathcal{N}$, we have $\alpha = 1$, and Z is also of cardinality $\aleph_1$ in $\mathcal{N}_1$. Let us notice that even though we shall well-order Z in $\mathcal{N}_1$ we cannot well-order any set of codes for members of Z in $\mathcal{N}_1$, since in $\mathcal{N}_1$ every well ordered set of reals is countable.

To prove that Z has a well-ordering in $\mathcal{N}_1$ we have to show that Z has a well-ordering which is $\mathcal{M}$-definable in $\mathcal{N}$ from a sequence of ordinals. Our construction of the sets B_γ at the end of § 2 depends only, except for $\mathcal{M}$ itself, on the sets Q and S of $\mathcal{M}$ (Q^* is defined directly from Q). S was defined using the members u, Q and p_0 of $\mathcal{M}$. Q was defined in $\mathcal{M}$ using the parameters ξ and A, but A does not occur in our present setup. Thus the sequence B_γ of length Ω which we defined was defined in $\mathcal{N}$ by refering to $\mathcal{M}$ (by means of a unary predicate) and by using the parameters u, p_0, ξ which belongs to $\mathcal{M}$, and therefore we write B_γ as

$B_{\xi,p_0,u,\gamma}$. Since $\mathfrak{N}$ is a model of ZF, and all the B's are members of the power set of ${}^{\omega}\omega$ in $\mathfrak{N}$ one can easily see that there is an ordinal μ such that all the $B_{\xi,p_0,u,\gamma}$'s are obtained for u's with set-theoretical rank $< \mu$. Since the axiom of choice holds in $\mathfrak{M}$, $\mathfrak{M}$ contains a well-ordering w of the set $R(\mu)$ of all members of $\mathfrak{M}$ of rank $< \mu$. u can be defined as the α^{th} member of w for an appropriate α. Thus we can write $B_{\xi,p_0,u,\gamma}$ as $B'_{w,\xi,p_0,\alpha,\gamma}$. We define a well-ordering of the Borel subsets $B'_{w,\xi,p_0,\alpha,\gamma}$ of U, with w fixed, by a lexicographic ordering of $\langle \xi, p_0, \alpha, \gamma \rangle$ (it is easy to define a well-ordering of the members p_0 of Q by some lexicographic method). This well ordering is definable in the parameter $w \in \mathfrak{M}$.

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