Continuous-time optimization problems involving invex functions

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Abstract

In [D.H. Martin, The essence of invexity, J. Optim. Theory Appl. 47 (1985) 65–76] Martin introduced the notions of KKT-invexity and WD-invexity for mathematical programming problems. These notions are relaxations of invexity. In this work we generalize these concepts for continuous-time nonlinear optimization problems. We prove that the notion of KKT-invexity is a necessary and sufficient condition for global optimality of a Karush–Kuhn–Tucker point and that the notion of WD-invexity is a necessary and sufficient condition for weak duality.

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1. Introduction

Consider the continuous-time nonlinear programming problem

\[
\begin{align*}
\text{Minimize } & \phi(x) = \int_0^T f(x(t), t) \, dt \\
\text{subject to } & g(x(t), t) \leq 0 \quad \text{a.e. in } [0, T], \ x \in X.
\end{align*}
\]

(CNP)

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Here $X$ is a nonempty open convex subset of the Banach space $L^n_\infty[0,T]$, $\phi:X \rightarrow \mathbb{R}$, $f(x(t),t) = \xi(x)(t)$ and $g(x(t),t) = \gamma(x)(t)$, where $\xi$ is a mapping from $X$ into $\Lambda_1^n[0,T]$ and $\gamma$ is a mapping from $X$ into $\Lambda_1^m[0,T]$. $L^n_\infty[0,T]$ denotes the space of all $n$-dimensional vector-valued Lebesgue measurable functions defined on the compact interval $[0,T] \subset \mathbb{R}$, which are essentially bounded, with norm $\| \cdot \|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \leq j \leq n} \text{ess sup}\{|x_j(t)|, 0 \leq t \leq T\},$$

where for each $t \in [0,T]$, $x_j(t)$ is the $j$th component of $x(t) \in \mathbb{R}^n$ and $\Lambda_1^m[0,T]$ denotes the space of all $m$-dimensional vector-valued functions which are essentially bounded and Lebesgue measurable, defined on $[0,T]$, with the norm $\| \cdot \|_1$ defined by

$$\|y\|_1 = \max_{1 \leq j \leq m} \int_0^T |y_j(t)| \, dt.$$

A certain class of continuous-time optimization problems was introduced in 1953 by Bellman [4] in connection with production-inventory “bottleneck processes.” He considered a type of optimization problem, which is now known as a continuous-time linear programming problem, formulated its dual and provided duality relations. He also suggested some computational procedures. Since then, several authors have extended his theory to wider classes of continuous-time linear problems (e.g. [2,3,8,9,11,14,15,19,20]). On the other hand, optimality conditions of Karush–Kuhn–Tucker-type for continuous nonlinear problems were first investigated by Hanson and Mond [10]. They considered a class of linearly constrained nonlinear programming problems. Assuming a twice differentiable cost function, they linearized the cost function and applied Levinson’s duality theory [11] to obtain the Karush–Kuhn–Tucker optimality conditions. Also applying linearization, Farr and Hanson [7] obtained necessary and sufficient optimality conditions for a more general class of continuous-time nonlinear problems in which both the cost and constraint functions were nonlinear. Assuming some kind of constraint qualifications and using direct methods, further generalizations of the theory of optimality conditions and duality for continuous-time nonlinear problems were given in Abraham and Buie [1], Reiland and Hanson [16], Scott and Jefferson [18] and Zalmai [21–25]. The development of nonsmooth necessary optimality conditions for (CNP) was given in [5]. Some sufficient optimality conditions for the nonsmooth case were given in [17]. Related results can be found in Craven [6]. However, his arguments are via approximation by smooth functions rather than alternative theorems.

In the works cited above were given necessary conditions and sufficient conditions for a KKT point to be a global solution of the continuous-time problem. Although, none of them established a condition that was, at the same time, necessary and sufficient. The same situation occurs with weak duality. We observe that in the case of mathematical programming these results were given by Martin [13]. In this work we obtain similar results for the continuous-time case.

This work is organized as follows. In Section 2, we give some preliminaries. In Section 3, we recall the notion of invexity for (CNP) and give the generalization for (CNP) of the notion of KKT-invexity introduced by Martin [13] for the mathematical programming case. Also we prove our first main result and give an example. In Section 4, we introduce the notion of WD-invexity for the continuous-time problem (also introduced by Martin [13] for the mathematical programming case). Further, we prove our second main result and give an example.
2. Preliminaries

Let $F$ be the set of all feasible solutions of (CNP) (which we suppose nonempty), i.e., let
\[ F = \{ x \in X : g(x(t), t) \leq 0 \ \text{a.e. in } [0, T] \} . \]

Let $V$ be an open subset of $\mathbb{R}^n$ containing the set \{ $x(t) \in \mathbb{R}^n : x \in X, t \in [0, T]$ \}. We assume that $f$ and $g_i$ (the $i$th component of $g$), $i \in I = \{ 1, 2, \ldots, m \}$, are real-valued functions defined on $V \times [0, T]$. The functions $t \mapsto f(x(t), t)$ and $t \mapsto g(x(t), t)$ are assumed to be Lebesgue measurable and integrable for all $x \in X$. We assume also that the functions $f$ and $g$ are continuously differentiable with respect to their first arguments throughout $[0, T]$.

Given $x \in F$, for each $i \in I$ we denote by $A_i(x)$ the subset of $[0, T]$ where the $i$th constraint is active, i.e.,
\[ A_i(x) = \{ t \in [0, T] : g_i(x(t), t) = 0 \} . \]

In this paper, all vectors are column vectors. We use a prime to denote transposition. Besides, given $w \in \mathbb{R}^p$, $w \leq 0$ means that $w_i \leq 0$ for $i = 1, 2, \ldots, p$, and $w < 0$ means that $w_i < 0$ for $i = 1, 2, \ldots, p$.

In what follows, we state a Motzkin-type theorem of the alternative which will be useful for the proof of our results. This theorem is the continuous-time analogue of the theorem given on p. 66 of the book by Mangasarian [12]. Its proof is almost identical to the one given in Mangasarian’s book.

**Theorem 2.1.** Let $Z \subseteq L^{n}_\infty[0, T]$ be a nonempty convex subset. Let $p : W \times [0, T] \to \mathbb{R}^m$ and $q : W \times [0, T] \to \mathbb{R}^k$ be mappings given by $p(z(t), t) = \pi(z)(t)$ and $q(z(t), t) = B(t)z(t) - b(t)$, respectively, where $W \subseteq \mathbb{R}^n$ is an open subset, $\pi$ is a mapping from $Z$ into $A^m_1[0, T]$. $B(t)$ is a $k \times n$ matrix and $b(t) \in \mathbb{R}^k$. We assume that $p$ is convex with respect to its first argument in $W$ throughout $[0, T]$ and that there does not exist $v \in L^k_\infty[0, T] \setminus \{0\}$, $v(t) \geq 0$ a.e. in $[0, T]$, such that
\[ B'(t)v(t) = 0 \quad \text{a.e. in } [0, T]. \]

Then exactly one of the following systems is consistent:

(I) $p(z(t), t) < 0$, $B(t)z(t) \leq b(t)$ a.e. in $[0, T]$ has solution $z \in Z$;

(II) \[ \int_0^T \{ u'(t)p(z(t), t) + v'(t)(B(t)z(t) - b(t)) \} \, dt \geq 0 \text{ for all } z \in Z, \text{ for some } u \in L^m_\infty[0, T], u(t) \geq 0, u(t) \neq 0 \text{ a.e. in } [0, T] \text{ and for some } v \in L^k_\infty[0, T], v(t) \geq 0 \text{ a.e. in } [0, T]. \]

**Proof.** Similar to the proof of Theorem 3.4 of Zalmai [22, p. 137]. □

3. KKT-invexity and optimality conditions

For the continuous-time nonlinear programming problem where the functions are differentiable or nonsmooth, the Karush–Kuhn–Tucker conditions provide necessary conditions for an optimum, given certain qualifications on the constraints. See [23] for the differentiable case and [5] for the nonsmooth case.
A problem that continues to evoke a great deal of interest is that of finding sufficient conditions for an optimum. Work [24] gives some results in this direction via pseudoconvexity in the differentiable case and work [17] studies the nonsmooth case via the notion of invexity.

We recall the notion of invexity for (CNP) in the case that the functions are differentiable with respect to their first arguments.

**Definition 3.1.** We say that (CNP) is invex if there exists a function \( \eta : V \times V \times [0, T] \to \mathbb{R}^n \) such that \( t \mapsto \eta(x(t), y(t), t) \in L^\infty_{\infty}[0, T] \) and

\[
\phi(x) - \phi(y) \geq \int_0^T \nabla f'(y(t), t) \eta(x(t), y(t), t) dt,
\]

\[
g_i(x(t), t) - g_i(y(t), t) \geq \nabla g'_i(y(t), t) \eta(x(t), y(t), t) \quad \text{a.e. in } [0, T], \quad i \in I,
\]

for all \( x, y \in X \).

**Definition 3.2.** We say that a feasible solution \( y \) of (CNP) satisfies the Karush–Kuhn–Tucker conditions (we write KKT-conditions) if there exists \( \lambda \in L^m_{\infty}[0, T] \) such that

\[
\int_0^T \left[ \nabla f' + \sum_{i \in I} \lambda_i(t) \nabla g'_i \right] z(t) dt = 0 \quad \forall z \in L^\infty_{\infty}[0, T],
\]

\[
\lambda_i(t) g_i(y(t), t) = 0 \quad \text{a.e. in } [0, T], \quad i \in I,
\]

\[
\lambda_i(t) \geq 0 \quad \text{a.e. in } [0, T], \quad i \in I.
\]

In such a case, we say that \( y \) is a Karush–Kuhn–Tucker point (we write KKT-point) of (CNP).

**Definition 3.3.** We say that \( y \in \mathbb{F} \) is a global minimizer of (CNP) if

\[
\phi(x) \geq \phi(y) \quad \forall x \in \mathbb{F}.
\]

Now we repeat the argument used in [17].

Let \( y \) be a feasible solution for (CNP) that satisfies the KKT-conditions and suppose that (CNP) is invex. From (2) and (5), we have

\[
\int_0^T [f(x(t), t) - f(y(t), t)] dt - \int_0^T \nabla f'(y(t), t) \eta(x(t), y(t), t) dt
\]

\[
+ \int_0^T \sum_{i \in I} \lambda_i(t) [g_i(x(t), t) - g_i(y(t), t) - \nabla g'_i(y(t), t) \eta(x(t), y(t), t)] dt \geq 0,
\]

i.e.,

\[
\int_0^T [f(x(t), t) - f(y(t), t)] dt
\]
\[
\begin{align*}
\geq & \int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g_i'(y(t), t) \right] \eta(x(t), y(t), t) \, dt \\
& - \int_0^T \left[ \sum_{i \in I} \lambda_i(t) \left[ g_i(x(t), t) - g_i(y(t), t) \right] \right] \, dt.
\end{align*}
\]

Hence, by using (3) and (4) we obtain
\[
\int_0^T \left[ f(x(t), t) - f(y(t), t) \right] \, dt \geq - \int_0^T \left[ \sum_{i \in I} \lambda_i(t) g_i(x(t), t) \right] \, dt.
\]

Finally, it follows from (5) that
\[
\int_0^T \left[ f(x(t), t) - f(y(t), t) \right] \, dt \geq 0 \quad \forall x \in \mathbb{F}.
\]

Therefore \( \phi(x) \geq \phi(y) \, \forall x \in \mathbb{F} \), that is, \( y \) is a global minimizer of (CNP).

If we carefully examine this proof, we can see that the inequalities in (2) need only hold for feasible solutions of (CNP), i.e., only for \( x, y \in \mathbb{F} \), and that it is not necessary to have
\[
g_i(x(t), t) - g_i(y(t), t) \geq \nabla g_i'(y(t), t) \eta(x(t), y(t), t)
\]
for \( t \in A_i(y), \; i \in I \), because of (4). Also, it is easy to see that the omission of the terms \( g_i(x(t), t), \; i \in I \), in (2) does not affect the conclusion (6). With this in mind, we introduce a relaxation of invexity, which will be called KKT-invexity (see [13] for the mathematical programming case).

**Definition 3.4.** The problem (CNP) is called Karush–Kuhn–Tucker invex (or KKT-invex) if there exists a function \( \eta : V \times V \times [0, T] \to \mathbb{R}^n \) such that \( t \mapsto \eta(x(t), y(t), t) \in L_\infty^\infty[0, T] \) and
\[
\begin{align*}
\phi(x) - \phi(y) & \geq \int_0^T \nabla f'(y(t), t) \eta(x(t), y(t), t) \, dt, \\
- \nabla g_i'(y(t), t) \eta(x(t), y(t), t) & \geq 0 \quad \text{a.e. in } A_i(y), \; i \in I,
\end{align*}
\]
for all \( x, y \in \mathbb{F} \).

The next example shows a problem that is not invex but that is KKT-invex, that means, the notion of KKT-invexity is weaker than the notion of invexity. Further, it shows that invexity is not a necessary condition for the property that all Karush–Kuhn–Tucker points are global minimizers.

**Example 3.5.** We consider the following continuous-time nonlinear programming:

Minimize \( \phi(x) = \int_0^T \left[ 1 - \exp(-x(t)) \right] \, dt \)

subject to \( x(t) \geq 0 \quad \text{a.e. in } [0, T] \),
where \( x \in L_\infty[0, T] \). Take \( f(x(t), t) = 1 - \exp(-x(t)) \) and \( g(x(t), t) = -x(t) \). We have that \( y \equiv 0 \) is the only point that satisfies the KKT-conditions. In fact, setting \( \lambda(t) = \nabla f(y(t), t) = 1 \) we have

\[
\int_0^T \left[ \nabla f(y(t), t) + \lambda(t) \nabla g(y(t), t) \right] z(t) \, dt = 0
\]

for all \( z \in L_\infty[0, T] \). Then \( y \equiv 0 \) satisfies the KKT-conditions. Now, we show that it is the only one. Let \( x \) be such that \( x(t) > 0 \) a.e. in \( [0, T] \). Assume that \( x \) satisfies the KKT-conditions. So there exists \( \lambda(t) \in L_\infty[0, T] \) such that

\[
\int_0^T \left[ \nabla f(x(t), t) + \lambda(t) \nabla g(x(t), t) \right] z(t) \, dt = 0 \quad \forall z \in L_\infty[0, T],
\]

\[
\lambda(t) g(x(t), t) = 0 \quad \text{a.e. in } [0, T],
\]

\[
\lambda(t) \geq 0 \quad \text{a.e. in } [0, T].
\]

Consequently,

\[
\int_0^T \left[ \nabla f(x(t), t) + \lambda(t) \nabla g(x(t), t) \right] z(t) \, dt = 0 \quad \forall z \in L_\infty[0, T],
\]

\[
\lambda(t) = 0 \quad \text{a.e. in } [0, T],
\]

i.e.,

\[
\int_0^T \nabla f(x(t), t) z(t) \, dt = 0 \quad \forall z \in L_\infty[0, T].
\]

Therefore, \( \nabla f(x(t), t) = \exp(-x(t)) = 0 \) a.e. in \([0, T] \). This is absurd.

It is easy to see that \( \phi(x) \geq \phi(0) \) \( \forall x \in \mathbb{R} \), that is, 0 is a global minimizer of the problem. Thus, every point that satisfies the KKT-conditions is a global minimizer.

This problem is not invex. Indeed, if it was invex, there would exist a function \( \eta : V \times V \times [0, T] \rightarrow \mathbb{R} \) such that for \( x, y \in X \)

\[
\int_0^T \left[ f(x(t), t) - f(y(t), t) \right] dt \geq \int_0^T \nabla f(y(t), t) \eta(x(t), y(t), t) \, dt,
\]

\[
-x(t) + y(t) \geq -\eta(x(t), y(t), t) \quad \text{a.e. in } [0, T].
\]

But, this implies

\[
\int_0^T \left[ f(x(t), t) - f(y(t), t) \right] dt \geq \int_0^T \nabla f(y(t), t) \left[ x(t) - y(t) \right] dt
\]

\[
\geq \int_0^T \left[ f(x(t), t) - f(y(t), t) \right] dt - \int_0^T \nabla f(y(t), t) \eta(x(t), y(t), t) dt \geq 0
\]
since $\nabla f(y(t), t) = \exp(-y(t)) > 0$ a.e. in $[0, T]$. Hence
\[
\phi(x) - \phi(y) \geq D\phi(y)(x - y),
\]
where $D\phi(y)$ denotes the Fréchet derivative of $\phi$ at $y$. Then the problem is invex if and only if it is convex. But this problem is not convex, so that it is not invex.

Therefore invexity is not a necessary condition for the property that all KKT-points are global minimizers.

At last, we show that the problem is KKT-invex. Define $\eta: V \times V \times [0, T] \to \mathbb{R}$ by
\[
\eta(x(t), y(t), t) = [\phi(x) - \phi(y)] \left[ \int_0^T \exp(-y(t)) \, dt \right]^{-1}.
\]
So
\[
\phi(x) - \phi(y) - \int_0^T \exp(-y(t)) \eta(x(t), y(t), t) \, dt
\]
\[
= \phi(x) - \phi(y) - \left[ \phi(x) - \phi(y) \right] \left[ \int_0^T \exp(-y(t)) \, dt \right]^{-1} \left[ \int_0^T \exp(-y(t)) \, dt \right] = 0.
\]
Now take $x, y \in \mathbb{F}$, that is, take $x, y$ such that $x(t), y(t) \geq 0$ a.e. in $[0, T]$. For $t \in A(y) = \{ t \in [0, T]: y(t) = 0 \}$, we have
\[
-\nabla g(y(t), t) \eta(x(t), y(t), t) = \eta(x(t), y(t), t) = \eta(x(t), 0, t)
\]
\[
= [\phi(x) - \phi(0)] \left[ \int_0^T \exp(0) \, dt \right]^{-1} \geq 0
\]
since $y \equiv 0$ is the global minimizer.

Below we state a constraint qualification that will be needed to establish our result.

**Definition 3.6.** We say that $g$ satisfies (CQ1) at $y \in \mathbb{F}$ if there do not exist $v_i \in L_\infty[0, T]$, $v_i(t) \geq 0$ a.e. in $[0, T]$, $i \in I$, not all zero, such that
\[
\sum_{i \in I} \int_{A_i(y)} v_i(t) \nabla g'_i(y(t), t) z(t) \, dt \geq 0 \quad \text{for all } z \in L^n_\infty[0, T].
\]

**Lemma 3.7.** Let $y \in \mathbb{F}$ and assume that $g$ satisfies (CQ1) at $y$. If $y$ does not satisfy the KKT-conditions, then there exists $z \in L^n_\infty[0, T]$ such that
\[
\int_0^T \nabla f'(y(t), t) z(t) \, dt < 0,
\]
\[
\nabla g'_i(y(t), t) z(t) \leq 0 \quad \text{a.e. in } A_i(y), \quad i \in I.
\]
Proof. If the system in (8), (9) has no solution, then clearly the system

$$\int_0^T \nabla f'(y(t), t)z(t) dt < 0,$$

$$I_i(t)\nabla g_i'(y(t), t)z(t) \leq 0 \text{ a.e. in } [0, T], \ i \in I,$$

has no solution, where $I_i : [0, T] \to \mathbb{R}$ is defined, for each $i \in I$, by

$$I_i(t) = \begin{cases} 1 & \text{if } t \in A_i(y), \\ 0 & \text{if } t \notin A_i(y). \end{cases}$$

Let us verify that condition (1) in Theorem 2.1 is verified. Suppose that there exists a nonzero $v \in L^\infty_0 [0, T], v(t) \geq 0$ a.e. in $[0, T]$ such that

$$\left[ I_1(t)\nabla g_1(y(t), t) \cdots I_m(t)\nabla g_m(y(t), t) \right] v(t) = 0 \text{ a.e. in } [0, T].$$

So,

$$\int_0^T \sum_{i \in I} v_i(t)I_i(t)\nabla g_i'(y(t), t)z(t) dt = 0 \ \forall z \in L^n_\infty [0, T],$$

and using the definition of $I_i$, we obtain

$$\sum_{i \in I} \int_{A_i(y)} v_i(t)\nabla g_i'(y(t), t)z(t) dt = 0 \ \forall z \in L^n_\infty [0, T],$$

which contradicts (CQ1). Therefore, it follows from Theorem 2.1 that there exist $u_0 \in \mathbb{R}$ and $u_i \in L^n_\infty [0, T], i \in I$, with $u_0 > 0, u_i(t) \geq 0$ a.e. in $[0, T], i \in I$, such that

$$\int_0^T \left[ u_0\nabla f'(y(t), t) + \sum_{i \in I} u_i(t)I_i(t)\nabla g_i'(y(t), t) \right] z(t) dt \geq 0 \ \forall z \in L^n_\infty [0, T].$$

Dividing the expression above by $u_0$ and defining $\lambda_i = u_iI_i/u_0, i \in I$, it becomes

$$\int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t)\nabla g_i'(y(t), t) \right] z(t) dt \geq 0 \ \forall z \in L^n_\infty [0, T].$$

So we have

$$\int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t)\nabla g_i'(y(t), t) \right] z(t) dt = 0 \ \forall z \in L^n_\infty [0, T],$$

$$\lambda_i(t)g_i(y(t), t) = 0 \text{ a.e. in } [0, T], \ i \in I,$$

$$\lambda_i(t) \geq 0 \text{ a.e. in } [0, T], \ i \in I.$$

Therefore, $y$ satisfies the KKT-conditions, which contradicts the hypothesis. Thus there exists $z \in L^n_\infty [0, T]$ satisfying (8) and (9). □

Theorem 3.8. We assume that $g$ satisfies (CQ1) at each $y \in \mathbb{F}$. Then, every KKT-point of (CNP) is a global minimizer if and only if (CNP) is KKT-invex.
**Proof.** Sufficiency. We assume that (CNP) is KKT-invex. Let \( y \in \mathbb{F} \) be a KKT-point of (CNP). Then, there exist \( \lambda_i \in L_\infty[0, T], i \in I \), such that

\[
\frac{1}{T} \int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g'_i(y(t), t) \right] z(t) \, dt = 0 \quad \forall z \in L_\infty^n[0, T], \tag{10}
\]

\[
\lambda_i(t) g_i(y(t), t) = 0 \quad \text{a.e. in } [0, T], \quad i \in I, \tag{11}
\]

\[
\lambda_i(t) \geq 0 \quad \text{a.e. in } [0, T], \quad i \in I. \tag{12}
\]

It follows from (11) that \( \lambda_i(t) = 0, t \in [0, T] \setminus A_i(y), i \in I \). So, from (7) and (12) we obtain

\[
\frac{1}{T} \int_0^T \left[ f(x(t), t) - f(y(t), t) \right] \, dt - \frac{1}{T} \int_0^T \nabla f'(y(t), t) \eta(x(t), y(t), t) \, dt
\]

\[
- \frac{1}{T} \int_0^T \sum_{i \in I} \lambda_i(t) \nabla g'_i(y(t), t) \eta(x(t), y(t), t) \, dt \geq 0
\]

for all \( x \in \mathbb{F} \). Therefore,

\[
\frac{1}{T} \int_0^T \left[ f(x(t), t) - f(y(t), t) \right] \, dt
\]

\[
\geq \frac{1}{T} \int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g'_i(y(t), t) \right] \eta(x(t), y(t), t) \, dt
\]

for all \( x \in \mathbb{F} \), which in view of (10) reduces to

\[
\frac{1}{T} \int_0^T \left[ f(x(t), t) - f(y(t), t) \right] \, dt \geq 0 \quad \forall x \in \mathbb{F},
\]

and hence \( \phi(x) \geq \phi(y) \ \forall x \in \mathbb{F} \), that is, \( y \) is a global minimizer of (CNP).

**Necessity.** Suppose that every KKT-point of (CNP) is a global minimizer, and consider any pair of feasible solutions \( x, y \in \mathbb{F} \).

If \( \phi(x) < \phi(y) \), then \( y \) is not a global minimizer, and so, by hypothesis, \( y \) is not a KKT-point of (CNP). Hence, by Lemma 3.7, it follows that there exists \( z \in L_\infty^n[0, T] \) such that

\[
\frac{1}{T} \int_0^T \nabla f'(y(t), t) z(t) \, dt < 0, \tag{13}
\]

\[
\nabla g'_i(y(t), t) z(t) \leq 0 \quad \text{a.e. in } A_i(y), \quad i \in I. \tag{14}
\]

Define

\[
\eta(x(t), y(t), t) = \left[ \phi(x) - \phi(y) \right] \left[ \frac{1}{T} \int_0^T \nabla f'(y(t), t) z(t) \, dt \right]^{-1} z(t).
\]
We have

$$
\phi(x) - \phi(y) - \int_0^T \nabla f'(y(t), t) \eta(x(t), y(t), t) \, dt = 0.
$$

(15)

Since $\phi(x) < \phi(y)$, it follows from (13) that

$$
\left[ \phi(x) - \phi(y) \right] \left[ \int_0^T \nabla f'(y(t), t) z(t) \, dt \right]^{-1} > 0.
$$

Then, from (14) we have

$$
\nabla g_i^t(y(t), t) \eta(x(t), y(t), t) \leq 0 \quad \text{a.e. in } A_i(y), \ i \in I.
$$

(16)

From (15) and (16) it follows that (CNP) is KKT-invex.

If $\phi(x) \geq \phi(y)$, we consider $\eta(x(t), y(t), t) = 0$ so that

$$
\phi(x) - \phi(y) - \int_0^T \nabla f'(x(t), t) \eta(x(t), y(t), t) \, dt \geq 0
$$

(17)

and

$$
\nabla g_i^t(y(t), t) \eta(x(t), y(t), t) = 0 \quad \text{a.e. in } A_i(y), \ i \in I.
$$

(18)

From (17) and (18) we obtain that (CNP) is KT-invex.

In the cases above we do not define $\eta$ for $x, y \notin \mathbb{F}$. But we can take $\eta(x(t), y(t), t) = 0$ when $x$ or $y$ is not feasible. \(\square\)

4. WD-invexity and duality

In this section we formulate a Lagrangian dual of (CNP) and introduce the notion of WD-invexity. WD-invexity is another relaxation of invexity for (CNP). This relaxation arises when we observe the proof that invexity implies weak duality.

When the functions $t \mapsto \nabla f(x(t), t)$ and $t \mapsto \nabla g_i^t(x(t), t) z(t), i \in I$, are Lebesgue integrable in $[0, T]$ for all $x \in X$ and for all $z \in L^n_{\infty}[0, T]$, the dual problem is formulated as follows:

Maximize $\psi(x, \lambda) = \int_0^T \left[ f(x(t), t) + \sum_{i \in I} \lambda_i(t) g_i(x(t), t) \right] \, dt$

subject to $\int_0^T \left[ \nabla f'(x(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g_i^t(x(t), t) \right] z(t) \, dt = 0$

\(\forall z \in L^n_{\infty}[0, T], \ \lambda_i(t) \geq 0 \quad \text{a.e. in } [0, T], \ i \in I, \ x \in X \quad \text{and} \quad \lambda \in L^n_{\infty}[0, T].\)

(WDP)

This dual problem may be viewed as the continuous-time analogue of Wolfe’s duality formulation. See Zalmai [21].
We denote by \( \mathbb{G} \) the set of all feasible solutions of (WDP), i.e.,
\[
\mathbb{G} = \left\{ (x, \lambda) \in X \times L^m_t[0, T]: \lambda_i(t) \geq 0 \text{ a.e. in } [0, T], \ i \in I, \right. \\
\left. \quad \int_0^T \left[ \nabla f'(x(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g_i'(x(t), t) \right] z(t) \, dt = 0 \quad \forall z \in L^n_t[0, T] \right\}.
\]

**Definition 4.1.** We say that weak duality holds between the problems (CNP) and (WDP) if
\[
\phi(x) \geq \psi(y, \lambda)
\]
for all \( x \in \mathbb{F} \) and all \( (y, \lambda) \in \mathbb{G} \).

Next we prove that invexity implies weak duality.

Assume that the problem (CNP) is invex. Then, there exists a function \( \eta: V \times V \times [0, T] \rightarrow \mathbb{R}^n \) such that \( t \mapsto \eta(x(t), y(t), t) \in L^n_t[0, T] \) and
\[
\phi(x) - \phi(y) \geq \int_0^T \nabla f'(y(t), t) \eta(x(t), y(t), t) \, dt,
\]
\[
g_i(x(t), t) - g_i(y(t), t) \geq \nabla g'_i(y(t), t) \eta(x(t), y(t), t) \quad \text{a.e. in } [0, T], \ i \in I,
\]
for all \( x, y \in X \). Let \( x \in \mathbb{F} \) and \( (y, \lambda) \in \mathbb{G} \). Combining and rearranging these inequalities, we obtain
\[
\int_0^T f(x(t), t) \, dt - \int_0^T \left[ f(y(t), t) + \sum_{i \in I} \lambda_i(t) g_i(y(t), t) \right] \, dt
\]
\[
\geq \int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g'_i(y(t), t) \right] \eta(x(t), y(t), t) \, dt
\]
\[
- \int_0^T \sum_{i \in I} \lambda_i(t) g_i(x(t), t) \, dt.
\]
Since \( x \in \mathbb{F} \) and \( (y, \lambda) \in \mathbb{G} \), the integral in the second line above is zero and the integral in the third line is nonnegative. Consequently, the term in the first line is nonnegative. Therefore,
\[
\phi(x) \geq \psi(y, \lambda)
\]
for all \( x \in \mathbb{F} \) and all \( (y, \lambda) \in \mathbb{G} \), that is, weak duality holds between the problems (CNP) and (WDP).

From this proof we see that the inequalities in (19) should hold only for feasible solutions of (CNP), i.e., only for \( x \in \mathbb{F} \). Also, it is easy to see that the omission of the terms \( g_i(x(t), t), i \in I \), in (19) does not affect the proof. With this in mind, we introduce a second relaxation of invexity, which will be called WD-invexity (see [13] for the mathematical programming case).
**Definition 4.2.** The problem (CNP) is called *weak duality invex* (or WD-invex) if there exists a function \( \eta : V \times V \times [0, T] \to \mathbb{R}^n \) such that \( t \mapsto \eta(x(t), y(t), t) \in L^\infty_\infty[0, T] \) and

\[
\phi(x) - \phi(y) \geq T \int_0^T \nabla f'((y(t), t)) \eta(x(t), y(t), t) \, dt,
\]

\[
-g_i(y(t), t) \geq \nabla g'_i(y(t), t) \eta(x(t), y(t), t) \quad \text{a.e. in } [0, T], \ i \in I,
\]

for all \( x \in \mathbb{F} \) and \( y \in \mathbb{X} \).

Below we state another constraint qualification which is assumed as a hypothesis in our result.

**Definition 4.3.** We say that \( g \) satisfies (CQ2) if there do not exist \( v_i \in L^\infty_\infty[0, T], v_i(t) \geq 0 \) a.e. in \([0, T] , \ i \in I \), not all zero, such that

\[
T \int_0^T \sum_{i \in I} v_i(t) g_i(x(t), t) \, dt \geq 0 \quad \forall x \in \mathbb{X}.
\]

**Theorem 4.4.** We assume that \( g \) satisfies (CQ2). Then, weak duality holds for (CNP) if and only if it is WD-invex.

**Proof.** *Sufficiency.* Suppose that (CNP) is WD-invex. Let \( x \in \mathbb{F} \) and \( (y, \lambda) \in \mathbb{G} \). From (20) we have

\[
\int_0^T f(x(t), t) \, dt - \int_0^T f(y(t), t) + \sum_{i \in I} \lambda_i(t) g_i(y(t), t) \, dt \geq \int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g'_i(y(t), t) \right] \eta(x(t), y(t), t) \, dt.
\]

Since \( (y, \lambda) \in \mathbb{G} \), the integral in the second line above is zero, and hence we conclude that weak duality holds between (CNP) and (WDP).

*Necessity.* We assume that weak duality holds. Then, for all \( x \in \mathbb{F} \) and all \( y \in \mathbb{X} \), the system

\[
\int_0^T \left[ \nabla f'(y(t), t) + \sum_{i \in I} \lambda_i(t) \nabla g'_i(y(t), t) \right] z(t) \, dt = 0 \quad \forall z \in L^\infty_\infty[0, T],
\]

\( \lambda_i(t) \geq 0 \) a.e. in \([0, T] , \ i \in I \),

\[
\int_0^T \left[ f(x(t), t) - f(y(t), t) - \sum_{i \in I} \lambda_i(t) g_i(y(t), t) \right] \, dt < 0,
\]

has no solution \( \lambda \in L^m_\infty[0, T] \). It is easy to see that this is equivalent to asserting that the system

\[
\int_0^T \begin{bmatrix} 0 & 1 \\ \nabla f'(y(t), t) & f(x(t), t) - f(y(t), t) \end{bmatrix} \begin{bmatrix} z(t) \\ \alpha \end{bmatrix} \, dt < 0
\]
\[
+ \begin{bmatrix}
\lambda_1(t) & \cdots & \lambda_m(t)
\end{bmatrix}
\begin{bmatrix}
\nabla g_1'(y(t), t) & -g_1(y(t), t) \\
\vdots & \vdots \\
\nabla g_m'(y(t), t) & -g_m(y(t), t)
\end{bmatrix}
\begin{bmatrix}
z(t) \\
\alpha
\end{bmatrix}
\end{bmatrix}
\int_0^T dt = 0
\]

\(\forall z \in L^m_\infty[0, T], \forall \alpha \in \mathbb{R},
\quad v, \mu > 0 \quad \text{and} \quad \lambda(t) \geq 0 \quad \text{a.e. in } [0, T],
\)

has no solution \((v, \mu, \lambda)\). Let us verify that condition (1) in Theorem 2.1 is satisfied. Suppose that there exists \(v \in L^m_\infty[0, T] \setminus \{0\}, \quad v(t) \geq 0 \quad \text{a.e. in } [0, T],\)

such that
\[
\begin{bmatrix}
\nabla g_1'(y(t), t) & \cdots & \nabla g_m'(y(t), t) \\
-g_1(y(t), t) & \cdots & -g_m(y(t), t)
\end{bmatrix}
\begin{bmatrix}
v_1(t) \\
\vdots \\
v_m(t)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\quad \text{a.e. in } [0, T].
\]

But the equality above is equivalent to
\[
\sum_{i \in I} v_i(t) \nabla g_i(y(t), t) = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\text{ a.e. in } [0, T],
\]

so that
\[
\sum_{i \in I} v_i(t) g_i(y(t), t) = 0 \quad \text{a.e. in } [0, T],
\]

and hence
\[
\int_0^T \sum_{i \in I} v_i(t) g_i(y(t), t) dt = 0,
\]

which contradicts (CQ2). Therefore, condition (1) in Theorem 2.1 holds. It follows from this theorem that the system
\[
\begin{bmatrix}
0 & 1 \\
\nabla f'(y(t), t) & f(x(t), t) - f(y(t), t)
\end{bmatrix}
\begin{bmatrix}
z(t) \\
\alpha
\end{bmatrix}
\leq 0 \quad \text{a.e. in } [0, T],
\]

\[
\begin{bmatrix}
\nabla g_i'(y(t), t) & -g_i(y(t), t) \\
\vdots \\
\nabla g_m'(y(t), t) & -g_m(y(t), t)
\end{bmatrix}
\begin{bmatrix}
z(t) \\
\alpha
\end{bmatrix}
\leq 0 \quad \text{a.e. in } [0, T],
\]

has solution \((z, \alpha) \in L^m_\infty[0, T] \times \mathbb{R}\) with \(\alpha < 0\) and so we can set \(\alpha = -1\). Defining
\[
\eta(x(t), y(t), t) = z(t),
\]

we have
\[
\nabla f'(y(t), t) \eta(x(t), y(t), t) - f(x(t), t) + f(y(t), t) < 0 \quad \text{a.e. in } [0, T],
\]

\[
\nabla g_i'(y(t), t) \eta(x(t), y(t), t) + g_i(y(t), t) \leq 0 \quad \text{a.e. in } [0, T], \quad i \in I.
\]

Consequently,
\[
\phi(x) - \phi(y) > \int_0^T \nabla f'(y(t), t) \eta(x(t), y(t), t) dt,
\]

\[
-g_i(y(t), t) \geq \nabla g_i'(y(t), t) \eta(x(t), y(t), t) \quad \text{a.e. in } [0, T], \quad i \in I,
\]

that is, the problem (CNP) is WD-invex.
Above, we do not define $\eta$ for $x \notin \mathbb{F}$. But in this case we can take $\eta(x(t), y(t), t) = 0$ for all $y \in X$. □

The next example shows that WD-invexity is a notion weaker than invexity and stronger than KKT-invexity.

**Example 4.5.** We consider again the problem of Example 3.5; we can easily show that weak duality holds if

$$
\phi(0) \geq \phi(y) - D\phi(y)y \quad \forall y \in X. \tag{21}
$$

In fact, let $x \in \mathbb{F}$. We saw that $\phi(x) \geq \phi(0)$. Hence, from (21) we have

$$
\phi(x) \geq \phi(y) - \int_{0}^{T} \exp(-y(t))y(t) dt \quad \forall y \in X. \tag{22}
$$

Let $(y, \lambda) \in \mathbb{G}$. Then $\lambda(t) \geq 0$ a.e. in $[0, T]$ and

$$
\int_{0}^{T} \left[ \nabla f(y(t), t) + \lambda(t) \nabla g(y(t), t) \right] z(t) dt = 0 \quad \forall z \in L_{\infty}[0, T]
$$

$$
\iff \int_{0}^{T} \left[ \exp(-y(t)) - \lambda(t) \right] z(t) dt = 0 \quad \forall z \in L_{\infty}[0, T].
$$

Consequently, $\exp(-y(t)) = \lambda(t)$ a.e. in $[0, T]$. From (22) we have

$$
\phi(x) \geq \phi(y) - \int_{0}^{T} \lambda(t)y(t) dt.
$$

Recalling that $g(y(t), t) = -y(t)$, we obtain

$$
\int_{0}^{T} f(x(t), t) \geq \int_{0}^{T} \left[ f(y(t), t) + \lambda(t)g(y(t), t) \right] dt.
$$

Therefore, weak duality holds.

Thus under condition (21) it follows from Theorem 4.4 ((CQ2) is trivially satisfied) that this problem is WD-invex. We saw in Example 3.5 that this problem is KKT-invex and that it is not invex. Therefore, we observe that WD-invexity is stronger than KKT-invexity and weaker than invexity.

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References