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On certain cohomological invariants of groups

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Abstract

Let R be a left and right $\aleph_0$-Noetherian ring. We show that if all projective left and all projective right R-modules have finite injective dimension, then all injective left and all injective right R-modules have finite projective dimension. Using this result, we prove that the invariants $\operatorname{silp}\mathbb{Z}G$ and $\operatorname{spli}\mathbb{Z}G$, which were introduced by Gedrich and Gruenberg (1987) [15], are equal for any group G. As an application of the latter equality, we show that a group G is finite if and only if $\underline{\operatorname{cd}}\,G = 0$, where $\underline{\operatorname{cd}}$ is the generalized cohomological dimension of groups introduced by Ikenaga (1984) [21].

Keywords: Group rings; Cohomological dimension; Projective dimension; Injective dimension; Flatness; Mittag–Leffler condition

Contents

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0. Introduction

The classical Tate cohomology theory of finite groups (cf. [6, Chapter VI] and [7, Chapter XII]) was generalized by Farrell in [14] to the class of all groups with finite virtual cohomological dimension. In his attempt to extend the definition of Farrell cohomology to an even bigger class of groups, Ikenaga introduced in [21] the generalized cohomological dimension $\underline{\mathrm{cd}}\, G$ of a group G, by defining $\underline{\mathrm{cd}}\, G$ to be the supremum of all integers n, for which there exist a $\mathbb{Z}$-free $\mathbb{Z}G$-module M and a projective $\mathbb{Z}G$-module P, such that $\mathrm{Ext}^n_{\mathbb{Z}G}(M, P) \neq 0$. Ikenaga showed that if the group G has finite virtual cohomological dimension, then $\underline{\mathrm{cd}}\, G = \mathrm{vcd}\, G$. In particular, if G is a finite group, then $\underline{\mathrm{cd}}\, G = 0$. We prove that the converse of the latter assertion is also true:

Theorem A. *A group G is finite if and only if $\underline{\mathrm{cd}}\, G = 0$.*

This characterization of finiteness was conjectured in [22] by Ikenaga, who proved that the vanishing of $\underline{\mathrm{cd}}$ implies finiteness under the presence of an additional homological finiteness hypothesis. We are able to remove that hypothesis by translating the vanishing of the groups $\mathrm{Ext}^1_{\mathbb{Z}G}(M, P)$, where M and P are as above, into a Mittag–Leffler condition on certain inverse systems of Hom-groups. We note that this technique has been also used in [20], in order to establish a projectivity criterion, which was conjectured by Kaplansky, for modules over commutative domains.

The generalized cohomological dimension $\underline{\mathrm{cd}}\, G$ of G is closely related to the invariants $\mathrm{spli}\, \mathbb{Z}G$ and $\mathrm{silp}\, \mathbb{Z}G$, which were introduced by Gedrich and Gruenberg in [15], in connection with the existence of complete cohomological functors in the category of $\mathbb{Z}G$-modules. Here, $\mathrm{spli}\, \mathbb{Z}G$ is the supremum of the projective lengths of injective $\mathbb{Z}G$-modules, whereas $\mathrm{silp}\, \mathbb{Z}G$ is the supremum of the injective lengths of projective $\mathbb{Z}G$-modules. The relation between these two invariants has been studied by several authors in various contexts: Using the Hopf algebra structure of the group ring, Gedrich and Gruenberg have shown in [15, 1.6 and Corollary 5.9] that for any group G we have $\mathrm{silp}\, \mathbb{Z}G \leqslant \mathrm{spli}\, \mathbb{Z}G$, with equality if $\mathrm{spli}\, \mathbb{Z}G$ is finite. The equality $\mathrm{silp}\, \mathbb{Z}G = \mathrm{spli}\, \mathbb{Z}G$ has been established for groups in the class $\mathbf{H}\mathfrak{F}$ of Kropholler (cf. [8, §6]) and, more generally, for groups in $\mathbf{LH}\mathfrak{F}$ (cf. [32, Corollary 1]). On the other hand, the equality between $\mathrm{silp}\, \mathbb{Z}G$ and $\mathrm{spli}\, \mathbb{Z}G$ is also known for groups G that have periodic cohomology after some steps (cf. [31, Theorem 3.2]); in that case, a result of Adem and Smith [1] shows that the finiteness of these invariants implies the existence of a free action of G on a finite-dimensional CW-complex, which is homotopy equivalent to a sphere. We shall prove that the equality between $\mathrm{silp}\, \mathbb{Z}G$ and $\mathrm{spli}\, \mathbb{Z}G$ holds for any group G:

Theorem B. *If G is any group, then* $\mathrm{silp}\, \mathbb{Z}G = \mathrm{spli}\, \mathbb{Z}G$.

One may prove Theorem A using Theorem B and the main result of [9], where it is shown that the finiteness of G is equivalent to the equality $\mathrm{spli}\, \mathbb{Z}G = 1$. As far as the proof of Theorem B is concerned, it suffices to establish the inequality $\mathrm{spli}\, \mathbb{Z}G \leqslant \mathrm{silp}\, \mathbb{Z}G$; indeed, as we mentioned above, the reverse inequality has been proved by Gedrich and Gruenberg in [15]. We can easily reduce the proof to the case of a countable group. Then, the integral group ring $\mathbb{Z}G$ is a countable ring and the proof of the inequality $\mathrm{spli}\, \mathbb{Z}G \leqslant \mathrm{silp}\, \mathbb{Z}G$ follows from a more general result, which we now describe.

Let R be any ring. Then, as in the special case where R is the integral group ring of a group, we may consider the injective lengths of projective left R-modules and the projective lengths of injective left R-modules, in order to define the left invariants $\text{l-silp}\, R$ and $\text{l-spli}\, R$ respectively. In the same way, we may consider right R-modules and define the right invariants $\text{r-silp}\, R$ and $\text{r-spli}\, R$. We note that the distinction between left and right R-modules is superfluous if, for example, R is a commutative ring or the group algebra of a group. Gedrich and Gruenberg note in [15] that the relation between $\text{l-spli}\, R$ and $\text{l-silp}\, R$ is unclear for a general ring R and ask whether the finiteness of one implies that of the other. A positive answer to the latter question would imply the equality $\text{l-spli}\, R = \text{l-silp}\, R$; indeed, it is easily seen that if both invariants are finite, then they are equal. As shown by Faith and Walker in [12,13], the class of quasi-Frobenius rings is characterized by the vanishing of any one of the four invariants l-silp, r-silp, l-spli and r-spli. On the other hand, Jensen has proved in [26, 5.9] that if R is a commutative Noetherian ring, then we always have $\text{silp}\, R = \text{spli}\, R$. In the special case where R is an Artin algebra, the equality $\text{l-spli}\, R = \text{l-silp}\, R$ is equivalent to a long-standing conjecture in representation theory, the so-called Gorenstein Symmetry Conjecture, which appears as conjecture 13 at the end of [2] (see also [3, §11] and [4, Chapter VII]). As shown by Happel in [19], the latter conjecture is closely related to the existence of Serre duality in the homotopy category of perfect complexes over R [29], which is itself a key hypothesis in Kontsevich's formalism of Non-commutative Algebraic Geometry [27].

The ring R is called left $\aleph_0$-Noetherian if all left ideals of it are countably generated. In the same way, one defines the class of right $\aleph_0$-Noetherian rings. For example, any countable ring is both left and right $\aleph_0$-Noetherian. We prove the following result:

Theorem C. *Let R be a ring which is both left and right $\aleph_0$-Noetherian. If both* $\text{l-silp}\, R$ *and* $\text{r-silp}\, R$ *are finite, then* $\text{l-spli}\, R = \text{l-silp}\, R$ *and* $\text{r-spli}\, R = \text{r-silp}\, R$.

As an immediate consequence of Theorem C, it follows that $\text{spli}\, R \leqslant \text{silp}\, R$ for any commutative $\aleph_0$-Noetherian ring R, with equality if $\text{silp}\, R < \infty$; this result may be viewed as a partial generalization of the result of Jensen mentioned above. As another consequence of Theorem C, we conclude that $\text{spli}\, \mathbb{Z}G \leqslant \text{silp}\, \mathbb{Z}G$ for any countable group G; as we noted above, the proof of Theorem B follows easily from this.

Instead of considering projective resolutions, we may consider flat resolutions and define the invariant $\text{l-sfli}\, R$ (resp. $\text{r-sfli}\, R$) as the supremum of the weak dimensions of injective left (resp. right) R-modules. Since projective modules are flat, it is clear that $\text{l-sfli}\, R \leqslant \text{l-spli}\, R$ and $\text{r-sfli}\, R \leqslant \text{r-spli}\, R$. Using a duality argument, we prove that we also have $\text{r-sfli}\, R \leqslant \text{l-silp}\, R$ if R is a left $\aleph_0$-Noetherian ring and $\text{l-sfli}\, R \leqslant \text{r-silp}\, R$ if R is a right $\aleph_0$-Noetherian ring. The use of flat modules and weak dimension in this setting enables us to use a surprising result of Jensen, which states that flat modules have finite projective dimension, provided that the finitistic dimension of the ring is finite (cf. [25, Proposition 6]). The proof of Theorem C is a consequence of these two results.

The contents of the paper are as follows: In Section 1, we present a few equivalent descriptions of the Mittag–Leffler condition on inverse systems of abelian groups and establish a relation between the vanishing of the Ext^1-group and the Mittag–Leffler condition on certain inverse systems of Hom-groups. In the following section, we examine a natural transformation (which was introduced by Cartan and Eilenberg) and use it in order to prove a key duality result. The latter result is then applied in Section 3, in order to relate the invariants silp and spli for $\aleph_0$-Noetherian rings. Finally, in Section 4, we establish the equality between silp and spli for integral group rings and prove that the finiteness of a group is equivalent to the vanishing of Ikenaga's generalized cohomological dimension.

1. Preliminaries on the Mittag–Leffler condition

The goal of this preliminary section is to record a few properties of inverse systems of abelian groups that are related to the Mittag–Leffler condition. All direct and inverse systems will be indexed by the ordered set $\mathbb{N}$ of natural numbers.

Let $(A_n)_n$ be an inverse system of abelian groups with structural maps $\sigma_{n,m} : A_m \longrightarrow A_n$, $n \leqslant m$, and consider the inverse limit $A = \varprojlim_n A_n$, which is endowed with canonical maps $s_n : A \longrightarrow A_n$, $n \geqslant 0$. The right derived functors of the inverse limit functor were introduced by Roos in [30]. In that paper, Roos showed that the higher inverse limits $\varprojlim^i$ vanish for all $i \geqslant 2$; the assumption that the inverse systems are indexed by $\mathbb{N}$ is crucial here. For all $n \leqslant m$ we shall denote by $A_{n,m}$ the image of $\sigma_{n,m} : A_m \longrightarrow A_n$; it is clear that $\operatorname{im} s_n \subseteq A_{n,m}$. We consider for all $n \geqslant 0$ the decreasing filtration

$$A_n = A_{n,n} \supseteq A_{n,n+1} \supseteq A_{n,n+2} \supseteq A_{n,n+3} \supseteq \cdots$$

of A_n by the $A_{n,m}$'s. The inverse system $(A_n)_n$ is said to satisfy the Mittag–Leffler condition if these filtrations are eventually constant. In other words, $(A_n)_n$ satisfies the Mittag–Leffler condition if for all $n \in \mathbb{N}$ there exists a suitable integer $N = N(n)$ with $N \geqslant n$, such that

$$A_{n,N} = A_{n,N+1} = A_{n,N+2} = A_{n,N+3} = \cdots.$$

Assuming that the inverse system $(A_n)_n$ satisfies the Mittag–Leffler condition, we shall refer to the subgroup $A'_n = A_{n,N} \subseteq A_n$ (where $N = N(n)$ as above) as the stable image. It is clear that the structural morphisms $\sigma_{n,m} : A_m \longrightarrow A_n$ map the stable image $A'_m \subseteq A_m$ onto the stable image $A'_n \subseteq A_n$ for all $n \leqslant m$. Therefore, we may consider the subsystem of stable images $(A'_n)_n \subseteq (A_n)_n$. It is easily seen that the inclusion $(A'_n)_n \hookrightarrow (A_n)_n$ induces an isomorphism between the corresponding inverse limits $\varprojlim_n A'_n \simeq \varprojlim_n A_n$.

Lemma 1.1. *Let $(A_n)_n$ be an inverse system of abelian groups with structural maps denoted by $\sigma_{n,m} : A_m \longrightarrow A_n$, $n \leqslant m$, and consider the inverse limit $A = \varprojlim_n A_n$. Then, the following conditions are equivalent:*

(i) *The inverse system $(A_n)_n$ satisfies the Mittag–Leffler condition.*
(ii) *For all $n \in \mathbb{N}$ there exists an integer $N = N(n) \geqslant n$, such that the subgroup $A_{n,N} \subseteq A_n$ coincides with the image of the canonical map $s_n : A \longrightarrow A_n$.*
(iii) *For all $n \in \mathbb{N}$ there exists an integer $N = N(n) \geqslant n$, such that for any abelian group B the kernel of the additive map*

$$\operatorname{Hom}_{\mathbb{Z}}(A_n, B) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(A, B), \tag{1}$$

which is induced by the canonical map $s_n : A \longrightarrow A_n$, coincides with the kernel of the additive map

$$\mathrm{Hom}_{\mathbb{Z}}(A_n, B) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A_N, B), \tag{2}$$

which is induced by $\sigma_{n,N} : A_N \longrightarrow A_n$.

Proof. Even though the equivalent formulation of the Mittag–Leffler condition that appears in (ii) is certainly well known, we shall provide a proof of the lemma because it is condition (iii) that we shall use in the sequel.

(i) → (ii): Assuming that the inverse system $(A_n)_n$ satisfies the Mittag–Leffler condition, let $(A'_n)_n$ be the subsystem of stable images of $(A_n)_n$ and consider the inverse limit $A' = \varprojlim_n A'_n$. As we have already noted, the inclusions $A'_n \hookrightarrow A_n$ induce an isomorphism $A' \simeq A$. The structural maps of the inverse system $(A'_n)_n$ being surjective, it is easily seen that the canonical maps $s'_n : A' \longrightarrow A'_n$ are surjective as well. Therefore, the commutativity of the diagram

$$\begin{array}{ccc} A' & \xrightarrow{\sim} & A \\ {\scriptstyle s'_n}\downarrow & & \downarrow{\scriptstyle s_n} \\ A'_n & \hookrightarrow & A_n \end{array}$$

shows that $\mathrm{im}\, s_n = A'_n$. This completes the proof, since the stable image $A'_n \subseteq A_n$ is the image of $\sigma_{n,N} : A_N \longrightarrow A_n$ for some $N \geqslant n$.

(ii) → (iii): Let B be an abelian group. Then, the kernel of (1) consists of those additive maps $f : A_n \longrightarrow B$ which vanish when restricted to the image of the canonical map $s_n : A \longrightarrow A_n$. Similarly, the kernel of (2) consists of those additive maps $f : A_n \longrightarrow B$ which vanish when restricted to the image of $\sigma_{n,N} : A_N \longrightarrow A_n$. Therefore, (iii) follows readily from (ii).

(iii) → (i): We fix $n \in \mathbb{N}$ and choose an integer $N = N(n) \geqslant n$ as in (iii). We consider the cokernel of the canonical map $s_n : A \longrightarrow A_n$ and note that the projection $\pi : A_n \longrightarrow \mathrm{coker}\, s_n$ is contained in the kernel of the additive map

$$\mathrm{Hom}_{\mathbb{Z}}(A_n, \mathrm{coker}\, s_n) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{coker}\, s_n).$$

Therefore, in view of our assumption, π is also contained in the kernel of the additive map

$$\mathrm{Hom}_{\mathbb{Z}}(A_n, \mathrm{coker}\, s_n) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A_N, \mathrm{coker}\, s_n).$$

In other words, π vanishes in the image $A_{n,N}$ of $\sigma_{n,N} : A_N \longrightarrow A_n$, i.e. we have $A_{n,N} \subseteq \mathrm{im}\, s_n$. Then, the chain of inclusions

$$A_{n,N} \supseteq A_{n,N+1} \supseteq A_{n,N+2} \supseteq A_{n,N+3} \supseteq \cdots \supseteq \mathrm{im}\, s_n \supseteq A_{n,N}$$

shows that we actually have equalities

$$A_{n,N} = A_{n,N+1} = A_{n,N+2} = A_{n,N+3} = \cdots = \mathrm{im}\, s_n.$$

Since this is the case for all $n \in \mathbb{N}$, we conclude that the inverse system $(A_n)_n$ satisfies the Mittag–Leffler condition, as needed. □

The Mittag–Leffler condition was introduced by Grothendieck in [17], as a sufficient condition for the vanishing of $\varprojlim^1$. Even though this sufficient condition turns out to be necessary in the case of inverse systems of countable abelian groups (cf. [16]), in general, the vanishing of $\varprojlim^1$ does not imply the Mittag–Leffler condition. In order to formulate a condition in terms of the functor $\varprojlim^1$, which is equivalent to the Mittag–Leffler condition, one may proceed as follows: The category of inverse systems of abelian groups has arbitrary direct sums, which are computed pointwise. In particular, for any inverse system $(A_n)_n$ of abelian groups we may consider the direct sum of an infinite countable number of copies of itself. The latter is the inverse system $(A_n^{(\mathbb{N})})_n$, whose structural maps $A_m^{(\mathbb{N})} \longrightarrow A_n^{(\mathbb{N})}$ are those induced by the structural maps $A_m \longrightarrow A_n$ of $(A_n)_n$ for all $n \leqslant m$. In [11, Corollary 6], it is proved that the inverse system $(A_n)_n$ satisfies the Mittag–Leffler condition if and only if $\varprojlim_n^1 A_n^{(\mathbb{N})} = 0$. We shall use that characterization of the Mittag–Leffler condition below.

We note that for any ring R and any direct system $(M_n)_n$ of left R-modules a contravariant functor $\mathfrak{F}$ from the category of left R-modules to that of abelian groups induces an inverse system of abelian groups $(\mathfrak{F}M_n)_n$, whose structural maps $\mathfrak{F}M_m \longrightarrow \mathfrak{F}M_n$ are induced by the structural maps $M_n \longrightarrow M_m$ of the direct system $(M_n)_n$ for all $n \leqslant m$.

Proposition 1.2. *(Cf. [20, Example 2.4(4)] and the discussion in pp. 208–213 of [18].) Let R be a ring and $(M_n)_n$ a direct system of finitely generated left R-modules with direct limit $M = \varinjlim_n M_n$. We consider a left R-module P and assume that $\operatorname{Ext}^1_R(M, P^{(\mathbb{N})}) = 0$, where $P^{(\mathbb{N})}$ is the direct sum of an infinite countable number of copies of P. Then, the inverse system of abelian groups $(\operatorname{Hom}_R(M_n, P))_n$, whose structural maps are induced by the structural maps of the direct system $(M_n)_n$, satisfies the Mittag–Leffler condition.*

Proof. It is well known that one may express the Ext-groups of M in terms of the Ext-groups of the M_n's, by means of short exact sequences

$$0 \longrightarrow \varprojlim_n^1 \operatorname{Ext}_R^{*-1}(M_n, \_) \longrightarrow \operatorname{Ext}_R^*(M, \_) \longrightarrow \varprojlim_n \operatorname{Ext}_R^*(M_n, \_) \longrightarrow 0.$$

In particular, there is a short exact sequence of abelian groups

$$0 \longrightarrow \varprojlim_n^1 \operatorname{Hom}_R\big(M_n, P^{(\mathbb{N})}\big) \longrightarrow \operatorname{Ext}^1_R\big(M, P^{(\mathbb{N})}\big) \longrightarrow \varprojlim_n \operatorname{Ext}^1_R\big(M_n, P^{(\mathbb{N})}\big) \longrightarrow 0.$$

Our assumption about the vanishing of the group $\operatorname{Ext}^1_R(M, P^{(\mathbb{N})})$ therefore implies that

$$\varprojlim_n^1 \operatorname{Hom}_R\big(M_n, P^{(\mathbb{N})}\big) = 0. \tag{3}$$

Since the left R-module M_n is finitely generated, the abelian group $\operatorname{Hom}_R(M_n, P^{(\mathbb{N})})$ can be identified with the direct sum $\operatorname{Hom}_R(M_n, P)^{(\mathbb{N})}$ of an infinite countable number of copies of the group $\operatorname{Hom}_R(M_n, P)$. Then, the inverse system $(\operatorname{Hom}_R(M_n, P^{(\mathbb{N})}))_n$ is identified with the direct sum of an infinite countable number of copies of the inverse system $(\operatorname{Hom}_R(M_n, P))_n$ and we may rewrite (3) into the form

$$\varprojlim_n^1 \operatorname{Hom}_R(M_n, P)^{(\mathbb{N})} = 0.$$

Using [11, Corollary 6], we conclude that the inverse system of abelian groups $(\operatorname{Hom}_R(M_n, P))_n$ satisfies the Mittag–Leffler condition, as needed. □

2. The natural transformation Φ

Let R be a ring. If P is a left R-module and D an abelian group, then the abelian group $\mathrm{Hom}_{\mathbb{Z}}(P, D)$ of all additive maps from P to D can be endowed with the structure of a right R-module, by using the left R-module structure of P. If M is another left R-module, then we may consider the tensor product $\mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M$ and define the map

$$\Phi : \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M, P), D\big), \tag{4}$$

by letting $\Phi(f \otimes m)$ be the operator which is given by $g \mapsto f(g(m))$, $g \in \mathrm{Hom}_R(M, P)$, for all $f \in \mathrm{Hom}_{\mathbb{Z}}(P, D)$ and $m \in M$. It is easily seen that Φ is a well-defined additive map, which is natural in P, D and M. As we are mainly interested in the dependence of Φ upon M, we shall denote the map (4) above by Φ_M.

The natural transformation Φ was introduced by Cartan and Eilenberg in [7, Chapter VI, §5], in order to obtain certain duality isomorphisms. In the special case where R is the group ring $\mathbb{Z}G$ of a group G and $P = \mathbb{Z}G$, the natural transformation Φ was used in a crucial way by Ikenaga [22], in his attempt to prove that the vanishing of the generalized cohomological dimension $\underline{\mathrm{cd}}\, G$ implies the finiteness of G.

Our first goal is to examine some properties of Φ. In particular, we shall obtain conditions under which Φ is a monomorphism of abelian groups. We begin with a preliminary result, whose easy proof is omitted (cf. [5, Chapter II, §4, Exercise 6]).

Lemma 2.1. *Let P, M be left R-modules and assume that M is finitely presented. Then, the additive map*

$$\Phi_M : \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M, P), D\big)$$

defined above is an isomorphism for any divisible abelian group D.

Proposition 2.2. *Let $(M_n)_n$ be a direct system of finitely presented left R-modules with direct limit $M = \varinjlim_n M_n$. We consider a left R-module P and assume that $\mathrm{Ext}^1_R(M, P^{(\mathbb{N})}) = 0$, where $P^{(\mathbb{N})}$ is the direct sum of an infinite countable number of copies of P. Then, the additive map*

$$\Phi_M : \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M, P), D\big)$$

is injective for any divisible abelian group D.

Proof. Since the group $\mathrm{Ext}^1_R(M, P^{(\mathbb{N})})$ is trivial, we may apply Proposition 1.2 and conclude that the inverse system of abelian groups $(\mathrm{Hom}_R(M_n, P))_n$, whose structural maps are induced by the structural maps of the direct system $(M_n)_n$, satisfies the Mittag–Leffler condition. We consider a divisible abelian group D and let $t_n : M_n \longrightarrow M$ be the canonical map for all $n \in \mathbb{N}$. In view of the naturality of Φ, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(P,D)\otimes_R M_n & \xrightarrow{\Phi_{M_n}} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M_n,P),D) \\ \downarrow{\scriptstyle 1\otimes t_n} & & \downarrow{\scriptstyle t_{n*}} \\ \mathrm{Hom}_{\mathbb{Z}}(P,D)\otimes_R M & \xrightarrow{\Phi_M} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M,P),D) \end{array}$$

is commutative for all $n \in \mathbb{N}$. In order to prove the injectivity of Φ_M, it suffices to show that

$$\ker(t_{n*} \circ \Phi_{M_n}) \subseteq \ker(1 \otimes t_n) \tag{5}$$

for all $n \in \mathbb{N}$. Indeed, let us assume that (5) holds and fix an element $\xi \in \ker \Phi_M$. Since the abelian group $\mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M$ is the direct limit of the direct system $(\mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M_n)_n$, there exists $n \in \mathbb{N}$ and an element $\xi_n \in \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M_n$, such that $\xi = (1 \otimes t_n)(\xi_n)$. Since

$$(t_{n*} \circ \Phi_{M_n})(\xi_n) = \big(\Phi_M \circ (1 \otimes t_n)\big)(\xi_n) = \Phi_M(\xi) = 0,$$

it follows that $\xi_n \in \ker(t_{n*} \circ \Phi_{M_n})$. In view of (5), we conclude that $\xi_n \in \ker(1 \otimes t_n)$ and hence $\xi = (1 \otimes t_n)(\xi_n) = 0$. This shows that $\ker \Phi_M = 0$, proving the injectivity of Φ_M.

Therefore, it only remains to prove that (5) holds for all $n \in \mathbb{N}$. Having fixed the non-negative integer n, we note that the abelian group $\mathrm{Hom}_R(M, P)$ can be naturally identified with the inverse limit of the system $(\mathrm{Hom}_R(M_n, P))_n$, in such a way that the canonical map from the inverse limit to $\mathrm{Hom}_R(M_n, P)$ is identified with the map

$$\mathrm{Hom}_R(M, P) \longrightarrow \mathrm{Hom}_R(M_n, P),$$

which is induced by $t_n : M_n \longrightarrow M$. Since the inverse system $(\mathrm{Hom}_R(M_n, P))_n$ satisfies the Mittag–Leffler condition, we may use Lemma 1.1 in order to find an integer $N = N(n) \geqslant n$, such that the kernel of

$$t_{n*} : \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M_n, P), D\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M, P), D\big)$$

coincides with the kernel of the additive map

$$\tau_{n,N*} : \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M_n, P), D\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M_N, P), D\big),$$

which is induced by the structural map $\tau_{n,N} : M_n \longrightarrow M_N$. We now consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(P,D)\otimes_R M_n & \xrightarrow{\Phi_{M_n}} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M_n,P),D) \\ \downarrow{\scriptstyle 1\otimes \tau_{n,N}} & & \downarrow{\scriptstyle \tau_{n,N*}} \\ \mathrm{Hom}_{\mathbb{Z}}(P,D)\otimes_R M_N & \xrightarrow{\Phi_{M_N}} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M_N,P),D). \end{array}$$

The R-module M_N being finitely presented, the additive map Φ_{M_N} is bijective (Lemma 2.1) and hence

$$\ker(\tau_{n,N*} \circ \Phi_{M_n}) = \ker(1 \otimes \tau_{n,N}). \tag{6}$$

Since $t_N \circ \tau_{n,N} = t_n$, it follows that $(1 \otimes t_N) \circ (1 \otimes \tau_{n,N}) = 1 \otimes t_n$ and hence we conclude that

$$\ker(1 \otimes \tau_{n,N}) \subseteq \ker(1 \otimes t_n). \tag{7}$$

On the other hand, in view of our choice of $N \in \mathbb{N}$, we have $\ker t_{n*} = \ker \tau_{n,N*}$ and hence

$$\ker(t_{n*} \circ \Phi_{M_n}) = \Phi_{M_n}^{-1}(\ker t_{n*}) = \Phi_{M_n}^{-1}(\ker \tau_{n,N*}) = \ker(\tau_{n,N*} \circ \Phi_{M_n}). \tag{8}$$

Combining (6), (7) and (8), it follows that $\ker(t_{n*} \circ \Phi_{M_n}) \subseteq \ker(1 \otimes t_n)$, as needed. □

A left R-module is called countably presented if it is the cokernel of a linear map between countably generated free left R-modules. It is well known that the class of countably presented modules coincides with the class of those modules that may be expressed as direct limits of finitely presented modules; see, for example, [18, Lemma 1.2.8]. (We remind the reader of our convention that all direct systems be indexed by the ordered set of natural numbers.) We may therefore restate Proposition 2.2 as follows:

Proposition 2.3. *Let P, M be left R-modules and assume that M is countably presented. If $\mathrm{Ext}^1_R(M, P^{(\mathbb{N})}) = 0$, where $P^{(\mathbb{N})}$ is the direct sum of an infinite countable number of copies of P, then the additive map*

$$\Phi_M : \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M, P), D\big)$$

is injective for any divisible abelian group D.

Having fixed a left R-module P and a divisible abelian group D, we consider for any left R-module M a projective resolution

$$F_* \longrightarrow M \longrightarrow 0.$$

Then, the natural transformation Φ induces a chain map

$$\Phi_{F_*} : \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R F_* \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(F_*, P), D\big).$$

By applying homology, we obtain additive maps

$$\Phi_M^{(n)} : \mathrm{Tor}_n^R\big(\mathrm{Hom}_{\mathbb{Z}}(P, D), M\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Ext}^n_R(M, P), D\big),$$

$n \geqslant 0$, which do not depend upon the particular choice of the projective resolution of M. It is clear that $\Phi_M^{(0)}$ can be identified with the map Φ_M studied before. Moreover, the $\Phi_M^{(n)}$'s are natural in M and commute with the connecting homomorphisms, which are associated with any short exact sequence of left R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

In other words, Φ induces a morphism of homological exact ∂-functors

$$\Phi^{(*)} : \mathrm{Tor}_*^R\big(\mathrm{Hom}_{\mathbb{Z}}(P, D), \_\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Ext}_R^*(\_, P), D\big).$$

Let n be a non-negative integer and assume that the projective resolution $F_* \longrightarrow M \longrightarrow 0$ is such that F_i is finitely generated for $i = n, n+1$. (Such a resolution exists if, for example, M is a module of type FP_{n+1}.) Then, the map $\Phi_M^{(n)}$ defined above is bijective; this is essentially shown by Ikenaga in [22, Proposition 1.7]. In the following result, we show that one may relax this homological finiteness condition on M and still be able to prove the injectivity of $\Phi_M^{(n)}$, under the additional assumption that a certain Ext-group vanishes. In this way, we obtain a generalization of certain well-known duality isomorphisms (cf. [7, Chapter VI, §5] and [18, Lemma 1.2.11]).

Proposition 2.4. *Let n be a non-negative integer and consider two left R-modules P and M. We assume that:*

(i) *M admits a free resolution $F_* \longrightarrow M \longrightarrow 0$, such that F_i is countably generated for $i = n, n+1$ and*
(ii) *$\mathrm{Ext}_R^{n+1}(M, P^{(\mathbb{N})}) = 0$, where $P^{(\mathbb{N})}$ is the direct sum of an infinite countable number of copies of P.*

Then, the additive map

$$\Phi_M^{(n)} : \mathrm{Tor}_n^R\big(\mathrm{Hom}_{\mathbb{Z}}(P, D), M\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Ext}_R^n(M, P), D\big)$$

is injective for any divisible abelian group D.

Proof. Let K be the cokernel of the map $F_{n+1} \longrightarrow F_n$. In view of our assumption (i), K is a countably presented left R-module which fits to an exact sequence

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0. \tag{9}$$

Since $\mathrm{Ext}_R^1(K, P^{(\mathbb{N})}) = \mathrm{Ext}_R^{n+1}(M, P^{(\mathbb{N})}) = 0$, we may use Proposition 2.3 and conclude that the additive map

$$\Phi_K : \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R K \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(K, P), D\big)$$

is injective. By applying dimension shifting to the exact sequence (9), we obtain (composing connecting homomorphisms) injective additive maps

$$\mathrm{Tor}_n^R\big(\mathrm{Hom}_{\mathbb{Z}}(P, D), M\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(P, D) \otimes_R K$$

and

$$\mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Ext}^n_R(M,P),D\big)\longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(K,P),D\big).$$

Since $\Phi^{(*)}$ commutes with connecting homomorphisms, we conclude that the following diagram of abelian groups (whose top raw is exact) is commutative

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Tor}^R_n(\mathrm{Hom}_{\mathbb{Z}}(P,D),M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(P,D)\otimes_R K \\ & \Big\downarrow \Phi^{(n)}_M & & \Big\downarrow \Phi^{(0)}_K \\ & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}^n_R(M,P),D) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(K,P),D). \end{array}$$

Therefore, the injectivity of $\Phi^{(n)}_M$ follows from the injectivity of $\Phi^{(0)}_K=\Phi_K$. □

Corollary 2.5. *Let n be a non-negative integer and consider a left R-module M. We assume that:*

(i) *M admits a free resolution $F_*\longrightarrow M\longrightarrow 0$, such that F_i is countably generated for $i=n,n+1$,*
(ii) $\mathrm{Ext}^{n+1}_R(M,R^{(\mathbb{N})})=0$, *where $R^{(\mathbb{N})}$ is the direct sum of an infinite countable number of copies of R, and*
(iii) $\mathrm{Ext}^n_R(M,R)=0$.

Then, $\mathrm{Tor}^R_n(I,M)=0$ *for any injective right R-module I.*

Proof. It is well known that $\mathrm{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})$ is an injective cogenerator of the category of right R-modules, i.e. any right R-module N embeds as a submodule of a suitable direct product of copies of the injective right R-module $\mathrm{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})$. In fact, there is a natural monomorphism of right R-modules

$$N\hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})^X\simeq \mathrm{Hom}_{\mathbb{Z}}\big(R,(\mathbb{Q}/\mathbb{Z})^X\big),$$

where X is the set of all additive maps form N to $\mathbb{Q}/\mathbb{Z}$. Of course, the above monomorphism splits if the right R-module N is injective.

Therefore, it follows that any injective right R-module I embeds as a direct summand of a right R-module of the form $\mathrm{Hom}_{\mathbb{Z}}(R,D)$, where D is a divisible abelian group. Hence, in order to show that $\mathrm{Tor}^R_n(I,M)=0$ for any injective right R-module I, it suffices to show that $\mathrm{Tor}^R_n(\mathrm{Hom}_{\mathbb{Z}}(R,D),M)=0$ for any divisible abelian group D. The latter equality follows from Proposition 2.4 (by letting $P=R$ therein). □

3. The invariants silp and spli for $\aleph_0$-Noetherian rings

Let R be a ring. In connection with the existence of complete cohomological functors in the category of left R-modules, Gedrich and Gruenberg have defined in [15] the invariant l-silp R as the supremum of the injective lengths of projective left R-modules and the invariant l-spli R as the supremum of the projective lengths of injective left R-modules. In the same way, we may consider right R-modules and define the right invariants r-silp R and r-spli R. We note that, instead of considering projective resolutions, we may consider flat resolutions and define the invariants l-sfli R (resp. r-sfli R) as the supremum of the weak dimensions of injective left (resp. right) R-modules. Since projective modules are flat, it is clear that l-sfli $R \leqslant$ l-spli R and r-sfli $R \leqslant$ r-spli R.

Remark 3.1. Let R be a left $\aleph_0$-Noetherian ring, i.e. a ring all of whose left ideals are countably generated. Then, arguing by induction on n, one can show that any submodule of the left R-module R^n is countably generated for all $n \geqslant 1$. It follows that the same is true for all submodules of the direct sum $R^{(\mathbb{N})}$ of an infinite countable number of copies of the left R-module R. Hence, we conclude that any submodule of a countably generated left R-module is countably generated (cf. [24, Lemma 1]). Therefore, a left R-module M is countably presented if and only if it is countably generated. Moreover, such an M possesses a resolution by countably generated free left R-modules.

If R is a left Noetherian ring, then, as shown by Iwanaga in [23], the injective dimension of the left R-module R is equal to r-sfli R. We shall now partly generalize Iwanaga's result, as follows:

Proposition 3.2. *Let R be a left $\aleph_0$-Noetherian ring. Then,* r-sfli $R \leqslant$ l-silp R.

Proof. We note that there is nothing to prove if l-silp $R = \infty$ and hence we may assume that l-silp $R = n < \infty$. We have to show that any injective right R-module has weak dimension $\leqslant n$. Let I be an injective right R-module.

In view of the hypothesis made on R, we know that any countably generated left R-module M admits a resolution by countably generated free modules (cf. Remark 3.1). Our assumption about the value of l-silp R implies that the functors $\mathrm{Ext}_R^{n+1}(\_, P)$ and $\mathrm{Ext}_R^{n+2}(\_, P)$ are identically zero for any projective left R-module P. Therefore, we may apply Corollary 2.5 and conclude that $\mathrm{Tor}_{n+1}^R(I, M) = 0$. Since this is the case for any countably generated left R-module M, the continuity of the Tor-functors with respect to filtered colimits implies that $\mathrm{Tor}_{n+1}^R(I, \_) = 0$. It follows that I has weak dimension $\leqslant n$, as needed. □

Recall that the left finitistic dimension l-fin.dim R of a ring R is defined as the supremum of the projective dimensions of those left R-modules that have finite projective dimension. Therefore, the finiteness of l-fin.dim R is equivalent to the assertion that there is a uniform upper bound on the projective dimension of those left R-modules that have finite projective dimension. In the same way, we may consider right R-modules and define the right finitistic dimension r-fin.dim R.

Proposition 3.3. *Let R be a ring, such that* r-sfli $R < \infty$. *Then,* r-spli $R \leqslant$ r-fin.dim R.

Proof. Assume that r-sfli $R = n < \infty$. The inequality to be proved is obvious if r-fin.dim $R = \infty$ and hence we may assume that r-fin.dim $R < \infty$. Then, in order to show that r-spli $R \leqslant$ r-fin.dim R, it suffices to show that any injective right R-module has finite projective dimension. (Indeed, it would then follow that any injective right R-module has projective dimension bounded by the right finitistic dimension r-fin.dim R.)

Let I be an injective right R-module and consider an exact sequence of right R-modules

$$0 \longrightarrow M \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0, \tag{10}$$

with F_i projective for all $i = 0, 1, \ldots, n-1$. Since r-sfli $R = n$, the right R-module I has weak dimension $\leqslant n$. The projective right R-module F_i is flat for all $i = 0, 1, \ldots, n-1$ and hence the right R-module M is also flat. As shown by Jensen in [25, Proposition 6], the finiteness of the right finitistic dimension r-fin.dim R of R implies that any flat right R-module has finite projective dimension; in particular, the right R-module M has finite projective dimension. In view of the exact sequence (10), we have $\mathrm{Ext}_R^{i+n}(I, \_) = \mathrm{Ext}_R^i(M, \_)$ for all $i \geqslant 1$ and hence the right R-module I has finite projective dimension as well. □

Corollary 3.4. *Let R be a left $\aleph_0$-Noetherian ring, such that* l-silp $R < \infty$. *Then,* r-spli $R \leqslant$ r-fin.dim R.

Proof. The assumption made on R implies that r-sfli $R \leqslant$ l-silp R (Proposition 3.2). Hence, r-sfli R is finite and the result follows from Proposition 3.3. □

By symmetry, i.e. by applying all of the above to the opposite R^{op} of a ring R, we obtain analogous results for right $\aleph_0$-Noetherian rings, i.e. for rings all of whose right ideals are countably generated. In particular, we have:

Corollary 3.4^{op}. *Let R be a right $\aleph_0$-Noetherian ring, such that* r-silp $R < \infty$. *Then,* l-spli $R \leqslant$ l-fin.dim R.

Our next goal is to combine Corollaries 3.4 and 3.4^{op}, in order to obtain a left–right symmetric assertion. To that end, we shall use the following well-known result, which appears for instance in [28, Lemma 6.4] and whose proof goes back to [8].

Lemma 3.5. *If R is a ring, then* l-fin.dim $R \leqslant$ l-silp R *and* r-fin.dim $R \leqslant$ r-silp R.

Proof. By symmetry, it suffices to prove the first of these inequalities. To that end, let M be a left R-module of finite projective dimension, say equal to m. Then, there exists a projective left R-module P, such that $\mathrm{Ext}_R^m(M, P) \neq 0$. In particular, such a module P has injective dimension $\geqslant m$ and hence l-silp $R \geqslant m$. The result follows since l-fin.dim R is the supremum of such m's. □

We can now prove the following result (Theorem C of the Introduction):

Theorem 3.6. *Let R be a ring which is both left and right $\aleph_0$-Noetherian. If both* l-silp R *and* r-silp R *are finite, then* l-spli R = l-silp R *and* r-spli R = r-silp R.

Proof. It follows from Corollaries 3.4 and 3.4^{op} that r-spli $R \leqslant$ r-fin.dim R and l-spli $R \leqslant$ l-fin.dim R. Using Lemma 3.5, we conclude that r-spli $R \leqslant$ r-silp R and l-spli $R \leqslant$ l-silp R. In particular, r-spli R and l-spli R are also finite. As noted by Gedrich and Gruenberg in [15, 1.6], the finiteness of both l-silp R and l-spli R (resp. of both r-silp R and r-spli R) implies that l-silp R = l-spli R (resp. that r-silp R = r-spli R). □

If a ring R is isomorphic with its opposite R^{op}, then any left R-module M may be identified with a right R-module M' and vice versa, in such a way that M and M' have the same homological properties (in particular, the same projective and injective dimensions) as left and right modules respectively. In that case, we have l-spli R = r-spli R(= spli R) and l-silp R = r-silp R(= silp R). The following result is an immediate consequence of Theorem 3.6:

Corollary 3.7. *Let R be a ring which is isomorphic with its opposite R^{op}. If R is left (and hence right) $\aleph_0$-Noetherian, then* $\operatorname{spli} R \leqslant \operatorname{silp} R$*, with equality if* $\operatorname{silp} R < \infty$.

Since commutative rings are obviously isomorphic with their opposites, we obtain the following partial generalization of a result of Jensen, who proved in [26, 5.9] that $\operatorname{spli} R = \operatorname{silp} R$ if R is a commutative Noetherian ring.

Corollary 3.8. *If R is a commutative $\aleph_0$-Noetherian ring, then* $\operatorname{spli} R \leqslant \operatorname{silp} R$*, with equality if* $\operatorname{silp} R < \infty$.

4. The case of group rings

Let k be a commutative ring, G a group and $R = kG$ the corresponding group ring. Then, R is isomorphic with the opposite ring R^{op} and hence the distinction between left and right modules is redundant. Assume that the group G is countable and the commutative ring k is $\aleph_0$-Noetherian; for example, the ring k may be countable or a field. Then, R is a left (and hence right) $\aleph_0$-Noetherian ring. Indeed, any left ideal of R is a k-submodule of the countably generated k-module R; hence, it is countably generated as a k-module and *a fortiori* as an R-module. Therefore, we may apply Corollary 3.7, in order to conclude that $\operatorname{spli} kG \leqslant \operatorname{silp} kG$.

We shall prove that the latter inequality is valid for any (not necessarily countable) group G. To that end, we consider the ordered set consisting of all countable subgroups H of G. The latter set is filtered and the inclusions $H \hookrightarrow G$ are easily seen to induce an isomorphism between G and the colimit of the H's. We also note that for any two $\mathbb{Z}G$-modules M and N, the abelian group $M \otimes_{\mathbb{Z}G} N$ is identified with the colimit of the abelian groups $M \otimes_{\mathbb{Z}H} N$, where H runs through the countable subgroups of G.

Lemma 4.1. *Let k be a commutative ring, G a group and M a kG-module. If M is flat as a kH-module for all countable subgroups H of G, then M is flat as a kG-module as well.*

Proof. Let $f : N' \longrightarrow N$ be an injective kG-linear map. We have to show that the additive map

$$1_M \otimes f : M \otimes_{kG} N' \longrightarrow M \otimes_{kG} N \tag{11}$$

is also injective. Let $\xi \in \ker(1_M \otimes f)$. We may express G as the filtered colimit of its countable subgroups as above and conclude that there is a countable subgroup H of G and an element $\xi_H \in M \otimes_{kH} N'$, which maps onto ξ under the canonical map

$$M \otimes_{kH} N' \longrightarrow M \otimes_{kG} N'$$

and is contained in the kernel of the additive map

$$1_M \otimes f : M \otimes_{kH} N' \longrightarrow M \otimes_{kH} N.$$

The latter map is injective, in view of our assumption about the flatness of M over kH, and hence $\xi_H = 0 \in M \otimes_{kH} N'$. It follows that $\xi = 0 \in M \otimes_{kG} N'$ and hence the additive map (11) is injective, as needed. □

Proposition 4.2. *Let k be a commutative $\aleph_0$-Noetherian ring and G a group. Then, we have* $\operatorname{sfli} kG \leqslant \operatorname{silp} kG$.

Proof. There is nothing to prove if $\operatorname{silp} kG = \infty$ and hence we may assume that $\operatorname{silp} kG = n < \infty$. We have to show that any injective kG-module has weak dimension $\leqslant n$. Let I be an injective kG-module and consider an exact sequence of kG-modules

$$0 \longrightarrow M \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0, \tag{12}$$

with F_i projective for all $i = 0, 1, \ldots, n-1$. If H is a countable subgroup of G, then kH is a left $\aleph_0$-Noetherian ring and hence we may apply Proposition 3.2 in order to conclude that $\operatorname{sfli} kH \leqslant \operatorname{silp} kH$. Since $\operatorname{silp} kH \leqslant \operatorname{silp} kG = n$ (cf. [15, 5.1(iii)]), it follows that $\operatorname{sfli} kH \leqslant n$. Therefore, the kH-module I being injective, it has weak dimension $\leqslant n$. Since the kH-module F_i is projective and hence flat for all $i = 0, 1, \ldots, n-1$, it follows that M is flat as a kH-module. This is the case for any countable subgroup H of G and hence Lemma 4.1 implies that the kG-module M is flat. Then, the exact sequence (12) is a resolution of I by flat kG-modules of length n, as needed. □

Proposition 4.3. *Let k be a commutative $\aleph_0$-Noetherian ring and G a group. Then, we have* $\operatorname{spli} kG \leqslant \operatorname{silp} kG$.

Proof. The inequality is obvious if $\operatorname{silp} kG = \infty$ and hence we may assume that $\operatorname{silp} kG < \infty$. Then, we also have $\operatorname{sfli} kG < \infty$ (Proposition 4.2) and hence $\operatorname{spli} kG \leqslant \operatorname{fin.dim} kG$ (Proposition 3.3). Since $\operatorname{fin.dim} kG \leqslant \operatorname{silp} kG$ (Lemma 3.5), it follows that $\operatorname{spli} kG \leqslant \operatorname{silp} kG$. □

We recall that the self-injective dimension of k is the injective dimension of k as a k-module.

Theorem 4.4. *Let k be a commutative $\aleph_0$-Noetherian ring of finite self-injective dimension and G a group. Then, we have* $\operatorname{silp} kG = \operatorname{spli} kG$.

Proof. If $\operatorname{silp} kG$ is finite, then $\operatorname{spli} kG$ is also finite, in view of Proposition 4.3. On the other hand, Gedrich and Gruenberg have proved in [15, Theorem 2.4] that the finiteness of $\operatorname{spli} kG$ implies that $\operatorname{silp} kG$ is finite. The equality to be proved therefore follows from [15, 1.6]. □

Since the self-injective dimension of $\mathbb{Z}$ is 1, the following result (which is precisely Theorem B of the Introduction) is an immediate consequence of Theorem 4.4.

Corollary 4.5. *If G is a group, then we have* $\operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G$.

Recall that Ikenaga has defined in [21] the generalized cohomological dimension $\underline{\operatorname{cd}}\, G$ of a group G to be the supremum of all integers n, for which there exist a $\mathbb{Z}$-free $\mathbb{Z}G$-module M and a projective $\mathbb{Z}G$-module P, such that $\operatorname{Ext}^n_{\mathbb{Z}G}(M, P) \neq 0$. As an application of Corollary 4.5, we shall now prove the following finiteness criterion for G, which complements Theorem A of the Introduction.

Theorem 4.6. *The following conditions are equivalent for a group G:*

(i) $\underline{\mathrm{cd}}\, G = 0$,
(ii) $\mathrm{silp}\, \mathbb{Z}G = 1$,
(iii) $\mathrm{spli}\, \mathbb{Z}G = 1$,
(iv) *G is finite.*

Proof. (i) → (ii): If $\underline{\mathrm{cd}}\, G = 0$, then it is easily seen that projective $\mathbb{Z}G$-modules have injective dimension $\leqslant 1$ (cf. [22, Lemma 1.8(a)]).

(ii) → (iii): This follows from Corollary 4.5.

(iii) → (iv): This is the main result of [9].

(iv) → (i): This is proved in [21, §II, Corollary 2]. □

Dembegioti and Talelli conjectured in [10, Conjecture A] that the invariants $\underline{\mathrm{cd}}\, G$ and $\mathrm{spli}\, \mathbb{Z}G$ of G are related by the equality $\mathrm{spli}\, \mathbb{Z}G = \underline{\mathrm{cd}}\, G + 1$. The results obtained above provide some evidence for the validity of that equality:

Corollary 4.7. *Let G be a group. Then:*

(i) *We always have* $\underline{\mathrm{cd}}\, G \leqslant \mathrm{spli}\, \mathbb{Z}G \leqslant \underline{\mathrm{cd}}\, G + 1$.
(ii) *The equality* $\mathrm{spli}\, \mathbb{Z}G = \underline{\mathrm{cd}}\, G + 1$ *holds true if* $\underline{\mathrm{cd}}\, G = 0$.
(iii) *The equality* $\mathrm{spli}\, \mathbb{Z}G = \underline{\mathrm{cd}}\, G + 1$ *holds true if* $\underline{\mathrm{cd}}\, G = 1$.

Proof. (i) As noted in [22, Lemma 1.8], we have $\underline{\mathrm{cd}}\, G \leqslant \mathrm{silp}\, \mathbb{Z}G \leqslant \underline{\mathrm{cd}}\, G + 1$ and hence the result follows from Corollary 4.5.

(ii) This follows from Theorem 4.6.

(iii) If $\underline{\mathrm{cd}}\, G = 1$, then $1 \leqslant \mathrm{spli}\, \mathbb{Z}G \leqslant 2$ (in view of (i) above). Therefore, if we assume that $\mathrm{spli}\, \mathbb{Z}G \neq 2$, then it would follow that $\mathrm{spli}\, \mathbb{Z}G = 1$. Using Theorem 4.6 once more, this would imply that $\underline{\mathrm{cd}}\, G = 0$, a contradiction. □

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